A Priori Bounds for Solutions of Quasi-Linear Elliptic Differential Equations

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1. Introduction. Let Ω be a region in R^n , $n = 1, 2, \dots, 5$, $\partial\Omega$ denote its boundary. We consider quasi-linear elliptic differential equations of the form

$$(1) \quad L[u] \equiv - \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta}(x) D^\beta u) = f(x, u), \quad x \in \Omega,$$

subject to the boundary conditions

$$(2) \quad u(x) = 0, \quad x \in \partial\Omega,$$

where we have freely used the standard multi-index notation, *cf.* [15]. For example, $x = (x_1, \dots, x_n) \in R^n$ and if $\alpha = (\alpha_1, \dots, \alpha_n)$ is any index whose components are non-negative integers, $|\alpha| \equiv \alpha_1 + \alpha_2 + \dots + \alpha_n$ and $D^\alpha \equiv D_1^{\alpha_1} \dots D_n^{\alpha_n} \equiv \partial^{\alpha_1 + \dots + \alpha_n} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$. A basic question, both in proving the existence of a solution of (1), (2) by means of the Schauder Fixed Point Theorem, *cf.* [1], [8], and in approximating that solution numerically, *cf.* [5], [6], [7], [11], [12], and [13] is whether or not we can obtain an *a priori* bound for classical solutions of (1), (2) in the uniform norm over Ω .

The special case of $n = 2$, $L \equiv -\Delta$ has been studied by many people. In [6] a uniform norm *a priori* bound was obtained for the case in which $\partial f / \partial u \leq \gamma < 1/\rho$, where $\rho \equiv \max_{x \in \Omega} |\psi(x)|$ and $\Delta\psi(x) = -1$, $x \in \Omega$, $\psi(x) = 0$, $x \in \partial\Omega$. In [9] such an *a priori* bound was obtained for the case in which

$$\liminf_{|u| \rightarrow \infty} f(x, u)/u \geq 0,$$

and in [11] such an *a priori* bound was obtained for the case in which there exists a positive constant u_0 such that $f(x, u)/u \leq -(1/a)^2$ for all $|u| \geq u_0$, where Ω is contained in the strip $|x_1| \leq a$. In this paper, we give new conditions on the problem (1), (2) which guarantee the existence of a uniform norm *a priori* bound.

2. Main results. We assume throughout this paper that the coefficients, $a_{\alpha\beta}(x)$, $|\alpha|, |\beta| \leq 1$, are real-valued, bounded, measurable functions in Ω and

that there exists a positive constant C such that if

$$(3) \quad a(u, v) \equiv \int_{\Omega} \sum_{|\alpha|, |\beta| \leq 1} a_{\alpha\beta}(x) D^{\alpha} u(x) D^{\beta} v(x) dx, \quad \text{then}$$

$$(4) \quad a(u, u) \geq C^2 \|u\|_{W^{1,2}(\Omega)}^2 \equiv C^2 \int_{\Omega} \sum_{|\alpha| \leq 1} |D^{\alpha} u(x)|^2 dx,$$

for all $u \in W_0^{1,2}(\Omega)$, the completion of the real-valued $C_0^{\infty}(\Omega)$ functions with respect to $\|\cdot\|_{W^{1,2}(\Omega)}$. We remark that if L is strongly elliptic, *i.e.*, there exists a positive constant θ such that $\sum_{|\alpha|, |\beta| \leq 1} a_{\alpha\beta}(x) \xi^{\alpha} \xi^{\beta} \geq \theta |\xi|^2$ for all $x \in \Omega$ and all real n -vectors ξ , then by Gårding's inequality, *cf.* [15, p. 175] there exists a constant S such that the quadratic form associated with the differential operator $L + SI$ satisfies (4) and we may consider the problem

$$(5) \quad (L + SI)[u] = f(x, u) + Su, \quad x \in \Omega,$$

$$(6) \quad u(x) = 0, \quad x \in \partial\Omega,$$

which is equivalent to (1), (2).

We define the real number

$$(7) \quad \Lambda \equiv \inf_{\substack{w \in W_0^{1,2}(\Omega) \\ w \neq 0}} \frac{\int_{\Omega} \sum_{|\alpha|, |\beta| \leq 1} a_{\alpha\beta}(x) D^{\alpha} w(x) D^{\beta} w(x) dx}{\int_{\Omega} [w(x)]^2 dx}.$$

Inequality (4) yields the result that $\Lambda > 0$.

We assume that $f(x, u)$ is a measurable function with respect to $x \in \Omega$, continuous with respect to u for almost all $x \in \Omega$ and there exists a constant $\gamma < \Lambda$ such that

$$(8) \quad \frac{f(x, u) - f(x, 0)}{u} \leq \gamma < \Lambda$$

for almost all $x \in \Omega$ and all $u \neq 0$. Clearly (8) is satisfied if $f(x, u)$ is continuously differentiable with respect to u for almost all $x \in \Omega$ and $(\partial f / \partial u)(x, u) \leq \gamma < \Lambda$ for almost all $x \in \Omega$. Our first result gives an *a priori* bound in the $\|\cdot\|_{W^{1,2}(\Omega)}$ -norm for generalized solutions of (1), (2). It improves and extends Lemma 4 of [7], which considers the case of $n = 1$.

Theorem 1. *Let $u(x)$ be a generalized solution of (1), (2), *i.e.*,*

$$(9) \quad \int_{\Omega} \sum_{|\alpha|, |\beta| \leq 1} a_{\alpha\beta}(x) D^{\alpha} u D^{\beta} \phi dx = \int_{\Omega} f(x, u) \phi dx$$

for all $\phi \in W_0^{1,2}(\Omega)$. Then

$$(10) \quad \|u\|_{W^{1,2}(\Omega)} \leq \frac{1}{C} \left(\int_{\Omega} \sum_{|\alpha|, |\beta| \leq 1} a_{\alpha\beta}(x) D^{\alpha} u D^{\beta} u dx \right)^{1/2} \equiv \frac{1}{C} [a(u, u)]^{1/2} \\ \leq \frac{1}{C(\Lambda - \gamma)^{1/2}} \left(\int_{\Omega} [f(x, 0)]^2 dx \right)^{1/2} \equiv M.$$

Proof. Setting $\phi = u$ in (9) and using inequality (8) we obtain

$$\begin{aligned} a(u, u) &= \int_{\Omega} f(x, u)u \, dx \leq \int_{\Omega} f(x, 0)u \, dx + \gamma \int_{\Omega} u^2(x) \, dx \\ &\leq \left(\int_{\Omega} u^2(x) \, dx \right)^{1/2} \left(\int_{\Omega} f^2(x, 0) \, dx \right)^{1/2} + \frac{\gamma}{\Lambda} a(u, u). \end{aligned}$$

Inequality (10) follows from the fact that $\gamma < \Lambda$.

QED.

The next result follows directly from Theorem 1 and the Sobolev Imbedding Theorem, cf. [15].

Corollary. *If $n = 1$ and $u(x)$ is a generalized solution of (1), (2) then $u(x) \in C^0[0, 1]$ and $\|u\|_{C^0[0, 1]} \equiv \max_{x \in [0, 1]} |u(x)| \leq \frac{1}{2}M$, where M is the positive constant defined in (8).*

The situation for $n > 1$ is not quite as simple. We say that the differential operator L given in (1) is *regular in Ω* if and only if the following condition is satisfied. If $g \in L^p(\Omega)$ ($1 < p < \infty$) and $u \in W_0^{1,2}(\Omega) \cap L^p(\Omega)$ ($1 < p < \infty$) is such that $a(u, v) = \int_{\Omega} g(x)v(x) \, dx$ for all $v \in W_0^{1,2}(\Omega)$, then $u \in W^{2,p}(\Omega)$ and

$$(11) \quad \|u\|_{W^{2,p}(\Omega)} \leq k\{\|g\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\},$$

where k is a positive constant independent of u . We remark that any differential operator, L , satisfying (4) is regular in Ω if Ω and the coefficients $a_{\alpha\beta}(x)$ are sufficiently smooth, cf. [2, Theorem 8.2].

Theorem 2. *Let $n = 2, \dots, 5$ and L be regular in Ω . If there exist positive constants A, k, t , and ϵ such that*

$$(12) \quad |f(x, u)| \leq A + k |u|^t,$$

if $n = 2$, for all $x \in \Omega_1 - \infty < u < \infty$, or

$$(13) \quad |f(x, u)| \leq A + k |u|^{4n/(n-2)(n+2\epsilon)},$$

if $n = 3, \dots, 5$, for all $x \in \Omega, -\infty < u < \infty$, and

$$(14) \quad \frac{n}{2} + \epsilon \leq \frac{2n}{n-2}$$

and $u(x)$ is a generalized solution of (1), (2), then $u(x) \in C^0(\Omega)$ and satisfies an a priori bound in the uniform norm.

Proof. We first consider the case $n = 2$. By Sobolev's Imbedding Theorem, it suffices to show that any generalized solution, $u(x)$, satisfies an a priori bound in $W^{2,2}(\Omega)$. Hence, by inequality (11) it suffices to show that u satisfies an a priori bound in $L^2(\Omega)$ and $g(x, u)$ satisfies an a priori bound in $L^2(\Omega)$. By Sobolev's Imbedding Theorem u satisfies such an a priori bound and by a result of Vainberg, cf. [14], $g(x, u)$ satisfies such an a priori bound if and only if inequality (12) holds.

For the cases $n = 3, 4$ and 5 , it suffices by Sobolev's Imbedding Theorem to show that u satisfies an *a priori* bound in $W^{2, (n/2)+\epsilon}(\Omega)$ for some $\epsilon > 0$. By inequality (11), it suffices to show that u satisfies an *a priori* bound in $L^{(n/2)+\epsilon}(\Omega)$ and $g(x, u)$ satisfies an *a priori* bound in $L^{(n/2)+\epsilon}(\Omega)$. u satisfies such an *a priori* bound since by Sobolev's Imbedding Theorem it satisfies such an *a priori* bound in $L^{2n/(n-2)}(\Omega)$ and $(n/2) + \epsilon \leq 2n/(n-2)$. $g(x, u)$ satisfies such an *a priori* bound by a result of Vainberg, cf. [14]. QED.

We remark that for the case $n = 2$, $L = -\Delta$, Theorem 2 extends the result of [6] since $1/\rho \leq \Delta$, cf. [3].

3. An Application.

Theorem 3. *If L and $f(x, u)$ satisfy the hypotheses of Theorem 2, then (1), (2) has a generalized solution.*

Proof. By Theorem 2, if u is a generalized solution of (1), (2) then $u \in C^0(\Omega)$ and $\|u\|_{C^0(\Omega)} \leq B$ for some positive constant B . Consider the modified boundary value problem

$$(15) \quad L[u] = \tilde{f}(x, u) = \begin{cases} f(x, B+1) & \text{if } u \geq B+1 \\ f(x, u) & \text{if } |u| \leq B+1, \quad x \in \Omega, \\ f(x, -B-1) & \text{if } u \leq -B-1 \end{cases}$$

$$(16) \quad u(x) = 0, \quad x \in \partial\Omega.$$

It is easy to see the generalized solutions of (15), (16) satisfy the same *a priori* bound as those of (1), (2), and hence it suffices to show that (15), (16) has a generalized solution. However, the existence of such a solution follows by applying the Schauder Fixed Point Theorem to the mapping $L^{-1}\tilde{f}$ of $W_0^{1,2}(\Omega)$ into itself, cf. [1], [8]. QED.

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