The Krein-Krasnoselskii-Kooi Condition for a Second Order Hyperbolic Partial Differential Equation with Complete Second Member

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1. Introduction. In 1956 Krasnoselskii and Krein [1] published a uniqueness criterium for the initial value problem for a first order ordinary differential equation. In the same paper they showed with an example the optimality of the conditions imposed. In 1958 Kooi [2] generalized this criterium. Furthermore, also in 1958 Luxenburg [3] proved that the Krasnoselskii-Krein-Kooi (K-K-K) condition also implied the convergence of the successive approximations to this unique solution. Moreover, he gave an example showing the optimality of the conditions on the parameters.

In 1964 Palczewski and Pawelski [4, 5] gave an analog of the Krasnoselskii-Krein condition for the problem $(\bar{P})u_{xy} = f(x, y, u)$, $u(x, 0) = \sigma(x)$, $u(0, y) = \tau(y)$, with suitable conditions on the functions involved. They showed uniqueness and convergence of the successive approximations and gave the corresponding examples of optimality. They also announced similar results for the equation in n variables, namely $u_{x_1 \dots x_n} = f(x_1, \dots, x_n, u)$.

In the present paper, we give the analogue of the K-K-K criterion, slightly generalized, for the "complete" equation, $u_{xy} = f(x, y, u, u_x, u_y)$, and give (without proof) the corresponding result for the complete equation in n variables. Also, we give examples showing the optimality of the conditions on the parameters.

A future paper will deal with the case where the order of the equation exceeds the number of variables involved, for ordinary as well as for partial differential equations.

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2. Definition. Let $f(x, y, u, v, w)$ be defined in

$$D = \begin{cases} (x, y) \in R = [0, a] \times [0, b] \\ -\infty < u, v, w < +\infty \end{cases}$$

We say that f satisfies in D a generalized K-K-K condition of order $(1, 1)$ if, for $(x, y, u, v, w), (x, y, \bar{u}, \bar{v}, \bar{w}) \in D$

- A) $xy |f(x, y, u, v, w) - f(x, y, \bar{u}, \bar{v}, \bar{w})|$
 $\leq K_1(x, y) |u - \bar{u}| + K_2(x, y)x |v - \bar{v}| + K_3(x, y)y |w - \bar{w}|$
 B) $(xy)^\beta |f(x, y, u, v, w) - f(x, y, \bar{u}, \bar{v}, \bar{w})|$
 $\leq C_1(x, y) |u - \bar{u}|^\alpha + C_2(x, y)x^\alpha |v - \bar{v}|^\alpha + C_3(x, y)y^\alpha |w - \bar{w}|^\alpha$
 C) $|f| \leq H,$

where

- a) $K_i(x, y) \in C(R), \quad K_i(x, y) \geq 0 \quad (i = 1, 2, 3)$
 $K = \max_R \{ \max_i [K_i(x, y)] \} > 1$
 b) $C_i(x, y) \in C(R), \quad C_i(x, y) \geq 0 \quad (i = 1, 2, 3)$
 $C = \max_R \{ \max_i [C_i(x, y)] \} > 0$
 c) $0 < \alpha < 1, \quad \beta < \alpha, \quad H > 0$
 d) $p(K)^{1/2}(1 - \alpha) < 1 - \beta, \quad p^2 = 1 + 2p(K)^{1/2}, \quad p > 0.$

We note that for $\beta = 0$ this definition neither contains nor is contained in the one in [4], in spite of the fact that in that paper f is restricted to depend only on (x, y, u) , and in A), B) they use simply K and C . For there $p = 1$, whereas in our case, $p > 1$, as seen by inspection.

3. Theorem *Let f be defined and continuous in D , where it satisfies a generalized K-K-K condition of order $(1, 1)$. Then the problem*

$$(P) \begin{cases} u_{xy} = f(x, y, u, u_x, u_y) \\ u(x, 0) = \sigma(x), \quad u(0, y) = \tau(y) \quad (\sigma(0) = \tau(0)) \\ \sigma(x) \in C^1([0, a]), \quad \tau(y) \in C^1([0, b]) \end{cases}$$

has one and only one solution in $\bar{C}(R) = \{u(x, y) \in C(R) : u_x(x, y), u_y(x, y), u_{xy}(x, y) \in C(R)\}$.

Furthermore, this solution can be obtained by successive approximations.

Again, we note that this theorem neither contains nor is contained in Theorem 1 of [4] or Theorem 2 of [5], for the same reasons stated before.

Along in the proof it will be seen that the continuity conditions on $\{K_i(x, y)\}$ and $\{C_i(x, y)\}$ could be relaxed to, say, boundedness and integrability.

4. *Proof.* Let $X = \{u(x, y) \in \bar{C}(R): u(x, 0) = \sigma(x), u(0, y) = \tau(y)\}$ and, for $u, v \in X$, let

$d(u, v)$

$$= \max_x \frac{|u(x, y) - v(x, y)| + x |u_x(x, y) - v_x(x, y)| + y |u_y(x, y) - v_y(x, y)|}{(xy)^{rp(K)^{1/2}}},$$

where $r > 1$ is such that $rp(K)^{1/2}(1 - \alpha) < 1 - \beta$.

Define for $u \in X$,

$$Tu = u_0(x, y) + \int_0^x \int_0^y f(s, t, u, u_x, u_y) ds dt,$$

where $u_0(x, y)$ is the solution of the homogeneous problem corresponding to (P). (The existence and uniqueness of such a solution follows the same lines as this proof, taking $u_0(x, y) \equiv 0$).

Clearly, T maps X into itself. We want to show that with these definitions Luxemburg's fixed point theorem [3] can be applied.

The fact that (X, d) is a generalized complete metric space is shown as in [3]. We still have to show

- 1) $d(u, v) < \infty \Rightarrow d(Tu, Tv) \leq \rho d(u, v), 0 < \rho < 1$
- 2) $u_0 \in X, u_n = Tu_{n-1} (n = 1, 2, \dots) \Rightarrow \exists N(u_0) \text{ s.t. } d(u_n, u_{n+h}) < \infty \text{ for } n \geq N, h = 1, 2, \dots$
- 3) $Tu = u, Tv = v \Rightarrow d(u, v) < \infty$.

Assume $d(u, v) < \infty$. Then

$$\begin{aligned} & |(Tu)(x, y) - (Tv)(x, y)| \\ & \leq \int_0^x \int_0^y |f(s, t, u, u_x, u_y) - f(s, t, v, v_x, v_y)| ds dt \\ & \leq K \int_0^x \int_0^y \frac{|u - v| + s |u_x - v_x| + t |u_y - v_y|}{st} ds dt \\ & \leq K d(u, v) \int_0^x \int_0^y (st)^{rp(K)^{1/2}-1} ds dt = K d(u, v) \frac{(xy)^{rp(K)^{1/2}}}{r^2 p^2 K}, \end{aligned}$$

and similarly

$$\begin{aligned} & | \langle Tu \rangle_x(x, y) - \langle Tv \rangle_x(x, y) | \leq K d(u, v) \frac{(xy)^{rp(K)^{1/2}}}{rp(K)^{1/2}} \\ & | \langle Tu \rangle_y(x, y) - \langle Tv \rangle_y(x, y) | \leq K d(u, v) \frac{(xy)^{rp(K)^{1/2}}}{rp(K)^{1/2}}. \end{aligned}$$

Hence

$$d(Tu, Tv) \leq K d(u, v) \left[\frac{1}{r^2 p^2 K} + \frac{2}{rp(K)^{1/2}} \right] < \frac{1}{r} d(u, v)$$

and 1) holds with $\rho = 1/r$.

Let $u_0 \in X$, and $u_n = Tu_{n-1}$ ($n = 1, 2, \dots$). Then, using C)

$$\begin{aligned} |u_{n+1}(x, y) - u_n(x, y)| &\leq 2Hxy \\ x |u_{n+1_x}(x, y) - u_{n_x}(x, y)| &\leq 2Hxy \\ y |u_{n+1_y}(x, y) - u_{n_y}(x, y)| &\leq 2Hxy. \end{aligned}$$

Using these estimates and B)

$$\left. \begin{aligned} &|u_{n+1}(x, y) - u_n(x, y)| \\ x |u_{n+1_x}(x, y) - u_{n_x}(x, y)| \\ y |u_{n+1_y}(x, y) - u_{n_y}(x, y)| \end{aligned} \right\} \leq 3C(2H)^\alpha (xy)^{1+\alpha-\beta}.$$

In general,

$$\left. \begin{aligned} &|u_{n+1}(x, y) - u_n(x, y)| \\ x |u_{n+1_x}(x, y) - u_{n_x}(x, y)| \\ y |u_{n+1_y}(x, y) - u_{n_y}(x, y)| \end{aligned} \right\} \leq (3C)^{1+\alpha+\dots+\alpha^{n-1}} (2H)^\alpha (xy)^{(1-\beta)(1+\dots+\alpha^{n-1})+\alpha^n} \quad n = 1, 2, \dots$$

Since $rp(K)^{1/2}(1-\alpha) < 1-\beta$, there exists $N(r)$ such that, if $n \geq N$, $(1-\beta) \cdot (1+\alpha+\dots+\alpha^{n-1}) + \alpha^n > rp(K)^{1/2}$. Let $A = (3C)^{1/(1-\alpha)} \max(2H, 1)$. Then, if $n \geq N$

$$\frac{|u_{n+1} - u_n| + x |u_{n+1_x} - u_{n_x}| + y |u_{n+1_y} - u_{n_y}|}{(xy)^{rp(K)^{1/2}}} \leq A(xy)^{\rho_n},$$

with $\rho_n = (1-\beta)(1+\alpha+\dots+\alpha^{n-1}) + \alpha^n - rp(K)^{1/2} > 0$. Therefore $d(u_{n+1}, u_n) < \infty$ when $n \geq N(r)$, and we have 2).

Assume $Tu = u$, $Tv = v$. Then, with the same procedure as before

$$d(u, v) = d(Tu, Tv) \leq A(xy)^{(1-\beta)/(1-\alpha)-rp(K)^{1/2}}$$

and 3) holds.

Therefore, by Luxemburg's Theorem, T has a unique fixed point; moreover, this fixed point can be obtained by successive approximations.

We note that the result is valid for the metric d which is clearly stronger than the one obtained when using uniform convergence.

5. Examples. (see [5]). We will show that the condition d) of our definition cannot be improved to get a stronger result.

I. Uniqueness. Let

$$f_1(x, y, u) = \begin{cases} 0 & u \leq 0 \\ \frac{ku}{xy} & 0 < u \leq \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \frac{1}{p(k)^{1/2}} \\ \frac{1}{(xy)^\beta} \left[\frac{(xy)^{1-\beta}}{k} \right]^{\alpha/(1-\alpha)} \frac{1}{p(k)^{1/2}} & u > \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \frac{1}{p(k)^{1/2}} \end{cases}$$

$$f_2(x, y, v) = \begin{cases} 0 & v \leq 0 \\ \frac{kv}{y} & 0 < v \leq \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \frac{1}{x} \\ \frac{1}{(xy)^\beta} \left[\frac{(xy)^{1-\beta}}{k} \right]^{\alpha/(1-\alpha)} & v > \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \frac{1}{x} \end{cases}$$

$$f_3(x, y, w) = f_2(y, x, w).$$

It can be shown that

$$(\#) \left\{ \begin{aligned} |f_1(x, y, u) - f_1(x, y, \bar{u})| &\leq \begin{cases} k \frac{|u - \bar{u}|}{xy} \\ |u - \bar{u}|^\alpha / (xy)^\beta \end{cases} \\ |f_2(x, y, v) - f_2(x, y, \bar{v})| &\leq \begin{cases} k \frac{|v - \bar{v}|}{y} \\ x^\alpha \frac{|v - \bar{v}|^\alpha}{(xy)^\beta} \end{cases} \\ |f_3(x, y, w) - f_3(x, y, \bar{w})| &\leq \begin{cases} k \frac{|w - \bar{w}|}{x} \\ y^\alpha \frac{|w - \bar{w}|^\alpha}{(xy)^\beta} \end{cases} \end{aligned} \right.$$

Let $f(x, y, u, v, w) = f_1(x, y, u) + f_2(x, y, v) + f_3(x, y, w)$. Then by (#),

$$\begin{aligned} xy |f(x, y, u, v, w) - f(x, y, \bar{u}, \bar{v}, \bar{w})| &\leq k(|u - \bar{u}| + x |v - \bar{v}| + y |w - \bar{w}|) \\ (xy)^\beta |f(x, y, u, v, w) - f(x, y, \bar{u}, \bar{v}, \bar{w})| &\leq (|u - \bar{u}|^\alpha + x^\alpha |v - \bar{v}|^\alpha + y^\alpha |w - \bar{w}|^\alpha). \end{aligned}$$

Clearly, f is bounded for $xy \geq 0$.

Consider the following problem:

$$\begin{aligned} u_{xy} &= f(x, y, u, u_x, u_y) \\ u(x, 0) &\equiv u(0, y) \equiv 0, \quad 0 \leq x, y \leq 1. \end{aligned}$$

Obviously, $u(x, y) \equiv 0$ is a solution.

We will show that if $p(k)^{1/2}(1 - \alpha) \geq 1 - \beta$, with $p^2 = 1 + 2p(k)^{1/2}$, $p > 0$, then $\hat{u}(x, y) = A(xy)^{p(k)^{1/2}}$, with

$$A = \frac{1}{p(k)^{1/2}} \frac{1}{k^{1/(1-\alpha)}}, \text{ is also a solution.}$$

We have

$$\begin{aligned} x\hat{u}_x(x, y) &= Ap(k)^{1/2}(xy)^{p(k)^{1/2}} = p(k)^{1/2}\hat{u}(x, y) \\ y\hat{u}_y(x, y) &= Ap(k)^{1/2}(xy)^{p(k)^{1/2}} = p(k)^{1/2}\hat{u}(x, y) \\ xy\hat{u}_{xy}(x, y) &= Ap^2k(xy)^{p(k)^{1/2}} = (1 + 2p(k)^{1/2})k\hat{u}(x, y) \\ &= k\hat{u}(x, y) + kx\hat{u}_x(x, y) + ky\hat{u}_y(x, y). \end{aligned}$$

Hence, if $xy \neq 0$,

$$\hat{u}_{xy}(x, y) = k \frac{\hat{u}(x, y)}{xy} + k \frac{\hat{u}_x(x, y)}{y} + k \frac{\hat{u}_y(x, y)}{x}.$$

Also, it is clear that $\hat{u}(x, 0) \equiv \hat{u}(0, y) \equiv 0$. It remains to be shown that $\hat{u}_{xy}(x, y) = f(x, y, \hat{u}, \hat{u}_x, \hat{u}_y)$, *i.e.*,

$$0 \leq \left\{ \begin{array}{c} p(k)^{1/2}\hat{u}(x, y) \\ x\hat{u}_x(x, y) \\ y\hat{u}_y(x, y) \end{array} \right\} \leq \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)}.$$

The left hand side inequalities are clear. Now

$$\begin{aligned} p(k)^{1/2}\hat{u}(x, y) &\leq \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \Leftrightarrow p(k)^{1/2}A(xy)^{p(k)^{1/2}} \leq \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \\ &\Leftrightarrow A^{1-\alpha}(p(k)^{1/2})^{1-\alpha}(xy)^{p(k)^{1/2}(1-\alpha)} \leq \frac{(xy)^{1-\beta}}{k} \\ &\Leftrightarrow A^{1-\alpha}(p(k)^{1/2})^{1-\alpha}k(xy)^{p(k)^{1/2}(1-\alpha)} = (xy)^{p(k)^{1/2}(1-\alpha)} \leq (xy)^{1-\beta} \end{aligned}$$

Since $p(k)^{1/2}(1 - \alpha) \geq 1 - \beta$ and $0 \leq xy \leq 1$, the last inequality holds. A simple computation shows that the other two inequalities reduce to the first one. Therefore, our assertion is true.

II. *Convergence of successive approximations.* Since $p < 3(k)^{1/2}$, choose $\beta < \alpha$ so that $p(1 - \beta)/(1 - \alpha) < 3(k)^{1/2}$, and let

$$f_1(x, y, u) = \begin{cases} \frac{1}{3} \frac{p}{(xy)^\beta} \left[\frac{(xy)^{1-\beta}}{k} \right]^{\alpha/(1-\alpha)} & u \leq 0 \\ \frac{1}{3} \frac{p}{(xy)^\beta} \left[\frac{(xy)^{1-\beta}}{k} \right]^{\alpha/(1-\alpha)} - \frac{1}{3} p(k)^{1/2} \frac{1-\beta}{1-\alpha} \frac{u}{xy} & 0 < u \leq (k)^{1/2} \frac{1-\alpha}{1-\beta} \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \\ 0 & u > (k)^{1/2} \frac{1-\alpha}{1-\beta} \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \end{cases}$$

$$f_2(x, y, v) = \begin{cases} \frac{1}{3} \frac{p}{(xy)^\beta} \left[\frac{(xy)^{1-\beta}}{k} \right]^{\alpha/(1-\alpha)} & v \leq 0 \\ \frac{1}{3} \frac{p}{(xy)^\beta} \left[\frac{(xy)^{1-\beta}}{k} \right]^{\alpha/(1-\alpha)} - \frac{1}{3} p(k)^{1/2} \frac{v}{y} & 0 < v \leq \frac{k}{x} \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \\ 0 & v > \frac{k}{x} \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \end{cases}$$

$$f_3(x, y, w) = f_3(y, x, w).$$

As we did with the previous example, it is possible to show that

$$|f_1(x, y, u) - f_1(x, y, \bar{u})| \leq \begin{cases} k \frac{|u - \bar{u}|}{xy} \\ C \frac{|u - \bar{u}|^\alpha}{(xy)^\beta} \end{cases}$$

$$|f_2(x, y, v) - f_2(x, y, \bar{v})| \leq \begin{cases} k \frac{|v - \bar{v}|}{y} \\ C x^\alpha \frac{|v - \bar{v}|^\alpha}{(xy)^\beta} \end{cases}$$

$$|f_3(x, y, w) - f_3(x, y, \bar{w})| \leq \begin{cases} k \frac{|w - \bar{w}|}{x} \\ C y^\alpha \frac{|w - \bar{w}|^\alpha}{(xy)^\beta}, \end{cases}$$

where

$$C = \max \left\{ \left(\frac{\gamma}{p} \right)^{1-\alpha}, \left(\frac{p}{3} \right)^{1-\alpha}, \frac{p}{3} \gamma^\alpha \right\} \quad \text{and} \quad \gamma = p(k)^{1/2} \frac{1-\alpha}{1-\beta};$$

hence, $\gamma \geq 1$.

Define $f(x, y, u, v, w) = f_1(x, y, u) + f_2(x, y, v) + f_3(x, y, w)$. Then

$$|f(x, y, u, v, w) - f(x, y, \bar{u}, \bar{v}, \bar{w})| \leq \begin{cases} k \left(\frac{|u - \bar{u}|}{xy} + \frac{|v - \bar{v}|}{y} + \frac{|w - \bar{w}|}{x} \right) \\ C \frac{|u - \bar{u}|^\alpha + x^\alpha |v - \bar{v}|^\alpha + y^\alpha |w - \bar{w}|^\alpha}{(xy)^\beta} \end{cases}.$$

Consider the same problem as in the previous example, and let $u_0(x, y) \equiv 0$ be a first approximation. Hence, also $u_{0*} \equiv u_{0*} = 0$. Using the integral representation, we get

$$\begin{aligned} u_1(x, y) &= \gamma(k)^{1/2} \frac{1 - \alpha}{1 - \beta} \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \\ u_{1*}(x, y) &= \gamma \frac{(k)^{1/2}}{x} \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \\ u_{1*}(x, y) &= \gamma \frac{(k)^{1/2}}{y} \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \\ f_1(x, y, u_1) &= f_2(x, y, u_{1*}) = f_3(x, y, u_{1*}) = 0. \end{aligned}$$

Therefore

$$\begin{aligned} u_{2n}(x, y) &\equiv 0 \\ u_{2n+1}(x, y) &= \gamma(k)^{1/2} \frac{1 - \alpha}{1 - \beta} \left[\frac{(xy)^{1-\beta}}{k} \right]^{1/(1-\alpha)} \quad n = 0, 1, 2, \dots \end{aligned}$$

and the successive approximations *do not* converge.

6. Case of n variables.

Definition¹. Let $f(x_1, \dots, x_n, \dots, q_{i_1 \dots i_n}, \dots)$ be defined in

$$D' = \begin{cases} (x_1, \dots, x_n) \in R' = \prod_{i=1}^n [0, a_i] \\ -\infty < q_{i_1 \dots i_n} < +\infty, \end{cases}$$

where i_h is either 0 or 1 ($h = 1, 2, \dots, n$), and $i_1 + \dots + i_n \leq n - 1$. We say that f satisfies in D' a generalized K-K-K condition of order $\underbrace{(1, 1, \dots, 1)}_n$ if,

for $(x_1, \dots, x_n, \dots, q_{i_1 \dots i_n}, \dots) \in D'$, $(x_1, \dots, x_n, \dots, \bar{q}_{i_1 \dots i_n}, \dots) \in D'$:

$$\begin{aligned} A') \quad & x_1 \dots x_n |f(x_1, \dots, x_n, \dots, q_{i_1 \dots i_n}, \dots) \\ & - f(x_1, \dots, x_n, \dots, \bar{q}_{i_1 \dots i_n}, \dots)| \\ & \leq \sum_{(i_1, \dots, i_n)} K_{i_1 \dots i_n}(x_1, \dots, x_n)((x_{i_1} \dots x_{i_n})) |q_{i_1 \dots i_n} - \bar{q}_{i_1 \dots i_n}| \end{aligned}$$

$$\begin{aligned}
B') \quad & (x_1 \cdots x_n)^\beta |f(x_1, \dots, x_n, \dots, q_{i_1 \dots i_n}, \dots) \\
& - f(x_1, \dots, x_n, \dots, \bar{q}_{i_1 \dots i_n})|, \\
& \leq \sum_{(i_1, \dots, i_n)} C_{i_1 \dots i_n}(x_1, \dots, x_n) ((x_{i_1} \cdots x_{i_n}))^\alpha |q_{i_1 \dots i_n} - \bar{q}_{i_1 \dots i_n}|^\alpha \\
C') \quad & |f| \leq H,
\end{aligned}$$

where $((x_{i_1} \cdots x_{i_n})) = x_{i_1} \cdots x_{i_n}$, with $x_{i_h} = 1$ for $i_h = 0$ and

$$\begin{aligned}
a') \quad & \{K_{i_1 \dots i_n}(x_1, \dots, x_n)\} \subset C(R'), \\
& K_{i_1 \dots i_n}(x_1, \dots, x_n) \geq 0 \quad \forall (i_1, \dots, i_n) \\
& K = \max_{R'} \{ \max_{(i_1, \dots, i_n)} [K_{i_1 \dots i_n}(x_1, \dots, x_n)] \} > 1 \\
b') \quad & \{C_{i_1 \dots i_n}(x_1, \dots, x_n)\} \subset C(R'), \\
& C_{i_1 \dots i_n}(x_1, \dots, x_n) \geq 0 \quad \forall (i_1, \dots, i_n) \\
& C = \max_{R'} \{ \max_{(i_1, \dots, i_n)} [C_{i_1 \dots i_n}(x_1, \dots, x_n)] \} > 0 \\
c') \quad & 0 < \alpha < 1, \\
d') \quad & pK^{1/n}(1 - \alpha) < 1 - \beta, \quad p^n = \sum_{j=0}^{n-1} \binom{n}{j} p^j K^{j/n},
\end{aligned}$$

and p is the smallest root of this equation greater than 1. (An elementary computation shows that such a $p > 1$ always exists.)

Theorem¹. *Let f be defined and continuous in D' , where it satisfies a generalized K-K-K condition of order $\underbrace{(1, \dots, 1)}_n$. Then the problem*

$$(P') \quad \begin{cases} u_{x_1 \dots x_n} = f(x_1, \dots, x_n, \dots, u_{x_{i_1} \dots x_{i_n}}, \dots) \\ u(x_1, \dots, x_n)|_{x_i=0} = \sigma_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \quad (j = 1, \dots, n) \\ \sigma_j \in C^{(n-1)} \left(\prod_{\substack{i=1 \\ i \neq j}}^n [0, a_i] \right) \end{cases}$$

has one and only one solution in

$$\begin{aligned}
C^\#(R') &= \{u(x_1, \dots, x_n) \in C(R') : u_{x_{i_1} \dots x_{i_n}} \in C(R') \\
&\quad i_h = \{1^0, \dots, i_1 + \dots + i_n \leq n\}
\end{aligned}$$

which can be obtained by successive approximations.

The proof follows the same pattern used for the case $n = 2$, with the natural modifications. Also, as in the former case, examples can be provided to show optimality of the conditions involving the parameters.

Note. After writing this paper, the author became acquainted with [6, 7]. However, although proved following the same lines, those results are also limited to the special case of a “reduced” second member, *i.e.* $f = f(x, y, u)$, and therefore do not imply and are not implied by our own ones.

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