On Trigonometric Series Associated with Weak Closed Subspaces of Continuous Functions*

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Communicated by HANS LEWY

If f is a complex Lebesgue integrable function on the unit circle, $\hat{f}(n)$, the n^{th} Fourier coefficient of f , is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta \quad (n = 0, \pm 1, \pm 2, \dots).$$

Let C be the Banach space of continuous functions on the unit circle, A those functions in C with absolutely convergent Fourier series, i.e., those f in C for which $\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$, and L^{∞} (resp. L^1) the Banach spaces of equivalence classes of bounded measurable (resp. integrable) functions on the circle; L^{∞} is endowed with the essential supremum norm, denoted $\|\cdot\|_{\infty}$; and we regard C as being a closed subspace of L^{∞} .

If $\mathfrak{B}$ is one of the spaces, C , A , L^{∞} or L^1 , and E is a subset of $\mathbf{Z}$, the integers, we define

$$\mathfrak{B}_E = \{f \in \mathfrak{B} : \hat{f}(n) = 0 \text{ for all } n \notin E\}.$$

We show here that there exists a set E such that $L_E^{\infty} \subset C_E$, but $C_E \not\subset A$. Indeed, examples of such sets may be given explicitly as follows:

Given S a subset of $\mathbf{Z}$, and $l \in \mathbf{Z}$, define $lS = \{ln : n \in S\}$. Let $E_n = \{1, 2, \dots, n\}$. Then $E^1 = \bigcup_{n=0}^{\infty} (19)^n n! E_{n+1}$ and $E^2 = \bigcup_{n=1}^{\infty} (2n)! E_{2n}$ both have this property.

In Theorem 3, we give a general class of sets E for which $L_E^{\infty} \subset C_E$; the above examples follow as a corollary to this result.

We wish to thank Professor Y. Katznelson for a stimulating discussion on a related topic; indeed, the techniques we use are similar to those shown us by him, concerning a different problem.

* This research was supported by National Science Foundation grant GP-5207.

1. Motivation. A set E is called a Sidon set if $C_E \subset A$. It is known that if E is a Sidon set, then $L_E^\infty \subset C_E$ (cf. p. 207 of [2]). Our results show that the converse to this is false.

If $f \in C$, $f \notin A$, then it is fairly easy to show that there exists a sequence $\{\epsilon_n\}$, $\epsilon_n = 0$ or 1 for $n = 0, \pm 1, \pm 2, \dots$ such that $\sum \epsilon_n \hat{f}(n) e^{in\theta}$ is not the Fourier series of a bounded measurable function; i.e., such that for no g in L^∞ is it true that $\hat{g}(n) = \epsilon_n \hat{f}(n)$ for all n . One might think that it would then be possible to find another sequence δ_n of zeros and ones, so that $\sum \delta_n \hat{f}(n) e^{in\theta}$ was the Fourier series of a bounded measurable function, but not of a continuous one. Our results show that this is not the case, in general.

Finally, we note that $L_E^\infty \subset C_E$ if and only if C_E is a weak* closed subspace of L^∞ . Since L_E^∞ may be isometrically identified with the conjugate space of a certain Banach space (namely $L^1/L_{\sim E}^1$), our examples show that there are non-Sidon sets E such that C_E is isometric to a conjugate space.

2. We first have need of a lemma which shows that the sum of the sups may be dominated by the sup of the sum, under certain conditions. (See théorème IV, page 155 of [1] for a related result.)

Lemma. 1. Let $p_1, p_2, \dots$ be a sequence of bounded real-valued periodic functions defined on the real line, and $K_1, K_2, \dots$ a sequence of positive constants, satisfying the following conditions for all numbers n :

- (1) $|p_n(x) - p_n(y)| \leq K_n \|p_n\|_\infty |x - y|$ for all real numbers x and y (where $\|p_n\|_\infty = \sup |p_n(x)|$).
- (2) Period $p_{n+1} \leq (1/3K_n)$. (By which we mean, there exists a positive real number τ , so that $p_{n+1}(x + \tau) = p_{n+1}(x)$ for all real x , and $\tau \leq (1/3K_n)$.)
- (3) p_n assumes the value zero.
- (4) $2K_n \leq K_{n+1}$

Conclusion. For all N ,

$$\sum_{n=1}^N \|p_n\|_\infty \leq 6 \left\| \sum_{n=1}^N p_n \right\|_\infty.$$

Proof. Fix N , and let $a_n = \|p_n\|_\infty$ for all n . Let G denote the set of those integers n , such that $1 \leq n \leq N$, and such that there exists a real number x with $p_n(x) = a_n$. Let B denote those integers n , $1 \leq n \leq N$, such that $n \notin G$.

Thus

$$\sum_{n \in G} a_n + \sum_{n \in B} a_n = \sum a_n;$$

hence either

$$\sum_{n \in G} a_n \quad \text{or} \quad \sum_{n \in B} a_n \geq \frac{1}{2} \sum a_n.$$

(We use the un-indexed symbol “ $\sum$ ” to mean, the indices run from 1 to N .) Suppose the former; then $\sum_{n \in B} a_n \leq \frac{1}{2} \sum a_n$.

Now, we claim that a sequence $I_1 \supset I_2 \supset \cdots \supset I_N$ of bounded closed intervals may be chosen so that for all n , $1 \leq n \leq N$,

$$(*) \quad \begin{cases} \text{if } n \in G, & \text{then } p_n(x) \geq \frac{2}{3}a_n & \text{for all } x \in I_n; \\ \text{if } n \in B, & \text{then } |p_n(x)| \leq \frac{1}{3}a_n & \text{for all } x \in I_n. \end{cases}$$

To see this, we observe first that if y belongs to an interval containing x as an end point, of length $1/3K_n$, then

$$(**) \quad \begin{cases} \text{if } p_n(x) = a_n, & \text{then } p_n(y) \geq \frac{2}{3}a_n \\ \text{if } p_n(x) = 0, & \text{then } |p_n(y)| \leq \frac{1}{3}a_n. \end{cases}$$

((**) follows immediately from (1).)

Now, if $1 \in G$, choose x so that $p_1(x) = a_1$; if $1 \in B$, choose x , so that $p_1(x) = 0$, which we can do by (3). Now let I_1 be a closed interval of length $1/3K_1$ with x as an end point. Then I_1 satisfies (*) by (**).

Suppose I_l chosen, with length $|I_l| = 1/3K_l$. Now since period $p_{l+1} \leq 1/3K_l$, all values of range $|p_{l+1}|$ are taken on in I_l (of course, p_{l+1} is continuous by (1)).

If $l+1 \in G$, choose $x_0 \in I_l$, with $p_{l+1}(x_0) = a_{l+1}$.

If $l+1 \in B$, choose $x_0 \in I_l$, with $p_{l+1}(x_0) = 0$. Now let I_{l+1} be one of the closed intervals of length $1/3K_{l+1}$, which has x_0 as an endpoint, and which is contained in I_l . (Since $1/3K_{l+1} \leq \frac{1}{2}$ length $I_l = (1/2)(1/3K_l)$ by (4), either

$$\left[x_0, x_0 + \frac{1}{3K_{l+1}} \right] \subset I_l \quad \text{or} \quad \left[x_0 - \frac{1}{3K_{l+1}}, x_0 \right] \subset I_l.$$

Again, I_{l+1} satisfies (*) by (**); and we have that length $|I_{l+1}| = (1/3K_{l+1})$.

Thus the nested sequence $I_1 \supset \cdots \supset I_N$ has been constructed satisfying (*). Now, let $y_0 \in I_N$. Then,

$$\begin{aligned} \sum p_n(y_0) &= \sum_{n \in G} p_n(y_0) + \sum_{n \in B} p_n(y_0) \geq \frac{2}{3} \sum_{n \in G} a_n - \frac{1}{3} \sum_{n \in B} a_n \\ &\geq \frac{2}{3} \cdot \frac{1}{2} \sum a_n - \frac{1}{3} \cdot \frac{1}{2} \sum a_n = \frac{1}{6} \sum a_n = \frac{1}{6} \sum \|p_n\|_\infty. \end{aligned}$$

Thus we have demonstrated that

$$\sum_{n=1}^N \|p_n\|_\infty \leq 6 \left\| \sum_{n=1}^N p_n \right\|_\infty$$

under the assumption that $\sum_{n \in G} \|p_n\|_\infty \geq \frac{1}{2} \sum \|p_n\|_\infty$. However, if $\sum_{n \in G} \|p_n\|_\infty < \frac{1}{2} \sum \|p_n\|_\infty$, we have merely to replace p_n by $-p_n$ throughout. Q.E.D.

Remark. The above argument is valid under a weaker hypothesis than (2); all that is really used, is the fact that $|p_{n+1}|$ achieves both its minimum and maximum on any interval of length $1/3K_n$. This won't be used in the sequel, however.

Corollary 2. Let $h_1, h_2, \dots$ be a sequence of bounded complex valued periodic functions defined on the real line, and $K_1, K_2, \dots$ a sequence of positive constants,

so that for both $p_n = \operatorname{Re} h_n = \text{Real part of } h_n$, and $p_n = \operatorname{Im} h_n = \text{Imaginary part of } h_n$, (1) through (4) are satisfied, for all n .

Conclusion: $\sum_{n=1}^N \|p_n\|_\infty \leq 12 \|\sum_{n=1}^N p_n\|_\infty$ for all N .

Proof. Fix N ; by Lemma 1, we have that

$$\sum \|\operatorname{Re} h_n\|_\infty \leq 6 \|\sum \operatorname{Re} h_n\|_\infty$$

and

$$\sum \|\operatorname{Im} h_n\|_\infty \leq 6 \|\sum \operatorname{Im} h_n\|_\infty.$$

But if f is any bounded complex function, then

$$\|f\|_\infty \leq \|\operatorname{Re} f\|_\infty + \|\operatorname{Im} f\|_\infty \leq 2 \|f\|_\infty.$$

Thus

$$\begin{aligned} 2 \|\sum h_n\|_\infty &\geq \|\operatorname{Re} \sum h_n\|_\infty + \|\operatorname{Im} \sum h_n\|_\infty \\ &\geq \frac{1}{6} \sum (\|\operatorname{Re} h_n\|_\infty + \|\operatorname{Im} h_n\|_\infty) \\ &\geq \frac{1}{6} \sum \|h_n\|_\infty; \end{aligned}$$

$$\text{i.e.,} \quad \sum_{n=1}^N \|h_n\|_\infty \leq 12 \left\| \sum_{n=1}^N h_n \right\|_\infty. \quad \text{Q.E.D.}$$

Our next result gives a class of sets E for which $L_E^\infty \subset C_E$; we first need the following notation: if S is a finite subset of the integers, let

$$\begin{aligned} M(S) &= \max \{|j| : j \in S\} \\ m(S) &= \min \{|j| : j \in S\}. \end{aligned}$$

Theorem 3. Let $F_1, F_2, \dots$ be a sequence of finite disjoint subsets of the integers, and $f_1, f_2, \dots$ a sequence of positive integers, satisfying the following two conditions for all numbers n :

- (a) $F_{n+1} \subset f_{n+1}\mathbf{Z}$ (where $\mathbf{Z}$ denotes the integers).
- (b) $f_{n+1} \geq 6\pi M(F_n)$.

Conclusion: Let $E = \bigcup_{n=1}^\infty F_n$; then $L_E^\infty \subset C_E$.

Proof. We may first suppose that $0 \notin F_n$ for all n . Indeed, our hypotheses imply that 0 belongs to at most one of the F_n 's; but if E' has the property that $L_{E'}^\infty \subset C_{E'}$, then the union of E' with any finite set has this property also. Next, suppose $q_1, q_2, q_3, \dots$ are given with $q_n \in C_{F_n}$ for all n . We claim that

$$\sum_{n=1}^N \|q_n\|_\infty \leq 12 \left\| \sum_{n=1}^N q_n \right\|_\infty$$

for all integers N .

Well, define $h_n(x) = q_n(e^{ix})$ for all real numbers x , and let $K_n = M(F_n)$. We have only to show that the h_n 's and K_n 's satisfy the hypotheses of Corollary 2. So, let p_n denote either $\operatorname{Re} h_n$ or $\operatorname{Im} h_n$. Then p_n is a trigonometric polynomial of degree at most K_n , whence by Bernstein's theorem (cf. page 11 of [3]),

$$|p'_n(x)| \leq K_n \|p_n\|_\infty \quad \text{for all } x;$$

thus (1) is satisfied.

By (a), h_{n+1} has $2\pi/f_{n+1}$ as a period, whence

$$\text{period } p_{n+1} \leq \frac{1}{3M(F_n)} = \frac{1}{3K_n}$$

by (b), so (2) is satisfied. Since $0 \notin F_n$ for all n by assumption,

$$0 = \int_0^{2\pi} q_n(e^{ix}) dx = 2\pi \hat{q}_n(0) = \int_0^{2\pi} h_n(x) dx.$$

Thus $\int_0^{2\pi} \operatorname{Re} h_n(x) dx = \int_0^{2\pi} \operatorname{Im} h_n(x) dx = 0$; thus (3) holds for p_n , since p_n is a continuous real function with $\int_0^{2\pi} p_n(x) dx = 0$.

Finally, again since $0 \notin F_n$ for all n , (a) implies that $m(F_{n+1}) \geq f_{n+1}$, whence by (b)

$$6\pi M(F_n) = 6\pi K_n \leq f_{n+1} \leq m(F_{n+1}) \leq M(F_{n+1}) = K_{n+1},$$

so (4) is satisfied.

We note for later use, that the last inequality shows that

$$M(F_n) < m(F_{n+1}) \quad \text{for all } n.$$

Now suppose that $f \in L_E^\infty$. Let f_n be defined by

$$f_n(e^{ix}) = \sum_{j \in F_n} \hat{f}(j) e^{ijx} \quad \text{for all } x.$$

Fix N an integer, and choose q a trigonometric polynomial such that

$$q \in C_E, \quad \|q\|_\infty \leq 2 \|f\|_\infty,$$

and

$$\hat{q}(j) = \hat{f}(j) \quad \text{for all } |j| \leq M(F_N).$$

(Indeed, if we let $K_n(x)$ be the n^{th} Fejer kernel defined by

$$K_n(x) = \sum_{j=-n}^n \left(1 - \frac{|j|}{n+1}\right) e^{ijx},$$

and set $V_n(x) = 2K_{2n-1}(x) - K_{n-1}(x)$, then q may be given as

$$q(e^{iy}) = \frac{1}{2\pi} \int_0^{2\pi} V_{M(F_N)}(y-x) f(e^{ix}) dx.$$

Now let

$$q_n(e^{ix}) = \sum_{j \in F_n} \hat{q}(j) e^{ijx}$$

for all n . We thus have that $q = \sum q_n$, with only finitely many terms being non-zero, since q is a trigonometric polynomial; moreover, $q_n \in C_{F_n}$ for all n . Finally, since $M(F_n) < m(F_{n+1})$, we have that

$$F_n \subset \{j: |j| \leq M(F_N)\}$$

for all $n \leq N$, whence $f_n = q_n$ for all $n \leq N$.

We thus have that

$$\sum_{n=1}^N \|f_n\|_{\infty} \leq \sum \|q_n\|_{\infty} \leq 12 \|\sum q_n\|_{\infty} = 12 \|q\|_{\infty} \leq 24 \|f\|_{\infty}.$$

Thus since N was arbitrary, we have that

$$\sum_{n=1}^{\infty} \|f_n\|_{\infty} < \infty.$$

So the series $\sum_{n=1}^{\infty} f_n(e^{ix})$ converges uniformly, to a continuous function h , say. Then $\hat{h}(j) = \hat{f}(j)$ for all integers j , whence $h = f$ by uniqueness of Fourier coefficients. Q.E.D.

Remark. We have actually shown that if $q \in C_E$, then

$$\sum_{n=1}^{\infty} \|q_n\|_{\infty} \leq 12 \|q\|_{\infty},$$

where E is the set given in the statement of the theorem, and

$$q_n = \sum_{i \in F_n} \hat{q}(i) e^{ix}$$

for all n . For at the beginning of the proof, we showed that this was true if $q \in C_E$, q a trigonometric polynomial; (for then there would exist an integer N so that $q = \sum_{n=1}^N q_n$); but the trigonometric polynomials in C_E are dense in C_E .

Corollary 4. Let $E^1 = \bigcup_{n=0}^{\infty} (19)^n n! E_{n+1}$, $E^2 = \bigcup_{n=1}^{\infty} (2n)! E_{2n}$. Then $L_E^{\infty} \subset C_E$, but E^i is not a Sidon set (i.e., $C_{E^i} \subsetneq A$), for $i = 1, 2$.

Proof. If $f_1, f_2, \dots$ is any sequence of integers with $\limsup |f_n| > 0$, then $E = \sum_{n=1}^{\infty} f_n E_n$ cannot be a Sidon set. Indeed, no Sidon set can contain finite arithmetic progressions of arbitrarily long length (cf. p. 214 of [2]). So, the f_n 's need only be chosen so that setting

$$F_{n+1} = f_n E_n,$$

condition (b) of Theorem 3 is satisfied. ((a) holds automatically, since $f_n E_n \subset f_n \mathbb{Z}$.)

This reduces to $f_{n+1} \geq 6\pi n f_n$. This condition is of course satisfied if $f_{n+1} = (19)^n n!$. For the case of E^2 , we simply let $F_n = (2n)! E_{2n}$ and $f_n = (2n)!$, and again apply Theorem 3. Q.E.D.

REFERENCES

- [1] KAHANE, J. P. & SALEM, R., *Ensembles parfaits et séries trigonométriques*, Hermann, Paris, 1963.
- [2] RUDIN, W., Trigonometric series with gaps, *J. Math. Mech.*, 9 (1960) 203-227.
- [3] ZYGMUND, A., *Trigonometric Series, II*, Cambridge University Press, 1959.

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Date Communicated: MAY 15, 1967