Quasi-Linear Elliptic Boundary Value Problems

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1. Introduction. Let Ω be a region in R^n and $\partial\Omega$ denote the boundary of Ω . We consider quasi-linear elliptic boundary value problems of the form

$$(1) \quad L[u] \equiv \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u) = f(x, u, \dots),$$

$$x \in \Omega, \quad a_{\alpha\beta}(x) = a_{\beta\alpha}(x), \quad |\alpha|, |\beta| \leq m,$$

$$(2) \quad L[u] \equiv \lambda f(x, u, \dots), \quad x \in \Omega,$$

subject to the boundary conditions

$$(3) \quad D^j u(x) = 0, \quad x \in \partial\Omega, \quad 0 \leq j \leq m-1,$$

where we have freely used multi-index notation, cf. [1], [2], or [3], and $f(x, u, \dots)$ denotes a function of x, u , and possibly all derivatives $D^\alpha u$ with $|\alpha| \leq m$.

This class of problems has been studied in [1], [2], and [3] under the restrictions that the coefficients $a_{\alpha\beta}(x)$ are measurable and uniformly bounded in Ω , that there exists a positive constant C such that

$$(4) \quad (L[w], w)_{L^2(\Omega)} \geq C \|w\|_{W^{m,2}(\Omega)}^2 \equiv C \left(\sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha w(x)|^2 dx \right)$$

for all $w \in W_0^{m,2}(\Omega)$, i.e., for all w in the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{m,2}(\Omega)}$, and that f depends on x and $D^\alpha u$ with $|\alpha| \leq m-1$, but *not* $D^\alpha u$ with $|\alpha| = m$.

In this paper, we extend the results of [1] to problems in which the differential operator L satisfies a weaker "positive definite" hypothesis than (4) and f depends on x and $D^\alpha u$ with $|\alpha| \leq m$. The price of this extension is a slightly stronger hypothesis on the smoothness of the coefficients $a_{\alpha\beta}(x)$.

2. Main results. Throughout this paper the coefficients $a_{\alpha\beta}(x)$ are assumed (I) to be bounded, measurable functions such that the domain of $L, \mathfrak{D}(L)$, in $L^2(\Omega)$ can be taken to be those C^2 functions satisfying the boundary condi-

tions (3) and (II) to be such that there exists a positive constant C such that

$$(5) \quad (L[u], u)_{L^2(\Omega)} \geq C \|u\|_{L^2(\Omega)}^2$$

for all $u \in \mathfrak{D}(L)$. We remark that for assumption (I), it suffices to assume that $D^j a_{\alpha\beta}(x) \in C^0(\Omega)$ for all $|j| \leq |\alpha|$.

Let H denote the Hilbert space which is the completion of $\mathfrak{D}(L)$ with respect to the norm

$$(6) \quad \|u\|_H = (L[u], u)_{L^2(\Omega)}^{1/2}.$$

It follows from Theorem 2, p. 323 of [5] that $H \subset L^2(\Omega)$.

Theorem 1. *If L satisfies assumptions (I) and (II) and is such that any set which is bounded in H is precompact in $L^2(\Omega)$, then L^{-1} is defined as a compact mapping from $L^2(\Omega)$ to H .*

Proof. It follows from Theorem 3, p. 222 of [5] that, under these hypotheses, L has a discrete spectrum and hence, from Theorem 2, p. 461 of [5] that L^{-1} is defined from $L^2(\Omega)$ to $H \subset L^2(\Omega)$ and is compact, when viewed as a mapping from $L^2(\Omega)$ to $L^2(\Omega)$. To show that L^{-1} is a compact mapping from $L^2(\Omega)$ to H , let S be any bounded set in $L^2(\Omega)$. Since $L^{-1}(S)$ is precompact in $L^2(\Omega)$, there exists a sequence $\{u_n\}_{n=1}^\infty \subset S$ such that $\{L^{-1}u_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\Omega)$, i.e., $\lim_{n,k \rightarrow \infty} \|L^{-1}u_n - L^{-1}u_k\|_{L^2(\Omega)} = 0$. But, since

$$\begin{aligned} \|L^{-1}u_n - L^{-1}u_k\|_H^2 &= (L[L^{-1}u_n - L^{-1}u_k], L^{-1}u_n - L^{-1}u_k)_{L^2(\Omega)} \\ &\leq \|u_n - u_k\|_{L^2(\Omega)} \|L^{-1}u_n - L^{-1}u_k\|_{L^2(\Omega)} \rightarrow 0 \end{aligned}$$

as $n, k \rightarrow \infty$, $\lim_{n,k \rightarrow \infty} \|L^{-1}u_n - L^{-1}u_k\|_H = 0$ and $L^{-1}(S)$ is precompact in H . QED.

Combining Theorem 1 with the Leray-Schander Fixed Point Theorem, we have

Theorem 2. *Let L satisfy the hypotheses of Theorem 1 and $f(x, u, \dots)$ be such that the mapping $F: u \rightarrow f(x, u, \dots)$ is defined from H to $L^2(\Omega)$ and such that F is a bounded mapping, i.e., F maps bounded subsets of H into bounded subsets of $L^2(\Omega)$. Then $T \equiv L^{-1}F$ is a compact mapping from H to H and if T satisfies the Leray-Schander condition on $S_R(H) \equiv \{x \in H \mid \|x\| = R\}$, i.e., no solutions of the equation $\lambda Tx = x$, $\lambda \in (0, 1]$, lie on the sphere $S_R(H)$, then T has a fixed point in $B_R(H) \equiv \{x \in H \mid \|x\| < R\}$.*

We remark that if there exists a continuous, non-negative function $g(r)$, for $0 \leq r < \infty$, with $\|F(u)\|_{L^2(\Omega)} \leq g(\|u\|_H)$ for all $u \in H$, then F is a bounded mapping from H to $L^2(\Omega)$.

The fixed point of T given by Theorem 2 is called a generalized solution of (1), (3). It may be shown as in [1] that if the coefficients $a_{\alpha\beta}(x)$, the function $f(x, u, \dots)$, and the domain Ω are sufficiently smooth, then the above generalized solution is a classical solution.

Corollary 1. *If the mapping $F: H \rightarrow L^2(\Omega)$ is uniformly bounded, i.e., there exists a positive constant K such that $\|F(u)\|_{L^2(\Omega)} \leq K$ for all $u \in H$, then T has a fixed point and (1), (3) has a generalized solution.*

The following two results for the eigenvalue problem (2), (3) are analogues of Theorem 3 of [1].

Theorem 3. *Let the hypotheses of Theorem 2 hold. If $F(0) \neq 0$, then for any $R > 0$, there exists a $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$, there exists a nontrivial solution of $u = \lambda L^{-1}F(u)$.*

Theorem 4. *Let the hypotheses of Theorem 2 hold. Given $\lambda_0 > 0$, either there exists $u \in H$ such that $u = \lambda_0 L^{-1}F(u)$ or for any $R > 0$ there exists $u \in S_R(H)$ such that $u = \lambda L^{-1}F(u)$ for some $\lambda < \lambda_0$.*

The λ and u given by either Theorem 3 or Theorem 4 are respectively called a generalized eigenvalue and eigenfunction of (2), (3).

By using the Sobolev Imbedding Theorem, cf. Theorem I.4.1 of [2], we can give some general conditions under which the hypotheses of Theorem 2 hold.

Theorem 5. *Let L satisfy assumption (I) and assume that there exists a positive constant C such that*

$$(7) \quad (L[u], u)_{L^2(\Omega)} \geq C \|u\|_{W^{j,2}(\Omega)}^2$$

for all $u \in \mathcal{D}(L)$, for some $0 \leq j \leq m$. Then $H \subset W_0^{j,2}(\Omega)$ and L^{-1} is a compact mapping from $L^2(\Omega)$ to H . If $f(x, u, \dots)$ depends on x and $D^\alpha u$ with $|\alpha| \leq j$ and there exists a continuous, non-negative function h on $\prod_{|\beta| < j-n/2} R_\beta^+$ and a positive real number t such that

$$(8) \quad |f(x, u, \dots)| \leq h(\dots, |D^\beta u|, \dots) \cdot \{1 + \sum_{|\beta|=j-n/2} |D^\beta u|^t + \sum_{j-n/2 < |\beta| \leq j-1} |D^\beta u|^\phi + \sum_{|\beta|=j} |D^\beta u|\},$$

for all $x \in \Omega$ where $\phi = n(n - 2j + 2|\beta|)^{-1}$, then $F: u \rightarrow f(x, u, \dots)$ is a bounded mapping from H to $L^2(\Omega)$.

Theorem 6. *Let L satisfy assumption (I) and assume that there exists a positive constant C such that*

$$(9) \quad (L[w], w)_{L^2(\Omega)} \geq C \|w\|_{C^j(\Omega)}^2 \equiv C \max_{x \in \Omega} \sum_{|\alpha| \leq j} |D^\alpha w(x)|$$

for all $w \in \mathcal{D}(L)$ and some $0 \leq j \leq m-1$. Then $H \subset C^j(\Omega)$ and L^{-1} is a compact mapping from $L^2(\Omega)$ to H . If $f(x, u, \dots)$ depends on x and $D^\alpha u$ with $|\alpha| \leq j$ and there exists a continuous, non-negative function h on $\prod_{|\beta| \leq j} R_\beta^+$ such that

$$(10) \quad |f(x, u, \dots)| \leq h(\dots, |D^\beta u|, \dots)$$

for all $x \in \Omega$, then $F: u \rightarrow f(x, u, \dots)$ is a bounded mapping from H to $L^2(\Omega)$.

3. Examples. As our first example, we consider the nonlinear two-point boundary value problem

$$(11) \quad L[u] \equiv \sum_{i=0}^m (-1)^{i+1} D^i [p_i(x) D^i u(x)] \\ = f(x, u(x), \dots, D^m u(x)), \quad 0 < x < 1,$$

subject to the boundary conditions

$$(12) \quad D^j u(0) = D^j u(1) = 0, \quad 0 \leq j \leq m-1.$$

We assume that $p_i(x) \in C^i(0, 1)$, $f(x, u, \dots, D^m u)$ is continuous with respect to $x, u, \dots, D^m u$, there exists a positive constant C such that

$$(13) \quad \int_0^1 \sum_{i=0}^m p_i(x) (D^i w(x))^2 dx \leq C \|w\|_{W^{m,2}[0,1]}^2$$

for all $w(x) \in C^{2m}(0, 1)$ and satisfying the boundary conditions (12), and there exists a continuous, non-negative function h on R^m such that

$$(14) \quad |f(x, u, \dots, D^m u)| \leq h(u, \dots, D^{m-1} u) \{1 + |D^m u|\}$$

for all $x \in [0, 1]$ and all $u, Du, \dots, D^m u \in R$. Then it follows from Theorem 5 that $H \equiv W_0^{m,2}[0, 1]$ and $L^{-1}F$ is compact in H . If, in addition $f(x, u, \dots, D^m u)$ is uniformly bounded, it follows from the Corollary of Theorem 2, that $L^{-1}F$ has a fixed point and hence (11), (12) has a generalized solution. We remark that under these hypotheses this generalized solution is a classical solution.

As our second example we consider the second order, nonlinear, two-point boundary value problem

$$(15) \quad L[u] \equiv -D[p_1(x) Du] + p_0(x)u(x) = f(x, u), \quad 0 < x < 1$$

subject to the boundary conditions

$$(16) \quad u(0) = u(1) = 0.$$

We assume that $p_1(x) \in C^1(0, 1)$, $p_0(x) \in C^0(0, 1)$, $p_1(x) \geq 0$ for all $x \in (0, 1)$, $p_0(x) \geq 0$ for all $x \in (0, 1)$, and $A \equiv \int_0^1 dx/p_1(x) < \infty$. For example, taking $p_1(x) \equiv x^\sigma$, $0 < \sigma < 1$, and $p_0(x) \equiv 0$, we obtain the singular differential operator considered in [4].

Since

$$u^2(x) = \left(\int_0^x Du(x) dx \right)^2 = \left(\int_0^x \frac{1}{p_1(x)^{1/2}} p_1(x)^{1/2} Du(x) dx \right)^2 \\ \leq \int_0^1 \frac{dx}{p_1(x)} \int_0^1 p_1(x) (Du(x))^2 dx \leq A \int_0^1 p(x) (Du(x))^2 dx,$$

for all $x \in [0, 1]$, we have $\|u\|_{C^0[0,1]} \leq \sqrt{A} \|u\|_H$ for all $u \in \mathfrak{D}(L)$ and hence $H \subset C^0[0, 1]$. Thus, by Theorem 6, if we assume that $f(x, u) \in C^0([0, 1] \times R)$,

then $F: u \rightarrow f(x, u)$ is a bounded mapping of H into $L^2[0, 1]$. Moreover, it follows from the discussion on p. 246 of [5] that every bounded subset of H is precompact in $L^2[0, 1]$ and hence L^{-1} is a compact mapping from $L^2[0, 1]$ to H . Thus, $L^{-1}F$ is a compact mapping in H . If in addition $f(x, u)$ is uniformly bounded, then it follows from the Corollary of Theorem 2 that $L^{-1}F$ has a fixed point and hence (15), (16) has a generalized solution. Moreover, under these hypotheses, this generalized solution is a classical solution.

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