Unsteady Flows Behind Hydromagnetic Shocks

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Summary. The differential effects of shock fronts on the flow and field parameters in 3-dimensional unsteady flows of conducting gases have been explicitly determined with no restriction on the magnetic field. The expressions for vorticity and current density generated by hydromagnetic shocks have been obtained and some interesting physical conclusions have been made.

1. Introduction. The problem of determining the differential effects of shock fronts on the flow variables has drawn the attention of several researchers striving for increasing generality. For plane shocks in non-conducting gases Thomas (1947) [1] developed a tensorial method to solve this problem and his results were extended by Kanwal (1958) [2, 3, 4] for 3-dimensional shocks in stationary, pseudo-stationary and unsteady flows of non-conducting gases. Ram and Mishra (1966) [5] further generalized their results for hydromagnetic shocks in 3-dimensional pseudo-stationary flows. Pant and Mishra (1965) [6] solved this problem in the case of stationary flows of conducting gases by applying a transformation of co-ordinates and obtained the expression for vorticity generated by the shock under the restriction that the magnetic field acts tangentially to the shock surface. The object of the present paper is to solve this problem in the case of unsteady flows of conducting gases with no restriction on the magnetic field.

The determination of vorticity generated by a shock has also been an interesting topic for researchers. For plane shocks in non-conducting gases Truesdell (1952) [7] concluded that *the magnitude of the vorticity generated by a shock of given strength and curvature depends only on the magnitude of the tangential component of the velocity and is independent of the form of the equation of state*. Hayes (1957) [8] observed that Truesdell's statement suggests that the result is purely dynamical and thus should be derivable without recourse to any thermodynamic law and to verify it he provided a proof based on the momentum equation alone. For plane shocks in stationary flows of conducting gases Kanwal (1960) [9] concluded that *the vorticity is no longer derivable from dynamical relations as in*

non-conducting gases and the energy equation is essentially needed in the process. Ram (1967) [10] extended this result to the general case of unsteady plane flows of conducting gases by employing an easier method. In the present paper I have shown that the vorticity and current density generated by an oblique hydro-magnetic shock in 3-dimensional unsteady flows depend upon dynamical as well as thermodynamical behavior of the fluid, whereas in some particular cases of hydromagnetic shocks the vorticity is derivable purely from dynamical relations.

Let the shock configuration in 3-dimensional unsteady flows be represented by continuously differentiable functions $x_i = x_i(y^\alpha, t)$ (In this and in what follows the range of Latin indices is 1, 2, 3 and that of Greek indices is I, II. A repeated index implies summation unless stated otherwise.) with respect to a frame of reference in which it is at rest, where x_i are the rectangular Cartesian co-ordinates of a point on the shock surface and y^α are its Gaussian co-ordinates. Let G be the 'instantaneous shock velocity' defined as the velocity of a point of the shock surface in a direction normal to the surface. A point with fixed co-ordinates y^α on the shock surface at every instant moves with the shock velocity GN_i , where N_i are the components of the unit normal to the shock surface at this point. Hence as observed from a frame of reference attached to a point with fixed y^α on the shock surface and moving with the shock velocity GN_i , the shock front may be treated as stationary. Any observation from such a frame of reference will be spoken of as apparent to an observer moving with the shock velocity GN_i .

The differential equations governing the continuous, adiabatic and unsteady flow of an ideal conducting gas are [11]

$$(1.1) \quad \partial_i H_i + u_i H_{i,i} - H_i u_{i,i} + H_i u_{,i} = 0,$$

$$(1.2) \quad \partial_i \rho + u_i \rho_{,i} + \rho u_{i,i} = 0,$$

$$(1.3) \quad \rho \partial_i u_i + \rho u_i u_{i,i} + p_{,i} + (H_i H_{i,i} - H_i H_{,i})/4\pi = 0,$$

$$(1.4) \quad \partial_i \eta + u_i \eta_{,i} = 0,$$

where H_i , u_i , p , ρ and η respectively represent magnetic field components, velocity components, pressure, density and specific entropy of the gas; ∂_i and a comma followed by an index i (say) stand for partial differentiation with respect to time and x_i respectively.

Let a quantity f , if evaluated upstream (downstream) from the shock be denoted by $f_1(f)$. Let $[f]$ denote the jump in the quantity enclosed as it crosses the shock surface. Then, the jump conditions expressing the values of flow variables just behind in terms of those just ahead of the shock surface, are [12] (The kernel letter $\cdot$ below a quantity denotes its value just at the shock surface and any index behind solidus does not obey summation convention.)

$$(1.5) \quad [u_i] = \delta \dot{A}_1 \dot{V}_{1n/} (\dot{H}_{1i} - \dot{H}_{1n/} \dot{N}_i) / \dot{H}_{1n/} (1 - \dot{A}_1 - \delta \dot{A}_1) \\ - (\delta \dot{V}_{1n/} \dot{N}_i) / (1 + \delta),$$

$$(1.6) \quad [H_i] = \delta (\dot{H}_{1i} - \dot{H}_{1n/} \dot{N}_i) / (1 - \dot{A}_1 - \delta \dot{A}_1),$$

$$(1.7) \quad [p] = (\delta \dot{\rho}_1 \dot{V}_{1n/}^2) / (1 + \delta) \\ - \delta (\dot{H}_1^2 - \dot{H}_{1n/}^2) (\delta + 2 - 2\dot{A}_1 - 2\delta \dot{A}_1) / 8\pi (1 - \dot{A}_1 - \delta \dot{A}_1)^2,$$

$$(1.8) \quad [\rho] = \delta \dot{\rho}_1,$$

where

$$\dot{V}_i = u_i - G \dot{N}_i, \quad \dot{V}_{1n/} = \dot{V}_{1i} \dot{N}_i, \quad \dot{H}_{1n/} = \dot{H}_{1i} \dot{N}_i, \quad \dot{A}_1 = \dot{H}_{1n/}^2 / 4\pi \dot{\rho}_1 \dot{V}_{1n/}^2$$

and δ is the density-strength of the shock, which in a perfect gas is given by [13]

$$(1.9) \quad 2\dot{c}_1^2 = \frac{(2 + \delta - \gamma\delta) \dot{V}_{1n/}^2}{1 + \delta} \\ + \left\{ \frac{\gamma\delta - 2\delta - 2 + \dot{A}_1(\delta^2 - \gamma\delta^2 + 3\delta - \gamma\delta + 2)}{4\pi \dot{\rho}_1 (1 - \dot{A}_1 - \delta \dot{A}_1)^2} \right\} (\dot{H}_1^2 \dot{H}_{1n/}^2),$$

where γ is the specific heat ratio and $H_\alpha = H_i x_{i\alpha}$ ($x_{i\alpha} \stackrel{\text{def}}{=} \partial x_i / \partial y^\alpha$).

2. Determination of the gradients of flow variables. Let x_i be the co-ordinates of a point P on the streak-line $y^\alpha = \text{const.}$ at a distance s measured along it from the point on the shock surface with the same y^α co-ordinates. By a 'streak-line' we mean the locus of all particles which at any time in the past have crossed the shock surface from the same point. The configuration of the family of streak lines may be represented by

$$(2.1) \quad x_i = x_i(y^\alpha, s), \quad \partial_s x_i = V_i/v \quad (\partial_s \equiv \partial/\partial s, v^2 = V_i V_i)$$

with the initial conditions

$$x_i(y^\alpha, 0) = x_i(y^\alpha), \quad (\partial_s x_i)_{s=0} = \dot{V}_i/v.$$

($x_i = x_i(y^\alpha)$ represents shock configuration at a particular instant in consideration.)

In view of the transformation (2.1) we have

$$(2.2) \quad f_{,i} = \partial_s f s_{,i} + y_i^\alpha f_{,\alpha},$$

$$(2.3) \quad x_{i,i} = \delta_{ii} = V_i s_{,i}/v + x_{i\alpha} y_{,i}^\alpha \quad (\delta_{ii} \text{ being Kronecker delta}).$$

Multiplying (2.3) by N_i and $\epsilon^{\alpha\beta} \epsilon_{i\ell k} V_i x_{k\beta}$, we obtain

$$(2.4) \quad s_{,i} = v N_i / V_{n/}, \quad y_{,i}^\alpha = \epsilon^{\alpha\beta} \epsilon_{i\ell k} V_i x_{k\beta} / V_{n/}, \quad V_i y_{,i}^\alpha = 0,$$

where $\epsilon^{\alpha\beta}$ and $\epsilon_{i\ell k}$ are permutation tensors of the surface and space respectively.

The equation of state of the gas may be represented by

$$(2.5) \quad \eta = \eta(p, \rho); \quad c^2 = -(\partial\eta/\partial\rho)/(\partial\eta/\partial p),$$

where c is the velocity of sound.

The δ_i -derivative of a quantity f is defined as [14]

$$(2.6) \quad \delta_i f = \partial_i f + G N_i f_{,i}.$$

Now by virtue of (2.5), (2.6), (2.2) and (2.4) the equations (1.1), (1.2), (1.3) and (1.4) can respectively be transformed into the following forms:

$$(2.8) \quad \delta_i H_i + v \partial_s H_i - H_i y_{,i}^\alpha u_{i,\alpha} + H_i u_{i,\alpha} y_{,i}^\alpha + (v H_i N_i \partial_s u_i - H_{n/} v \partial_s u_i) / V_{n/} = 0,$$

$$(2.9) \quad \delta_i \rho + v \partial_s \rho + \rho v N_i \partial_s u_i / V_{n/} + \rho u_{i,\alpha} y_{,i}^\alpha = 0,$$

$$(2.10) \quad \rho \delta_i u_i + \rho v \partial_s u_i + p_{,\alpha} y_{,i}^\alpha + v \partial_s p N_i / V_{n/} + H_i (H_{i,\alpha} y_{,i}^\alpha - H_{i,\alpha} y_{,i}^\alpha) / 4\pi + (H_i N_i v \partial_s H_i - H_{n/} v \partial_s H_i) / 4\pi V_{n/} = 0,$$

$$(2.11) \quad v \partial_s p - c^2 v \partial_s \rho + \delta_i p - c^2 \delta_i \rho = 0.$$

From the eqs. (2.8), (2.9), (2.10) and (2.11) we obtain

$$(2.12) \quad v \partial_s \rho = -\delta_i \rho + 4\pi \rho V_{n/}^2 \{ (A - 1) \delta_i p - H^\alpha x_{i\alpha} \delta_i H_i / 4\pi + \rho V_{n/} (N_i - A H_i / H_{n/}) \delta_i u_i + \rho V_{n/}^2 (A - 1) u_{i,\alpha} y_{,i}^\alpha - (A V_{n/} p_{,\alpha} / H_{n/} - H^\beta x_{i\beta} u_{i,\alpha} / 4\pi + V_{n/} N_i H_{i,\alpha} / 4\pi) H_i y_{,i}^\alpha + V_{n/} (p_{,\alpha} + H_i H_{i,\alpha} / 4\pi) y_{,i}^\alpha N_i \} / \{ (V_{n/}^2 - c^2) (4\pi \rho V_{n/}^2 - H_{n/}^2) - V_{n/}^2 H_\alpha H^\alpha \},$$

$$(2.13) \quad v \partial_s u_i = \{ [H_{n/} V_{n/}^2 H_i - V_{n/}^2 H^2 N_i + c^2 (H_{n/}^2 - 4\pi \rho V_{n/}^2) N_i] (v \partial_s \rho + \delta_i \rho) / \rho V_{n/} - H_{n/}^2 \delta_i p N_i / \rho V_{n/} + (H_{n/} \delta_i u_i + V_{n/} \delta_i H_i) H_i N_i - V_{n/} (H_{n/} \delta_i H_i + 4\pi \rho V_{n/} \delta_i u_i) + (H_{n/} p_{,\alpha} N_i / \rho + H_{n/} V_{n/} u_{i,\alpha} + V_{n/}^2 H_{i,\alpha} - V_{n/} H_k u_{k,\alpha} N_i) \cdot H_i y_{,i}^\alpha - V_{n/}^2 (4\pi p_{,\alpha} + H_i H_{i,\alpha}) y_{,i}^\alpha \} / \{ 4\pi \rho V_{n/}^2 - H_{n/}^2 \},$$

$$(2.14) \quad v \partial_s H_i = \{ [4\pi \rho V_{n/}^2 H_i - H_{n/} H^2 N_i + c^2 H_{n/} (H_{n/}^2 - 4\pi \rho V_{n/}^2) N_i / V_{n/}^2] (v \partial_s \rho + \delta_i \rho) / \rho - H_{n/}^3 \delta_i p N_i / \rho V_{n/}^2 + H_{n/} H_i N_i (H_{n/} \delta_i u_i + V_{n/} \delta_i H_i) / V_{n/} - 4\pi \rho V_{n/} (V_{n/} \delta_i H_i + H_{n/} \delta_i u_i) - H_{n/} V_{n/} (4\pi p_{,\alpha} + H_i H_{i,\alpha}) y_{,i}^\alpha + (H_{n/}^2 p_{,\alpha} N_i / \rho V_{n/} + 4\pi \rho V_{n/}^2 u_{i,\alpha} + H_{n/} V_{n/} H_{i,\alpha} - H_{n/} H_k u_{k,\alpha} N_i) H_i y_{,i}^\alpha \} / \{ 4\pi \rho V_{n/}^2 - H_{n/}^2 \}.$$

Hence by virtue of the transformation (2.2) we have the following theorem:

Theorem 2.1. *The gradients of flow variables at any point on the streak lines behind the shock surface are given by the following equations:*

$$(2.15) \quad u_{i,i} = (v \partial_s u_i N_i) / V_{n'} + u_{i,\alpha} y_{,i}^\alpha,$$

$$(2.16) \quad H_{i,i} = (v \partial_s H_i N_i) / V_{n'} + H_{i,\alpha} y_{,i}^\alpha,$$

$$(2.17) \quad p_{,i} = (c^2 v \partial_s \rho + c^2 \delta_i \rho - \delta_i p) N_i / V_{n'} + p_{,\alpha} y_{,i}^\alpha,$$

$$(2.18) \quad \rho_{,i} = (v \partial_s \rho N_i) / V_{n'} + \rho_{,\alpha} y_{,i}^\alpha,$$

where $y_{,i}^\alpha$, $v \partial_s \rho$, $v \partial_s u_i$ and $v \partial_s H_i$ are given by (2.4), (2.12), (2.13) and (2.14) respectively.

Under the initial conditions of the transformation (2.1), as $s \rightarrow 0$, the quantities u_i , H_i , V_i , p , ρ , N_i and c tend to $\dot{u}_i$, $\dot{H}_i$, $\dot{V}_i$, $\dot{p}$, $\dot{\rho}$, $\dot{N}_i$ and $\dot{c}$ respectively.

Accordingly we conclude the following theorem:

Theorem 2.2. *The gradients of flow and field parameters at any point just behind an oblique hydromagnetic shock are determined by the equations (2.15), (2.16), (2.17) and (2.18), if the kernel letter is placed below every quantity involved there and in the eqs. (2.4), (2.12), (2.13) and (2.14).*

It is obvious from the expressions for $v \partial_s \rho$, $v \partial_s u_i$ and $v \partial_s H_i$ that the gradients of flow and field parameters behind the shock are uniquely determinable provided that

$$V_{n'}(4\pi\rho V_{n'}^2 - H_{n'}^2)\{(V_{n'}^2 - c^2)(4\pi\rho V_{n'}^2 - H_{n'}^2) - V_{n'}^2 H_{\alpha} H^{\alpha}\} \neq 0,$$

which is exactly the same condition as obtained for the existence and uniqueness of flows behind a magnetogasdynamical shock wave [15].

3. Determination of vorticity and current density. The components of vorticity and current density generated behind the shock are given by

$$(3.1) \quad \omega_i = \epsilon_{ijk} u_{k,i}$$

and

$$(3.2) \quad J_i = \epsilon_{ijk} H_{k,i} / 4\pi$$

respectively. Substituting for $u_{k,i}$ and $H_{k,i}$ in (3.1) and (3.2), we obtain

$$(3.3) \quad \omega_i = (\epsilon^{\beta\theta} x_{k\beta} x_{i\theta}) \{v H_{n'} H_k \partial_s \rho + \rho (H_{n'} u_{k,\alpha} + V_{n'} H_{k,\alpha}) H_i y_{,i}^\alpha \\ - \rho V_{n'} (4\pi p_{,\alpha} + H_i H_{i,\alpha}) y_{,k}^\alpha - 4\pi \rho^2 V_{n'} \delta_i u_k - \rho H_{n'} \delta_i H_k \\ + H_{n'} H_k \delta_i \rho\} / \{\rho (4\pi \rho V_{n'}^2 - H_{n'}^2)\} + \epsilon_{ijk} y_{,i}^\alpha u_{k,\alpha},$$

$$(3.4) \quad J_i = (\epsilon^{\beta\theta} x_{k\beta} x_{i\theta}) \{4\pi V_{n'} H_k (v \partial_s \rho + \delta_i \rho) + (H_{n'} H_{k,\alpha} + 4\pi \rho V_{n'} u_{k,\alpha}) \\ \cdot H_i y_{,i}^\alpha - H_{n'} (4\pi p_{,\alpha} + H_i H_{i,\alpha}) y_{,k}^\alpha - 4\pi \rho V_{n'} \delta_i H_k \\ - 4\pi \rho H_{n'} \delta_i u_k\} / \{4\pi (4\pi \rho V_{n'}^2 - H_{n'}^2)\} + \epsilon_{ijk} y_{,i}^\alpha H_{k,\alpha} / 4\pi.$$

The presence of $v \partial_{,\rho}$ in (3.3) and (3.4) indicates that the energy equation (2.11) is essentially needed for the derivation of vorticity and current density in order to determine the value of $v \partial_{,\rho}$. This shows that *the vorticity and current density generated by an oblique hydromagnetic shock depend upon the dynamical as well as thermodynamical behavior of the fluid*. In the case of plane flows the same physical conclusion was made by a different approach [10].

Multiplying (3.3) and (3.4) by N_i , we get

$$(3.5) \quad \omega_{n/} = \epsilon^{\alpha\theta} x_{k\theta} u_{k,\alpha}, \quad J_{n/} = \epsilon^{\alpha\theta} x_{k\theta} H_{k,\alpha} / 4\pi.$$

The absence of $v \partial_{,\rho}$ in (3.5) indicates that *the normal vorticity and current density can be derived purely from dynamical considerations*. This physical fact has also been established by a different approach [16].

The quantities $\delta_i u_i$, $\delta_i H_i$, $\delta_i p$, $\delta_i \rho$, $u_{i,\alpha}$, $H_{i,\alpha}$ and $p_{,\alpha}$ involved in the expressions for gradients, vorticity and current density are expressible in terms of known quantities. Such quantities depend upon the shape, speed and strength of the shock and the flow and field parameters just in front of the shock as shown below:

We assume that the flow upstream from the shock is uniform and known and the lines of curvature of the shock surface are the Gaussian co-ordinate curves. Keeping in mind this assumption and making use of the relation [12] for the time-derivative of N_i as apparent to an observer moving with the shock velocity GN_i as given by

$$(3.6) \quad \delta_i N_i = -a^{\alpha\beta} G_{,\alpha} x_{i\beta},$$

where $a^{\alpha\beta}$ are the contravariant components of the metric tensor of the shock surface, we obtain

$$(3.7) \quad \delta_i V_{1n/} = -(u_1^\alpha G_{,\alpha} + \delta_i G), \quad \delta_i H_{1n/} = -H_1^\alpha G_{,\alpha}.$$

Now applying the operator δ_i on the equations (1.5), (1.6), (1.7) and (1.8) and using (3.6) and (3.7), we get

$$(3.8) \quad \begin{aligned} \delta_i u_i = & \left\{ -\frac{V_{1n/} N_i}{(1 + \delta)^2} + \frac{A_1(1 - A_1) V_{1n/} H_1^\alpha x_{i\alpha}}{H_{1n/}(1 - A_1 - \delta A_1)^2} \right\} \delta_i \delta \\ & + \left\{ \frac{\delta N_i}{1 + \delta} + \frac{\delta A_1(1 + A_1 + \delta A_1) H_1^\alpha x_{i\alpha}}{H_{1n/}(1 - A_1 - \delta A_1)^2} \right\} (u_1^\alpha G_{,\alpha} + \delta_i G) \\ & + \frac{\delta V_{1n/} a^{\alpha\beta} G_{,\alpha} x_{i\beta}}{(1 + \delta)(1 - A_1 - \delta A_1)} - \left\{ \frac{(1 + A_1 + \delta A_1) H_{1i} - 2 H_{1n/} N_i}{4\pi \rho_1 V_{1n/}(1 - A_1 - \delta A_1)^2} \right\} (\delta H_1^\alpha G_{,\alpha}), \end{aligned}$$

$$(3.9) \quad \delta_i H_i = \frac{(1 - A_1) H_1^\alpha x_{i\alpha} \delta_i \delta}{(1 - A_1 - \delta A_1)^2} + \left\{ \frac{2 A_1 \delta(1 + \delta) H_1^\alpha x_{i\alpha}}{V_{1n/}(1 - A_1 - \delta A_1)^2} \right\} (u_1^\alpha G_{,\alpha} + \delta_i G)$$

$$\begin{aligned}
& + \delta \left\{ \frac{(1 + A_1 + \delta A_1)H_{1n}N_i - 2A_1(1 + \delta)H_{1i}}{H_{1n}(1 - A_1 - \delta A_1)^2} \right\} (H_1^\alpha G_{,\alpha}) \\
& + \frac{\delta H_{1n} a^{\alpha\beta} G_{,\alpha} x_{i\beta}}{1 - A_1 - \delta A_1}, \\
(3.10) \quad & \delta_i p = \left\{ \frac{\rho_1 V_{1n}^2}{(1 + \delta)^2} - \frac{(1 + \delta)(1 - A_1)^2(H_1^2 - H_{1n}^2)}{4\pi(1 - A_1 - \delta A_1)^3} \right\} \delta_i \delta \\
& - \left\{ \frac{2\delta\rho_1 V_{1n}}{1 + \delta} + \frac{\delta(1 + \delta)^2 A_1(1 - A_1)}{2\pi V_{1n}(1 - A_1 - \delta A_1)^3} (H_1^2 - H_{1n}^2) \right\} (u_1^\alpha G_{,\alpha} + \delta_i G) \\
& + \left\{ \frac{A_1(1 - A_1)(1 + \delta)^2}{2\pi H_{1n}(1 - A_1 - \delta A_1)^3} (H_1^2 - H_{1n}^2) \right. \\
& \left. - \frac{H_{1n}(\delta + 2 - 2A_1 - 2\delta A_1)}{4\pi(1 - A_1 - \delta A_1)^2} \right\} (\delta H_1^\alpha G_{,\alpha}), \\
(3.11) \quad & \delta_i \rho = \rho_1 \delta_i \delta.
\end{aligned}$$

The value of $\delta_i \delta$ can be easily calculated for a perfect gas by applying the operator δ_i on the eq. (1.9) and making use of (3.6) and (3.7).

Using the Weingarten's formula for the surface derivative of N_i as given by

$$(3.12) \quad N_{i,\alpha} = -a^{\theta\gamma} b_{\beta\alpha} x_{i\theta},$$

where $b_{\beta\alpha}$ are the covariant components of the second fundamental tensor of the surface, we obtain

$$(3.13) \quad V_{1n,\alpha} = -(K_\alpha u_{1\alpha} + G_{,\alpha}), \quad H_{1n,\alpha} = -K_\alpha H_{1\alpha} \quad (\alpha \text{ unsummed}),$$

where K_α are the normal curvatures of the shock surface.

Now differentiating (1.5), (1.6), (1.7) and (1.8) with respect to y^α and making use of (3.12) and (3.13), we can easily obtain similar expressions for $u_{i,\alpha}$, $H_{i,\alpha}$, $p_{,\alpha}$ and $\rho_{,\alpha}$.

4. Flows behind special hydromagnetic shocks. The vortex lines generated by shocks in non-conducting gases are tangential to the shock surface and the magnitude of vorticity depends only on the tangential components of velocity [2]. But this conclusion is no longer true for vortex lines generated by an oblique hydromagnetic shock in which the magnetic field is neither normal nor tangential to the shock surface. It can easily be seen from (3.3) and (3.4) that *the vortex lines and current lines generated by an oblique hydromagnetic shock are oblique to the shock surface and the magnitudes of vorticity and current density depend upon both magnetohydrodynamic and thermodynamic parameters of the flow.*

Let us now consider the flow behind a 'perpendicular hydromagnetic shock' in which the magnetic field is perpendicular to the direction of propagation of

the shock. In such shocks the magnetic field acts as if glued to the matter and is compressed by the shock to the same extent as the fluid. In this case we have the condition

$$(4.1) \quad H_{n'} = H_{1n'} = 0 \quad \text{and} \quad H_\alpha = (1 + \delta)H_{1\alpha} \neq 0.$$

In consequence of (4.1) the jump conditions take simpler forms and the expressions for vorticity and current density generated by a perpendicular hydromagnetic shock assume the following forms:

$$(4.2) \quad \omega_i = - \left\{ \frac{\delta^2}{1 + \delta} \epsilon^{\alpha\theta} x_{i\theta} + \frac{T(1 + \delta)}{4\pi\rho_1 V_{1n'}} H_{1\beta} H_1^\alpha \epsilon^{\beta\theta} x_{i\theta} \right\} (K_\alpha u_{1\alpha} + G_{,\alpha}),$$

(This result agrees with that of Kanwal [4] and Hayes [8] in the absence of magnetic field.)

$$(4.3) \quad J_i = \frac{(1 + \delta)v}{4\pi\rho_1 V_{1n'}} \partial_\alpha \rho H_{1\beta} \epsilon^{\beta\theta} x_{i\theta} + \frac{\delta(1 + \delta)}{4\pi} K_\alpha H_{1\alpha} \epsilon^{\alpha\theta} x_{i\theta} - \frac{T(1 + \delta)}{4\pi V_{1n'}} \cdot (H_{1\theta} \epsilon^{\alpha\theta} V_i - H_1^\beta V_{1\beta} \epsilon^{\alpha\theta} x_{i\theta}) (K_\alpha u_{1\alpha} + G_{,\alpha}),$$

where

$$T = \{2(2 - \gamma \delta + \delta) V_{1n'}\} / \left\{ \frac{\gamma + 1}{1 + \delta} V_{1n'}^2 + (2 - \gamma)(1 + \delta) \frac{H_1^2}{4\pi\rho_1} \right\}.$$

The expressions for vorticity and current density generated by a perpendicular hydromagnetic shock show that *the vortex lines are tangential to the shock surface as in non-conducting gases and the magnitude of vorticity depends only on the magnetohydrodynamic parameters of the flow and does not depend on the thermodynamic behavior of the fluid, whereas the current lines are oblique to the shock surface and the magnitude of current density depends upon both dynamical and thermodynamical behavior of the fluid.*

It is also interesting to conclude here that *the coincidence of vortex lines with stream-lines and that of current lines with magnetic lines is impossible behind a perpendicular hydromagnetic shock.*

Next we consider the flow behind a 'parallel hydromagnetic shock' in which the magnetic field is parallel to the direction of propagation of the shock. In such shock the magnetic field has no influence on the hydrodynamical phenomenon caused by the shock and the situation may be completely described by the classical jump conditions of pure gas shocks. In this case we have the condition

$$(4.4) \quad H_\alpha = H_{1\alpha} = 0 \quad \text{and} \quad H_{n'} = H_{1n'} \neq 0.$$

In consequence of (4.4) the jump conditions (1.5), (1.6), (1.7) and (1.8) reduce to those for shocks in non-conducting gases. Under these conditions the expressions for vorticity and current density assume the following forms:

$$(4.5) \quad \omega_i = \left\{ \frac{4(1 + \delta)A_1}{\gamma + 1} - \frac{\delta(\delta + A_1 + \delta A_1)}{1 + \delta} \right\} \left\{ \frac{K_\alpha u_{1\alpha} + G_{,\alpha}}{1 - A_1 - \delta A_1} \right\} \epsilon^{\alpha\theta} x_{i\theta} ,$$

(In the absence of magnetic field it agrees with that of Kanwal [4].)

$$(4.6) \quad J_i = \left\{ \frac{(1 + \delta)(4 - \gamma\delta - \delta)H_{1n/}}{4\pi(\gamma + 1)V_{1n/}} \right\} \left\{ \frac{K_\alpha u_{1\alpha} + G_{,\alpha}}{1 - A_1 - \delta A_1} \right\} \epsilon^{\alpha\theta} x_{i\theta} .$$

Hence it can be easily concluded from (4.5) and (4.6) that *the vortex lines and current lines generated behind a parallel hydromagnetic shock are tangential to the shock surface and the magnitudes of vorticity and current density depend on magnetohydrodynamic parameters but do not depend on the thermodynamic behavior of the fluid. The coincidence of vortex lines with stream lines and that of current lines with magnetic lines is again impossible behind a parallel hydromagnetic shock.*

Lastly we consider the flow behind a 'switch-off hydromagnetic shock' in which the velocity vector and the magnetic field vector are oblique to the shock surface on the upstream side and normal to it on the downstream side. In such shocks the tangential components of gas velocity and magnetic field are switched off while crossing the shock surface. In this case we have the conditions

$$(4.7) \quad H_\alpha = 0, \quad H_{1\alpha} \neq 0, \quad u_\alpha = 0, \quad u_{1\alpha} \neq 0, \quad A_1 = 1, \quad u_1^\alpha u_{1\alpha} = \frac{H_1^\alpha H_{1\alpha}}{4\pi\rho_1}.$$

In consequence of (4.7) the jump conditions assume the following forms:

$$(4.8) \quad [u_i] = -(u_{1i} - u_{1n/}N_i) - \frac{\delta}{1 + \delta} V_{1n/}N_i ,$$

$$(4.9) \quad [H_i] = -(H_{1i} - H_{1n/}N_i),$$

$$(4.10) \quad [p] = \frac{\delta}{1 + \delta} \rho_1 V_{1n/}^2 + \frac{1}{2}\rho_1(u_1^2 - u_{1n/}^2),$$

$$(4.11) \quad [\rho] = \delta\rho_1 ,$$

where δ is given by

$$(4.12) \quad \delta^2 \{2c_1^2 + (\gamma - 1)(G^2 - 2Gu_{1n/} + u_1^2)\} \\ + \delta \{2c_1^2 - 2V_{1n/}^2 + (\gamma - 2)u_1^\alpha u_{1\alpha}\} - u_1^\alpha u_{1\alpha} = 0.$$

In view of the above relations the expressions for vorticity and current density generated behind a switch-off hydromagnetic shock are of the forms

$$(4.13) \quad \omega_i = \left\{ \frac{V_{1n/}}{\delta(1 + \delta)} \frac{\delta_{,\alpha}}{\delta} - \frac{K_\alpha u_{1\alpha} + G_{,\alpha}}{1 + \delta} - \frac{(\delta + 1)(2u_{1n/} - G)G_{,\alpha}}{\delta V_{1n/}} \right\} \epsilon^{\alpha\theta} x_{i\theta} ,$$

$$(4.14) \quad J_i = \left\{ \frac{(1 + \delta)H_{1n/}}{4\pi\delta V_{1n/}^2} [(\delta - 1)V_{1n/} - (\delta + 1)u_{1n/}]G_{,\alpha} \right\} \epsilon^{\alpha\theta} x_{i\theta} ,$$

$$+ \frac{H_{1n/} \delta_{,\alpha}}{4\pi \delta(1 + \delta)} - \frac{H_{1n/}(K_{\alpha} u_{1\alpha} + G_{,\alpha})}{2\pi V_{1n/}} + \frac{1}{4\pi} K_{\alpha} H_{1\alpha} \Big\} \epsilon^{\alpha\theta} x_{i\theta} .$$

Hence from the eqs. (4.13) and (4.14) we arrive at the same conclusion which has been established in flows behind parallel hydromagnetic shocks.

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