Global Solutions for Certain Systems of Quasi-Linear Hyperbolic Equations

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1. Introduction. In this paper we shall be concerned with systems of conservation laws of the form

$$(1.1) \quad \begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(v) &= 0 \\ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} g(u) &= 0 \end{aligned}$$

where u and v are functions of t and x , $t \geq 0$, $-\infty < x < \infty$, and f and g are C^2 functions of a single real variable. We assume that the system is hyperbolic and genuinely non-linear in the sense of Lax, [10]. We are interested in the Cauchy problem for (1.1), *i.e.*, we seek solutions of (1.1) defined in $t \geq 0$ and also satisfying

$$(1.2) \quad (u(0, x), v(0, x)) = (u_0(x), v_0(x)), \quad -\infty < x < \infty,$$

for some given initial functions $u_0(x)$ and $v_0(x)$. By a solution of this problem we mean a pair of bounded and measurable functions $u(t, x)$, $v(t, x)$ defined in $t \geq 0$ and satisfying

$$(1.3) \quad \iint_{t \geq 0} \left[u(t, x) \frac{\partial \phi}{\partial t} + f(v(t, x)) \frac{\partial \phi}{\partial x} \right] dx dt + \int_{t=0} u_0(x) \phi(0, x) dx = 0$$

$$(1.4) \quad \iint_{t \geq 0} \left[v(t, x) \frac{\partial \phi}{\partial t} + g(u(t, x)) \frac{\partial \phi}{\partial x} \right] dx dt + \int_{t=0} v_0(x) \phi(0, x) dx = 0$$

for all smooth (*i.e.*, C^1) functions $\phi(t, x)$ having compact support.

We assume that the initial data satisfies a certain order condition, the motivation of which we get from [11]. This condition can be described as follows: If we let $(u_i, v_i) = (u_0(x_i), v_0(x_i))$, $i = 1, 2$, for $x_1 < x_2$, then the Riemann

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problem for (1.1) with initial data (u_1, v_1) if $x < 0$ and (u_2, v_2) if $x > 0$ can be solved by a forward shock and a backward rarefaction wave. (The case of a forward rarefaction wave and a backward shock can be treated in an analogous way.) In the paper [11], the case $g''(u) = 0$ is considered and this condition is equivalent to a simple set of inequalities. In our case, we can only prove that our condition implies the validity of a similar set of inequalities. These take the form

$$-M[(v_1 - v_2)(f(v_2) - f(v_1))]^{1/2} \leq u_2 - u_1 \leq M \int_{v_1}^{v_2} [-f'(\xi)]^{1/2} d\xi$$

for some constant $M > 0$. Our main theorem is that if the initial data satisfies our order condition, and v_0 (or u_0) is bounded, then there exists a solution to the Cauchy problem (1.1), (1.2). This solution $(u(t, x), v(t, x))$ will be bounded and have the properties that $u(t, x)$ is an increasing function of t and is of bounded variation in x , and $v(t, x)$ is an increasing function of x and is of bounded variation in t .

The method of proof is an extension of the methods in [11]. We first show that through any point (v_0, u_0) in $v - u$ space there exist two curves $u = w(v)$ and $u = s(v)$ defined for all $v \geq v_0$, called the wave and shock curves respectively, which satisfy the Lax conditions for backward rarefaction waves and forward shocks, [10]. If we let $C(v_0, u_0)$ denote the region between these two curves, then if $(v_1, u_1) \in C(v_0, u_0)$, we prove that $C(v_1, u_1) \subset C(v_0, u_0)$. This result is used to study interactions of forward shocks and backward rarefaction waves. It implies in particular that the interaction of two forward shocks produces a forward shock and a backward rarefaction wave. (In a similar manner we can show that the interaction of two backward shocks produces a backward shock and a forward rarefaction wave. This implies that for this system hypothesis (c) of [5] is a consequence of genuine non-linearity.) The shock and wave curves also enable us to construct, for each n , an exact solution $(u_n(t, x), v_n(t, x))$ of (1.1) where the initial data are step functions $(u_0^n(x), v_0^n(x))$ which converge to $(u_0(x), v_0(x))$ and which also satisfy our order condition. In order to show that this sequence of approximating solutions is compact, we employ a device introduced in [2]. Namely, we show that the approximating solutions are uniformly bounded and have uniformly bounded variation in the sense of Tonelli-Cesari, [1], on every compact subset of the half-plane $t \geq 0$.

Throughout this paper we assume that the reader is familiar with the results of Lax in [10].

2. Construction of the wave and shock curves. We require that the system (1.1) should be hyperbolic. This means that $f'(v)g'(u) > 0$, and for definiteness, let us take

$$(2.1) \quad f'(v) < 0, \quad g'(u) < 0, \quad \text{for all } v > 0 \text{ and all } u.$$

According to Lax, [10], the system is genuinely non-linear if $r \cdot \text{grad } \lambda \neq 0$

for all $v > 0$ and all u . Here r is the right eigenvector associated with the eigenvalue λ of the matrix associated with (1.1). Thus the system is genuinely non-linear if $f''(v)[-g'(u)]^{3/2} \pm g''(u)[-f'(v)]^{3/2} \neq 0$, and for definiteness, we shall assume that

$$(2.2) \quad f''(v) > |g''(u)[f'(v)/g'(u)]^{3/2}|$$

for all $v > 0$ and for all u .

Now let (v_0, u_0) be any point in the $v - u$ plane, $v_0 > 0$. We wish to construct curves through (v_0, u_0) which represent the states in the $x - t$ plane which can be connected to the initial state (v_0, u_0) on the right by forward and backward shocks and rarefaction waves. We shall first consider the case of rarefaction waves.

The Riemann-invariants of the system (1.1) are the functions

$$\begin{aligned} r(u, v) &= \int^v [-f'(\xi)]^{1/2} d\xi - \int^u [-g'(\xi)]^{1/2} d\xi \\ s(u, v) &= \int^v [-f'(\xi)]^{1/2} d\xi + \int^u [-g'(\xi)]^{1/2} d\xi. \end{aligned}$$

Since we are concerned here only with backward rarefaction waves we shall only be interested in $r(u, v)$. From the above representation, we see that the curve $r(u, v) = \text{constant}$ is a solution of the differential equation

$$(2.3) \quad \frac{du}{dv} = [f'(v)/g'(u)]^{1/2}.$$

Through any point (v_0, u_0) , $v_0 > 0$ in the $v - u$ plane, there exists a unique solution $u = w(v)$ of (2.3) which is C^1 and is an increasing function of v . If we let $F(v, u) = [f'(v)/g'(u)]^{1/2}$ then by our assumption that the system (1.1) is genuinely non-linear we see that $|F_u(v, u)| \leq (1/2)(f''(v)/-f'(v))$. Therefore from [8, p. 485], we see that the curve $u = w(v)$ cannot escape and is defined for all $v \geq v_0 > 0$.

Next if $\phi(u, v) = [f'(v_0)g'(u_0)]^{1/2} - [f'(v)g'(u)]^{1/2}$, then since $(d/dv)\phi(v, w(v)) > 0$, we see that the Lax condition for backward rarefaction waves, (equation (8.7) of [10]) holds along this curve. We have therefore proved the following theorem.

Theorem 2.1. *Let the system (1.1) be hyperbolic and genuinely non-linear; i.e., let the system satisfy (2.1) and (2.2). Let (v_0, u_0) be a point in $v - u$ space, $v_0 > 0$. Then there exists a curve $u = w(v)$ through (v_0, u_0) defined for all $v \geq v_0$, satisfying (2.3) and having the property that any point (v_1, u_1) on the curve is a state which can be connected to (v_0, u_0) by a backward rarefaction wave on the right.*

(We remark that a similar theorem holds for forward rarefaction waves.)

We shall next show that we can construct analogous curves for forward (and backward) shocks. It is well known, [10], that if the discontinuities of

(1.1) lie on a curve $x = x(t)$, then this curve must satisfy the Rankine-Hugoniot conditions. Thus if (v_0, u_0) , $v_0 > 0$ is any state, then the states (v, u) which can be connected to (v_0, u_0) by a shock must satisfy $\sigma(u - u_0) = f(v) - f(v_0)$ and $\sigma(v - v_0) = g(u) - g(u_0)$, where $\sigma = \dot{x}$ is the shock speed. We can write this pair of equations as

$$(2.4) \quad (u - u_0)[g(u) - g(u_0)] = (v - v_0)[f(v) - f(v_0)].$$

We seek a and b so that we can write $u = u_0 + a(v - v_0) + (1/2)b(v - v_0)^2 + O[(v - v_0)^3]$. If we expand both sides of (2.4) about v_0 out to terms of second order in $v - v_0$, we obtain

$$a = \pm [f'(v_0)/g'(u_0)]^{1/2}$$

and

$$b = \frac{g''(u_0)[-f'(v_0)]^{3/2} \mp f''(v_0)[-g'(u_0)]^{3/2}}{2[g'(u_0)]^2[-f(v_0)]^{1/2}}.$$

Therefore equation (2.4) has two local solutions passing through (v_0, u_0) . Since we are interested in forward shocks we shall only consider the solution

$$(2.5) \quad u = u_0 - \left[\frac{f'(v_0)}{g'(u_0)} \right]^{1/2} (v - v_0) + \frac{(v - v_0)^2}{4[f'(v_0)g'(u_0)]^{1/2}} \\ \cdot \left\{ g''(u_0) \left[\frac{f'(v_0)}{g'(u_0)} \right]^{3/2} + f''(v_0) \right\} + O[(v - v_0)^3].$$

We shall refer to this as the shock curve. From (2.4) and (2.5) the differential equation of the shock curve is

$$(2.6) \quad \frac{du}{dv} = \begin{cases} \frac{f(v) - f(v_0) + (v - v_0)f'(v)}{g(u) - g(u_0) + (u - u_0)g'(u)} & v \neq v_0 \\ -[f'(v_0)/g'(u_0)]^{1/2} & v = v_0. \end{cases}$$

From equation (7.2) of [10], since we are considering forward shocks, we have $\sigma > 0$ for $v - v_0$ sufficiently small.

We shall next show that we can extend the curve $u = u(v)$, which is thus far only defined for v close to v_0 , to a global curve defined for all $v > v_0$, which still satisfies (2.4). We need only show that the solution of (2.6) through (v_0, u_0) does not escape for a finite $v > v_0$. But if there were a $v' > v_1$ such that

$$\lim_{v \rightarrow v'} s(v) = -\infty,$$

then since

$$\frac{s(v) - u_0}{v - v_0} = \frac{f(v) - f(v_0)}{g(s(v)) - g(u_0)},$$

we see that the left-hand side of this equation becomes infinite while the right-hand side stays finite. Hence we have constructed a curve, $u = s(v)$ defined for all $v > v_0$ which passes through (v_0, u_0) and satisfies (2.4).

We shall now need some notation. Let $u = s(v)$; then we define

$$\sigma = \sigma(u, v; u_0, v_0) = \frac{f(v) - f(v_0)}{u - u_0} = \frac{g(u) - g(u_0)}{v - v_0}$$

$$h = h(u, v; u_0, v_0) = \frac{f(v) - f(v_0) + f'(v)(v - v_0)}{g(u) - g(u_0) + g'(u)(u - u_0)} = \frac{\sigma(u - u_0) + f'(v)(v - v_0)}{\sigma(v - v_0) + g'(u)(u - u_0)}$$

$\frac{d}{d\mu}$ = differentiation along $u = s(v)$.

We shall now show that the curve $u = s(v)$ satisfies the Lax shock conditions (Eq. (7.1) of [10]), globally.

Theorem 2.2. *Let the system (1.1) be hyperbolic and genuine non-linear; i.e., let the system satisfy (2.1) and (2.2). Let (v_0, u_0) be a point in $v - u$ space, $v_0 > 0$. Then there exists a curve $u = s(v)$ through (v_0, u_0) defined for all $v \geq v_0$, satisfying (2.4) and also satisfying the condition that $s'(v) < 0$. Furthermore, the Lax shock condition $[f'(v_0)g'(u_0)]^{1/2} > \sigma > [f'(v)g'(u)]^{1/2}$, $v > v_0$ is also valid along this curve, so that every point (v_1, u_1) on this curve is a state which can be connected to (v_0, u_0) by a forward shock on the right.*

Proof. In view of our previous remarks we need only show that the Lax shock condition holds along the curve. Let $u = s(v)$, $v > v_0$. From [10], $\sigma > [f'(v)g'(u)]^{1/2}$ for v close to v_0 .

Now suppose that this inequality is not true for all $v > v_0$. Then there exists a first point $(v_1, u_1) \neq (v_0, u_0)$ for which equality holds; i.e., $\sigma(u_1, u_1; u_0, v_0) = [f'(v_1)g'(u_1)]^{1/2}$. Then $h(u_1, v_1; u_0, v_0) = -[f'(v_1)/g'(u_1)]^{1/2}$. But then

$$\left. \frac{d\sigma}{d\mu} \right|_{(v_1, u_1)} = \frac{(u_1 - u_0)f'(v_1) + (u_1 - u_0)[f'(v_1)g'(u_1)]^{1/2}[f'(v_1)/g'(u_1)]^{1/2}}{(u_1 - u_0)^2} = 0$$

so that if $\phi = \sigma - [f'(v)g'(u)]^{1/2}$, then

$$(d\phi/d\mu)_{(v_1, u_1)} = (d/d\mu)(-[f'(v)g'(u)]^{1/2})_{(v_1, u_1)} > 0$$

by (2.2). Hence $\sigma(u, v; u_0, v_0) < [f'(v)g'(u)]^{1/2}$ for some $v < v_1$. This contradicts the fact that (v_1, u_1) was the first place of equality. Next,

$$\begin{aligned} \frac{d\sigma}{d\mu} &= \frac{(u - u_0)f'(v) - [f(v) - f(v_0)]h(u, v; u_0, v_0)}{(u - u_0)^2} \\ &= \frac{1}{u - u_0} [f'(v) - \sigma h] \\ &= \left[\frac{f'(v)g'(u) - \sigma^2}{\sigma(v - v_0) + g'(u)(u - u_0)} \right] < 0 \end{aligned}$$

in view of what we have already shown. Thus $[f'(v_0)g'(u_0)]^{1/2} > \sigma$. This completes the proof of the theorem.

In order to be able to study interactions of forward shocks and backward rarefaction waves, we must obtain further properties of our curves $u = s(v)$. We shall state these in the following theorem.

Theorem 2.3. *Let the system (1.1) satisfy (2.1) and (2.2), and let $u = s(v)$ be the shock curve passing through (v_0, u_0) , $v_0 > 0$. Then along this curve, the following relations hold:*

$$(2.8) \quad -[f'(v)/g'(u)]^{1/2} > h(u, v; u_0, v_0) > \frac{u - u_0}{v - v_0}$$

$$(2.9) \quad \frac{d}{d\mu} \left(\frac{u - u_0}{v - v_0} \right) > 0$$

$$(2.10) \quad \frac{u_2 - u}{v_2 - v} > \frac{u_1 - u}{v_1 - v} \quad \text{if } v > v_2 > v_1.$$

Proof. We shall first prove that

$$(2.11) \quad -[f'(v)/g'(u)]^{1/2} > \frac{u - u_0}{v - v_0}.$$

It is easy to show that this holds locally for v close to v_0 . To prove it in general, we assume that there is a first place (v_1, u_1) , where equality holds: $(u_1 - u_0)/(v_1 - v_0) = -[f'(v_1)/g'(u_1)]^{1/2}$. Then $h(u_1, v_1; u_0, v_0) = -[f'(v_1)/g'(u_1)]^{1/2}$, so that $(d/d\mu)[(u - u_0)/(v - v_0)]|_{(v_1, u_1)} = 0$, and therefore

$$\frac{d}{d\mu} \left(\frac{u - u_0}{v - v_0} + [f'(v)/g'(u)]^{1/2} \right) \Big|_{(v_1, u_1)} = -\frac{f''(v_1) + (f'(v_1)/g'(u_1))^{3/2} g''(u_1)}{2[f'(v_1)g'(u_1)]^{1/2}} < 0.$$

Hence $(u - u_0)/(v - v_0) + [f'(v)/g'(u)]^{1/2} > 0$ for v slightly less than v_1 . This contradicts the fact that (v_1, u_1) was the first point of equality. We shall next prove that (2.9) holds. First, observe that by (2.5) this holds for v close to v_0 . As before, if

$$\frac{d}{d\mu} \left(\frac{u - u_0}{v - v_0} \right) = 0$$

for some (v, u) , $v > v_0$, then if we carry out the differentiation we get

$$h(u, v; u_0, v_0) = (u - u_0)/(v - v_0)$$

which violates (2.11). Now, from (2.9)

$$0 < \frac{d}{d\mu} \left(\frac{u - u_0}{v - v_0} \right) = \frac{(v - v_0)h(u, v; u_0, v_0) - (u - u_0)}{(v - v_0)^2}$$

and this gives the right-hand inequality in (2.8). Next, we fix $v > v_0$ and look upon $h = h(\sigma)$ as a function of σ . Then $h([f'(v)g'(u)]^{1/2}) = -[f'(v)/g'(u)]^{1/2}$

and $dh/d\sigma < 0$. Therefore since $\sigma > [f'(v)g'(u)]^{1/2}$, we have $h(\sigma) < h([f'(v)g'(u)]^{1/2})$, and this gives the left-hand inequality in (2.8). Finally, if we differentiate $(u_1 - u)/(v_1 - v)$, for $v > v_1$ with respect to v_1 along the curve we get

$$\frac{d}{dv_1} \left(\frac{u_1 - u}{v_1 - v} \right) = \frac{(v_1 - v)h(u_1, v_1; u_0, v_0) - (u_1 - u)}{(v_1 - v)^2}.$$

But

$$h(u_1, v_1; u_0, v_0) < -[f'(v_1)/g'(u_1)]^{1/2} = \lim_{\alpha \rightarrow v_1} \left(\frac{s(\alpha) - u_1}{\alpha - v_1} \right) < \frac{u_1 - u}{v_1 - v}$$

from (2.5), (2.8) and (2.9). Hence $(d/dv_1)[(u_1 - u)/(v_1 - v)] > 0$, $v > v_1$ which implies (2.10) and completes the proof of the theorem.

Fix a point (v_0, u_0) , $v_0 > 0$ and let $u = w(v) = w(v; v_0, u_0)$ and $u = s(v) = s(v; v_0, u_0)$ be the backward wave and forward shock curves respectively, passing through (v_0, u_0) . Let

$$C(v_0, u_0) = \{(v, u) : v \geq v_0; s(v; v_0, u_0) \leq u \leq w(v; v_0, u_0)\}.$$

We then have

Theorem 2.4. *If $(v_1, u_1) \in C(v_0, u_0)$, then $C(v_1, u_1) \subset C(v_0, u_0)$.*

Proof. It is clear that we may assume that $v_1 > v_0$. Let $(v, u) \in C(v_1, u_1)$. Since the wave curves $u = w(v; v_0, u_0)$ and $u = w(v; v_1, u_1)$ both satisfy (2.3), the uniqueness theorem for ordinary differential equations shows that the curve $u = w(v; v_1, u_1)$ cannot ever get above the curve $u = w(v; v_0, u_0)$. Therefore it remains to show that the shock curve $u = s(v; v_1, u_1)$ cannot ever go below the shock curve $u = s(v; v_0, u_0)$.

Suppose first that $u_1 = s(v_1; v_0, u_0)$. The left hand inequality in (2.8) shows that the curve $u = s(v; v_1, u_1)$ starts out above the curve $u = s(v; v_0, u_0)$. We shall show that $s(v; v_1, u_1) > s(v; v_0, u_0)$ if $v > v_1$. If this does not hold, then there is a first point (v, u) with $v > v_1$ such that $s(v; v_1, u_1) = s(v; v_0, u_0)$. Then we must have $h(u, v; u_0, v_0) \geq h(u, v; u_1, v_1)$, so that

$$\frac{\sigma(u - u_0) + f'(v)(v - v_0)}{\sigma(v - v_0) + g'(u)(u - u_0)} \geq \frac{\bar{\sigma}(u - u_1) + f'(v)(v - v_1)}{\bar{\sigma}(v - v_1) + g'(u)(u - u_1)},$$

where

$$\sigma = \frac{f(v) - f(v_0)}{u - u_0}, \quad \bar{\sigma} = \frac{f(v) - f(v_1)}{u - u_1}.$$

If we cross multiply and collect terms in the last inequality, we get

$$\begin{aligned} (2.12) \quad & \left[\left(\frac{u - u_0}{v - v_0} \right) - \left(\frac{u - u_1}{v - v_1} \right) \right] (\sigma \bar{\sigma} - f'(v)g'(u)) \\ & \geq (\bar{\sigma} - \sigma) \left[g'(u) \left(\frac{u - u_1}{v - v_1} \right) \left(\frac{u - u_0}{v - v_0} \right) - f'(v) \right]. \end{aligned}$$

Now from the Lax shock condition, (2.10) and (2.8), we see that $\sigma\bar{\sigma} > f'(v)g'(u)$, $(u - u_0)/(v - v_0) - (u - u_1)/(v - v_1) < 0$, and $g'(u)(u - u_1)/(v - v_1)(u - u_0)/(v - v_0) - f'(v) < 0$. Thus if we can show that $\bar{\sigma} - \sigma < 0$, then these will be inconsistent with (2.12) and the result will be proved in this case.

To show that $\bar{\sigma} - \sigma < 0$, we proceed as follows. Fix a state (v_2, u_2) on the right and let $(v, \tilde{u}(v))$ be the states which can be joined to (v_2, u_2) on the left by a forward shock. Let $\bar{\sigma}(v)$ be the corresponding shock slope and let $\tilde{h}(v) = \tilde{u}'(v)$. Then

$$\bar{\sigma}(v) = \frac{f(v) - f(v_2)}{\tilde{u}(v) - u_2}$$

and

$$\tilde{h}(v) = \frac{\bar{\sigma}(v)(\tilde{u} - u_2) + f'(v)(v - v_2)}{\bar{\sigma}(v)(v - v_2) + g'(\tilde{u})(\tilde{u} - u_2)}.$$

Now a calculation shows that

$$\begin{aligned} \frac{d\bar{\sigma}(v)}{dv} &= (\tilde{u} - u_2)^{-1} \left[f'(v) - \bar{\sigma}(v) \left(\frac{\bar{\sigma}(v)(\tilde{u} - u_2) + f'(v)(v - v_2)}{\bar{\sigma}(v)(v - v_2) + g'(\tilde{u})(\tilde{u} - u_2)} \right) \right] \\ &= \frac{f'(v)g'(\tilde{u}) - \bar{\sigma}(v)^2}{\bar{\sigma}(v)(v - v_2) + g'(\tilde{u})(\tilde{u} - u_2)} < 0; \end{aligned}$$

hence

$$\sigma = \bar{\sigma}(v_0) > \bar{\sigma}(v_1) = \bar{\sigma}.$$

For the general case, we first note that $(\partial/\partial u_0)h(u, v; u_0, v_0) > 0$. This implies that the curve $u = s(v; v_1, u_1)$ stays above the curve $u = s(v; v_1, u_2)$, where u_2 is defined by $u_2 = s(v_1; v_0, u_0)$. Therefore by what we have shown above, the curve $u = s(v; v_1, u_1)$ stays above the curve $u = s(v; v_0, u_0)$. This completes the proof of the theorem.

We remark that a similar theorem holds for forward rarefaction waves and backward shock waves. One consequence of these theorems and their analogues for negative shocks is that the interaction of two shocks of the same family always produces a shock of the same family plus a rarefaction wave of the other family. This shows that for our system (1.1) hypothesis (c) of [5], is a consequence of genuine non-linearity.

3. Solution of the system with step data. Let (v_0, u_0) be a point in the $v - u$ plane with $v_0 > 0$. Let $u = w(v) = w(v; v_0, u_0)$ and $u = s(v) = s(v; v_0, u_0)$ be the forward wave and backward shock curve through (v_0, u_0) . Let

$$R_c(v_0, u_0) = C(v_0, u_0) \cap \{(v, u) : v \leq c\}$$

for some $c > v_0$. It is clear that there is a constant d such that if $(v, u) \in R_c(v_0, u_0)$ then $|u| \leq d$. We let

$$M = \sup_{|\xi| \leq d} [-g'(\xi)]^{-1/2}.$$

Now for any $(v, u) \in R_c(v_0, u_0)$, we have

$$(3.1) \quad s(v) = s(v; v_0, u_0) \leq u \leq w(v; v_0, u_0) = w(v).$$

From Theorem 2.1 we have

$$\int_{u_0}^{w(v)} [-g'(\xi)]^{1/2} d\xi = \int_{v_0}^v [-f'(\xi)]^{1/2} d\xi,$$

so that using the mean-value theorem for integrals we obtain

$$(3.2) \quad w(v) \leq u_0 + M \int_{v_0}^v [-f'(\xi)]^{1/2} d\xi.$$

Next, from Theorem 2.2 we get

$$\frac{s(v) - u_0}{v - v_0} = \frac{f(v) - f(v_0)}{g(s(v)) - g(u_0)}$$

so that if we use the mean-value theorem we obtain

$$(3.3) \quad s(v) \geq u_0 - M[(v_0 - v)(f(v) - f(v_0))]^{1/2}.$$

Hence (3.1), (3.2) and (3.3) imply that

$$-M[(v_0 - v)(f(v) - f(v_0))]^{1/2} \leq u - u_0 \leq M \int_{v_0}^v [-f'(\xi)]^{1/2} d\xi$$

for all $(v, u) \in R_c(v_0, u_0)$.

We shall consider the initial-value problem for (1.1) with data $(u_0(x), v_0(x))$ where v_0 is bounded, say $|v_0(x)| \leq c$, and satisfying the following condition:

$$(C) \quad \begin{cases} \text{If } x_1 < x_2, \text{ and } (v_0(x_i), u_0(x_i)) = (v_i, u_i), & i = 1, 2, \text{ then} \\ (v_2, u_2) \in C(v_1, u_1). \end{cases}$$

This condition is equivalent to the fact that the Riemann problem for (1.1) with the initial data

$$(u_0(x), v_0(x)) = \begin{cases} (u_1, v_1), & \text{if } x < 0 \\ (u_2, v_2), & \text{if } x > 0 \end{cases}$$

is solvable by a backward rarefaction wave and a forward shock. From our above discussion, we have an $M > 0$ such that

$$(3.4) \quad -M \sqrt{(v_1 - v_2)(f(v_2) - f(v_1))} \leq u_2 - u_1 \leq M \int_{v_1}^{v_2} \sqrt{-f'(\xi)} d\xi$$

where $(v_i, u_i) = (v_0(x_i), u_0(x_i))$, $i = 1, 2$ for $x_1 < x_2$. (For $g'(\xi) = 1$, $M = 1$ and this then reduces to the order condition in [11].)

Lemma 3.1. $v_0(x)$ is an increasing function and $u_0(x)$ is a function of bounded variation.

Proof. It follows at once from (3.4) that v_0 is an increasing function. Moreover, the function

$$r_0(x) = u_0(x) - M \int_{v_-}^{v_0(x)} [-f'(\xi)]^{1/2} d\xi$$

where $v_- = v_0(-\infty)$ is a decreasing function so that u_0 is a function of bounded variation.

Let $\{u_0^n(x)\}$ and $\{v_0^n(x)\}$ be sequences of step functions converging pointwise to $u_0(x)$ and $v_0(x)$ respectively, which also satisfy condition (C) and therefore also satisfy (3.4) for each n . We shall solve the initial value problem for (1.1) with initial data $(u_0^n(x), v_0^n(x))$.

For each n , functions u_0^n and v_0^n have only a finite number of discontinuities. At each point of discontinuity, there occurs a backward rarefaction wave and a forward shock. The rarefaction wave is separated from constant states on both sides by backward characteristics, and the forward shock separates two constant states [10] (see Fig. 1).

We shall now consider the types of interactions that can occur. Our discussion will be short since we shall obtain essentially the same results as in [11]. However, we shall elucidate on some of the discussion in [11] which appears to us to be somewhat brief.

First suppose that a backward rarefaction wave interacts with a forward shock. We claim that this produces a backward rarefaction wave and a forward shock (see Fig. 2). To see this, first note that we can obtain the equation of the

FIGURE 1

FIGURE 2

shock in the region where the shock meets the backward rarefaction wave (P_1P_2 in Fig. 2) as follows. Let (t, x) be any point on the shock. We know the solution below the shock, $u=u_0$, $v=v_0$, depending on (t, x) , and we know the Riemann invariant r which is constant along characteristics of positive slope, above the shock; say $r=\rho$ in this region. Thus from the point (v_0, u_0) in the $v-u$ plane we draw the forward shock curve $u = \tilde{s}(v; v_0, u_0)$ which consists of those states which can be joined to the state (v_0, u_0) by a forward shock on the left. In the $v-u$ plane this curve meets the curve $r = \rho$ in a point (v_1, u_1) (see Fig. 3). Then the Rankine-Hugoniot conditions then give us $\sigma(t, x)$ and hence we solve the equation $dx/dt = \sigma(t, x)$ to get $x = x(t)$ the equation of the shock.

Now using the Rankine-Hugoniot conditions we can obtain the solution just above the shock. We can then obtain the characteristics of negative slope above the shock. Since the Riemann invariant s , which is constant along characteristics of negative slope, is therefore determined above the shock, we thus know r and s above the shock and hence we know u and v above the shock (region G in Fig. 2). Since r is constant in the region G , it follows that the solution is a backward rarefaction wave in region G . This solution is described in Fig. 4.

Next suppose that two (or more) forward shocks meet at a point. We claim that this produces a forward shock and backwards rarefaction wave (see Fig. 5).

FIGURE 3

FIGURE 4. The curves of positive slope are the backward wave curves and the curves of negative slope are the forward shock curves. (This will be true in all subsequent diagrams in the $v-u$ plane.)

FIGURE 5

FIGURE 6

This can be proved using Theorem 2.4 as follows. In the $v - u$ plane we draw the forward shock curve from the constant state ①. The constant state ② is a point on this curve. From this point we draw the forward shock curve. In view of Theorem 2.4, this curve lies above the first shock curve. The constant state ③ is on this curve. At the time of intersection we have a Riemann problem for (1.1) with constant data ① and ③ on the left and right sides of the point of intersections respectively. According to Lax, [10], this problem is resolvable by a backward rarefaction wave and a forward shock (see Fig. 6).

We observe that these are the only interactions which can occur. These interactions always produce backward rarefaction waves and forward shocks. No forward rarefaction waves or backward shocks can occur. Furthermore, no two backward rarefaction waves can ever catch up with each other. This follows since the rarefaction waves are bounded by characteristics. (In his dissertation, [7], the first author has obtained theorems which solve the problem of the interaction of forward and backward rarefaction waves.)

It follows from this discussion that for each n , the solution $(u_n(t, x), v_n(t, x))$ of the initial-value problem for (1.1) with the initial data $(u_0^n(x), v_0^n(x))$ can be constructed using the above analysis. The solution consists of finitely-many forward shocks, backward rarefaction waves, and constant states. A typical solution is given in Figures 7 and 8.

Since this solution satisfies the Rankine-Hugoniot condition across shocks and satisfies (1.1) in the regions where there are no shocks, and also assumes

the correct initial data, it follows from well-known facts (see [10], for example) that for each n

$$(3.5) \quad \iint_{t \geq 0} \left[u_n(t, x) \frac{\partial \phi}{\partial t} + f(v_n(t, x)) \frac{\partial \phi}{\partial x} \right] dt dx + \int_{t=0} u_0^n(x) \phi(0, x) dx = 0$$

$$(3.6) \quad \iint_{t \geq 0} \left[v_n(t, x) \frac{\partial \phi}{\partial t} + g(u_n(t, x)) \frac{\partial \phi}{\partial x} \right] dt dx + \int_{t=0} v_0^n(x) \phi(0, x) dx = 0$$

for all $\phi \in C^1$ having compact support.

Now from a consideration of the solution as represented in Fig. 8, we see that the functions $u_n(t, x)$ and $v_n(t, x)$ are bounded for all t and x , $t \geq 0$, $-\infty < x < \infty$ where the bound is independent of n . Furthermore, we can also show that for each fixed $t > 0$, our condition (C) holds for the pair $(u_n(t, x), v_n(t, x))$. This can be easily seen from Figures 7 and 8, where for each fixed $t > 0$, as

FIGURE 7

FIGURE 8

we move along in the direction of increasing x , we cross both backward rarefaction waves and forward shocks from left to right. Thus it follows that the following estimate is valid for each fixed $t > 0$:

$$(3.7) \quad -M[(v_1^n - v_2^n)(f(v_2^n) - f(v_1^n))]^{1/2} \leq u_2^n - u_1^n \leq M \int_{v_2^n}^{v_1^n} [-f'(\xi)]^{1/2} d\xi.$$

Here

$$(u_i^n, v_i^n) = (u_n(t, x_i), v_n(t, x_i)), \quad i = 1, 2, \quad x_1 < x_2.$$

Finally if we fix x and move along in the direction of increasing t , we see from Figure 7 that we cross backward rarefaction waves from left to right and we cross forward shocks from right to left. We therefore obtain for each fixed x and $t_1 < t_2$:

$$(3.8) \quad \bar{u}_2^n - \bar{u}_1^n \geq -M[(\bar{v}_1^n - \bar{v}_2^n)(f(\bar{v}_2^n) - f(\bar{v}_1^n))]^{1/2}$$

and

$$(3.9) \quad \bar{u}_2^n - \bar{u}_1^n \geq M \int_{\bar{v}_1^n}^{\bar{v}_2^n} [-f'(\xi)]^{1/2} d\xi.$$

Here $(\bar{u}_i^n, \bar{v}_i^n) = (u_n(t_i, x), v_n(t_i, x))$, $i = 1, 2$. The inequalities (3.7), (3.8), and (3.9) then yield the following lemma.

Lemma 3.2. *For each n , $v_n(t, x)$ is an increasing function of x , $u_n(t, x)$ is of bounded variation in x , $v_n(t, x)$ is of bounded variation in t and $u_n(t, x)$ is an increasing function of t .*

4. Convergence of the approximating solutions. We shall now show that the sequences $\{u_n(t, x)\}$ and $\{v_n(t, x)\}$ contain a subsequence which converges in the topology of L_1 -convergence on compact subsets of $t \geq 0$. To do this we shall show that $\{u_n(t, x)\}$ and $\{v_n(t, x)\}$ have uniformly bounded variation in the sense of Tonelli-Cesari on each compact subset of $t \geq 0$. A function $f(t, x)$ is said to have bounded variation in the sense of Tonelli-Cesari over the compact subset Q if there exists a set $Z \subset Q$ of measure zero such that the functions

$$v(t) = \text{var}_{Q-Z} f(t, \cdot)$$

and

$$w(x) = \text{var}_{Q-Z} f(\cdot, x)$$

are summable over the projections of Q on the t and x axes respectively [1]. This is equivalent to the fact that the distribution derivatives of f are measures having finite variation over the set Q , [9].

Let Q be a compact subset of $t \geq 0$ of the form $0 \leq t \leq T$, $-X \leq x \leq X$. We let $\pi_x = \{-X = x_1 < x_2 < \cdots < x_p = X\}$ and $\pi_t = \{0 = t_1 < t_2 < \cdots < t_q = T\}$ be partitions of $-X \leq x \leq X$ and $0 \leq t \leq T$ respectively. The variation of v_n over Q in the sense of Tonelli-Cesari, $V(v_n; Q)$, is the sup over all partitions π_x and π_t of sums of the form

$$\sum |v_n(t, x_i) - v_n(t, x_{i-1})| \Delta_k t + |v_n(t_k, x) - v_n(t_{k-1}, x)| \Delta_i x$$

where the summation is taken over $x_i \in \pi_x$, $t_k \in \pi_t$, $t_{k-1} \leq t \leq t_k$, $x_{i-1} \leq x \leq x_i$, and where $\Delta_k t = t_k - t_{k-1}$, $\Delta_i x = x_i - x_{i-1}$. $V(u_n; Q)$ is defined similarly.

Note that (3.8) and (3.9) imply that there is a constant k_1 independent of n, t_1, t_2, x such that for $t_1 < t_2$

$$(4.1) \quad |v_n(t_2, x) - v_n(t_1, x)| \leq k_1 [u_n(t_2, x) - u_n(t_1, x)].$$

Also (3.7) shows that there is a constant k_2 independent of n, x_1, x_2 and t such that for $x_1 < x_2$

$$(4.2) \quad |u_n(t, x_2) - u_n(t, x_1)| \leq k_2 |v_n(t, x_2) - v_n(t, x_1)|.$$

Then from Lemma 3.2 and (4.1), we get

$$\begin{aligned} & \sum |v_n(t, x_i) - v_n(t, x_{i-1})| \Delta_k t + |v_n(t_k, x) - v_n(t_{k-1}, x)| \Delta_i x \\ &= \sum_k \Delta_k t \sum_i |v_n(t, x_i) - v_n(t, x_{i-1})| + \sum_i \Delta_i x \sum_k |v_n(t_k, x) - v_n(t_{k-1}, x)| \\ &\leq \sum_k \Delta_k t [v_n(t, X) - v_n(t, -X)] + k_1 \sum_i \Delta_i x [u_n(T, x) - u_n(0, x)] \\ &\leq 2M_1 T + 4M_1 X k_1 \end{aligned}$$

where M_1 is a bound for $u_n(t, x)$ and $v_n(t, x)$. Hence $V(v_n; Q)$ is bounded independently of n . Next from Lemma 3.2 and (4.2), we get (where the summation is over the same set as before)

$$\begin{aligned} & \sum |u_n(t, x_i) - u_n(t, x_{i-1})| \Delta_k t + |u_n(t_k, x) - u_n(t_{k-1}, x)| \Delta_i x \\ &\leq \sum_k \Delta_k t \sum_i |u_n(t, x_i) - u_n(t, x_{i-1})| + \sum_i \Delta_i x \sum_k |u_n(t_k, x) - u_n(t_{k-1}, x)| \\ &\leq k_2 \sum_k \Delta_k t [v_n(t, X) - v_n(t, -X)] + \sum_i \Delta_i x [u_n(T, x) - u_n(0, x)] \\ &\leq 2M_1 T k_2 + 4M_1 X. \end{aligned}$$

Hence $V(u_n; Q)$ is also bounded independently of n . From [3, 6], this sequence forms a compact family so there is a subsequence of $(u_n(t, x), v_n(t, x))$ which converges in the sense of L_1 on Q . By a diagonal process we can find a subsequence, call it $(u_n(t, x), v_n(t, x))$ again, which converges in the sense of L_1 on any compact subset of the upper half-plane. Let the limit function be $(u(t, x), v(t, x))$. Then passing to the limit in (3.5) and (3.6), we see that $(u(t, x), v(t, x))$ is a solution of our problem (1.1), (1.2). Furthermore since L_1 convergence implies convergence a.e. of a subsequence, we see from (3.7) that for almost all (t, x) we have

$$(4.3) \quad -M[(v_1 - v_2)(f(v_2) - f(v_1))]^{1/2} \leq u_2 - u_1 \leq M \int_{v_1}^{v_2} [-f'(\xi)]^{1/2} d\xi$$

where $(u_i, v_i) = (u(t, x_i), v(t, x_i))$, $i = 1, 2$, $x_1 < x_2$ and

$$(4.4) \quad \bar{u}_2 - \bar{u}_1 \geq -M[(\bar{v}_1 - \bar{v}_2)(f(\bar{v}_2) - f(\bar{v}_1))]^{1/2}$$

$$(4.5) \quad \bar{u}_2 - \bar{u}_1 \geq M \int_{\bar{v}_1}^{\bar{v}_2} [-f'(\xi)]^{1/2} d\xi$$

where $(\bar{u}_i, \bar{v}_i) = (u(t_i, x), v(t_i, x))$, $i = 1, 2$, $t_1 < t_2$.

Therefore we see that u is an increasing function of t and is of bounded variation in x , while v is an increasing function of x and is of bounded variation in t .

We can summarize our results in the following theorem.

Theorem 4.1. *Let f and g be C^2 functions of a single real variable and let the system (1.1) be hyperbolic and genuinely-non linear, i.e., let (2.1) and (2.2) hold, and let the initial data (1.2) satisfy condition (C). Then there exists a global solution (u, v) of the Cauchy problem (1.1), (1.2). This solution has the property that u and v have locally bounded variation in the sense of Tonelli–Cesari. Furthermore the inequalities (4.3), (4.4) and (4.5) are valid.*

We remark that we can also show that our solution $(u(t, x), v(t, x))$ also satisfies, for every $t_1 > 0, t_2 > 0$,

$$\int_{-\infty}^{\infty} \{|u(t_2, x) - u(t_1, x)| + |v(t_2, x) - v(t_1, x)|\} dx \leq k |t_2 - t_1|$$

for some constant k independent of t_1 and t_2 . The proof of this is similar to a corresponding result in [11], and we omit the details.

Finally, we note that although there are bounded functions which have bounded variation in the sense of Tonelli–Cesari and are discontinuous almost everywhere, [C. Goffman, Two remarks on linearly continuous functions, *J. Math. Mech.*, **16** (1967) 1227–1228], the fact that u and v are each monotonic in one-variable implies that their discontinuities are sets of measure zero. (This fact was communicated to us orally by C. Goffman).

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