*A Method of Generating Classes of Huygens' Operators**

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1. Let $\mathcal{L}$ be a linear hyperbolic differential operator defined in some domain $\mathfrak{D} \subset E_n$. It is assumed that the coefficients of $\mathcal{L}$ are smooth enough in order to allow in some neighborhood $N(\xi) \subset \mathfrak{D}$ of every point $\xi \in \mathfrak{D}$ the existence of a special elementary solution $\mathcal{E}(x, \xi)$ (in the sense of L. Schwartz) whose support is in the closure of the interior of the backward characteristic conoid $\mathcal{C}h_{(\xi)}$ with vertex at ξ (considered in $N(\xi)$ only). $\mathcal{L}$ is called a *Huygens' operator* (in short, an *H-operator*) in a domain $\mathfrak{D}$ if, for each $\xi \in \mathfrak{D}$, the support of $\mathcal{E}(x, \xi)$ is restricted to the surface of C_ξ . This definition is equivalent to Hadamard's "minor premise" ([1], p. 54). *H-operators* are also called operators of pure wave type (Matthiesson [2], Günther [3]). If $\mathcal{L}$ is an *H-operator*, the solution $u(\xi)$ of every proper Cauchy initial value problem for the equation $\mathcal{L}u = 0$ depends on the initial data only on the intersection of the surface of C_ξ with the spacelike manifold which carries the initial data.

The Huygens' character of $\mathcal{L}$ is invariant under

- (i) proper coordinate transformations,
- (ii) "factor transformations" of $\mathcal{L}$ defined by

$$u = \lambda(x)\bar{u}, \quad \bar{\mathcal{L}}\bar{u} = \frac{1}{\lambda} \mathcal{L}(\lambda\bar{u}) \quad (\lambda \in C^2, \lambda \neq 0),$$

- (iii) multiplication of $\mathcal{L}$ by a scalar function, *i.e.*

$$\bar{\mathcal{L}} = \mu\mathcal{L} \quad (\mu \neq 0).$$

These three transformation groups are called *trivial transformations*. Two operators are called *equivalent* if each can be transformed to the other through trivial transformations. Operators equivalent to the ordinary wave operator $\partial^2/\partial t^2 - \sum_{i=1}^{n-1} \partial^2/\partial x_i^2$ are called *trivial*. *Hadamard's problem* consists of finding all equivalence classes of *H-operators*.

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Hadamard has shown that H -operators have necessarily an even number $n \geq 4$ of independent variables, and that the vanishing of the logarithmic part of his fundamental solution in the interior of C_ξ is necessary and sufficient for the Huygens' behavior of a hyperbolic operator (see Section 2). Matthisson proved in [2] that all H -operators in four independent variables with constant principal part are trivial, provided the coefficients are real-valued. If complex-valued coefficients are admitted for these operators, the question has been completely settled in [6] where large classes of nontrivial H -operators are found. Other nontrivial classes of real H -operators have been given in [4, 5] and in [7] for operators with constant principal part, and by Günther [8] for $n = 4$ and nonconstant principal part.

In this note we present an infinite number of new equivalence classes of nontrivial H -operators. These are obtained by constructing certain *nontrivial* transformations of one H -operator into another H -operator with a greater number of independent variables. By applying these transformations first to the wave operator and then to its transforms (in this sense), we obtain an infinite collection of classes of H -operators which contains in particular all known classes of real H -operators with the exception of those given in [8]. Thus all known classes of real H -operators with constant principal part are included.

2. We shall need the following facts from Hadamard's theory [1]: For an even number n of independent variables the hyperbolic differential operator

$$(2.1) \quad \mathfrak{L}_n = \square_n + \sum_{i=0}^{n-1} b^i(x) \frac{\partial}{\partial x_i} + c(x), \quad \square_n = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial x_0^2}$$

with coefficients analytic in a domain $\mathfrak{D}$ admits, in some neighborhood of each point $\xi \in \mathfrak{D}$, a fundamental solution of the form

$$(2.2) \quad V = \Gamma^{(2-n)/2} V(x, \xi) + W(x, \xi) \log \Gamma + R = K(x, \xi) + R.$$

Here $\Gamma = -\sum_{i=1}^{n-1} (x_i - \xi_i)^2 + (x_0 - \xi_0)^2$ and the analytic functions V and W are given by the developments

$$(2.3) \quad V = \sum_{r=0}^{(n-4)/2} U_r(x, \xi) \Gamma^r, \quad W = \sum_{r=(n-2)/2}^{\infty} U_r(x, \xi) \Gamma^{r-(n-2)/2}.$$

The $U_r(x, \xi)$ are well determined regular functions; R doesn't enter the theory of integration. Hadamard's theory establishes with (2.2) explicit formulas for the solution of Cauchy's problem ([1], pg. 233) from which the elementary solution $\mathfrak{E}(x, \xi)$, mentioned in the beginning of the present paper can be read off. $\mathfrak{L}_n$ is an H -operator in $\mathfrak{D}$ if the logarithmic part $W(x, \xi)$ vanishes for each $\xi \in \mathfrak{D}$ and all $x \in N(\xi) \subset \mathfrak{D}$ which lie in the interior of the backward characteristic cone $\Gamma(x, \xi) = 0$. As W is a regular solution of $\mathfrak{L}_n u = 0$, this can happen if and only if

$$(2.3a) \quad W|_{\Gamma=0} = V_{(n-2)/2}|_{\Gamma=0} = 0.$$

This condition (2.3a) is invariant under the transformations (i), (ii), (iii) mentioned in the beginning.

We shall be especially interested in operators $\mathfrak{L}_n$ of the form

$$(2.4) \quad \mathfrak{L}_n = \mathfrak{L}_m + \Delta_{n-m},$$

where $\mathfrak{L}_m$ is a linear hyperbolic operator in $x_0, x_1, \dots, x_{m-1}$ of the form (2.1) and $\Delta_{n-m} = \sum_{i=m}^{n-1} \partial^2 / \partial x_i^2$. In this case the coefficients U_ν in (2.3) depend only on the coordinates $x_0, x_1, x_2, \dots, x_{m-1}; \xi_0, \xi_1, \dots, \xi_{m-1}$ but not on the coordinates $x_m, x_{m+1}, \dots, x_{n-1}; \xi_m, \xi_{m+1}, \dots, \xi_{n-1}$ (compare Hadamard [1], page 108). If we introduce furthermore the notation

$$(2.4a) \quad U_\nu = k_\nu^{-1} V_\nu; \quad k_0 = 1; \quad k_\nu = \alpha_1 \alpha_2 \cdots \alpha_\nu; \quad \alpha_h = 4(h - (n - 2)/2)$$

if $2h \neq n - 2$; $\alpha_{(n-2)/2} = 4$, then the operators $\mathfrak{L}_n$ and $\mathfrak{L}_m$ in (2.4) will have the same coefficients $V_\nu(x, \xi)$. This independence of the coefficients V_ν of the number of dimensions is closely connected with Hadamard's method of descent. Furthermore it plays a decisive role in Marcel Riesz's construction of his kernel for a not constant metric ([13], page 182).

Since for $n > m$ the U_ν do not depend on all the coordinates, it follows that $U_{(n-2)/2}$ vanishes on $\Gamma = 0$ if and only if it vanishes identically, and this implies that $U_\nu = 0$ for $\nu \geq (n - 2)/2$. (If the U_ν depend on all the coordinates, it is possible that $U_{(n-2)/2}$ is zero on $\Gamma = 0$ but not identically zero. In this case we may have $W \equiv 0$ but a never terminating sequence $\{U_\nu\}$. [4]) Thus for $n > m$, $\mathfrak{L}_n$ in (2.4) is an H -operator if and only if $V_{(n-2)/2} \equiv 0$. If (2.1) is not an H -operator but the series development for W terminates after p terms (*i.e.*, $V_{(n-2)/2+p-1} \equiv 0$), then $\mathfrak{L}_{n+2p} = \mathfrak{L}_n + \Delta_{2p}$ will be an H -operator provided the coordinates appearing in Δ_{2p} do not appear in $\mathfrak{L}_n$. This last point will be essential in our considerations in the following sections.

3. Here we consider the still more specialized operator

$$(3.1) \quad \mathfrak{L}_n = \Delta_{n-1} - \left(\frac{\partial^2}{\partial t^2} + c(t) \right), \quad \Delta_{n-1} = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2},$$

where $c(t)$ is continuous on some interval I . The decomposition

$$\frac{\partial^2}{\partial t^2} + c(t) = \left(\frac{\partial}{\partial t} + b \right) \left(\frac{\partial}{\partial t} - b \right) = \frac{\partial^2}{\partial t^2} - \frac{\partial b}{\partial t} - b^2$$

is valid whenever we put $b = \mu_t / \mu$ and choose μ as a nonzero solution of

$$(3.2) \quad \mu_{tt} + c(t)\mu = 0 \quad \left(\mu_t = \frac{d\mu(t)}{dt} \right)$$

on an interval $I' \subset I$. We define the linear operators

$$(3.3) \quad l = \frac{\partial}{\partial t} - \frac{\mu_t}{\mu}, \quad l^* = -\frac{\partial}{\partial t} - \frac{\mu_t}{\mu}$$

and write

$$(3.4) \quad \mathfrak{L}_n = \Delta_{n-1} + l^* l.$$

is independent of n and

$$(3.5) \quad ll^* - l^*l = -2(\log \mu)_{,t} = c(t) + 2\left(\frac{\mu_t}{\mu}\right)^2.$$

From (3.4) it immediately follows that

$$(3.6) \quad l\mathfrak{L}_n = \tilde{\mathfrak{L}}_n l, \quad \tilde{\mathfrak{L}}_n = \Delta_{n-1} + ll^*.$$

The operator $\tilde{\mathfrak{L}}_n$ defined by (3.2), (3.3) and (3.6) is called the l -transform (on I') of $\mathfrak{L}_n$. The operators $\mathfrak{L}_n$ and $\tilde{\mathfrak{L}}_n$ will be identical if and only if (3.5) vanishes, and this can happen only if $c(t) \equiv \text{const}$. If u is three times differentiable and $\mathfrak{L}_n u = 0$, then $\tilde{\mathfrak{L}}_n(lu) = 0$.

With this simple formalism we present our main result.

Theorem 1. *Let $\mathfrak{L}_n$ be defined by (3.1) with $c(t)$ analytic on an interval I . Let $\tilde{\mathfrak{L}}_n$ be the l -transform of $\mathfrak{L}_n$ on some interval $I' \subset I$. If $\mathfrak{L}_n$ is an H -operator in $I' \times R^{n-1}$, then $\tilde{\mathfrak{L}}_{n+2}$ is an H -operator in $I' \times R^{n+1}$.*

Proof. Let $K = \Gamma^{(2-n)/2} V + W \log \Gamma$ and $\tilde{K} = \Gamma^{(2-n)/2} \tilde{V} + \tilde{W} \log \Gamma$ be the irregular parts of the fundamental solutions of $\mathfrak{L}_n$ and $\tilde{\mathfrak{L}}_n$ respectively. According to Section 2, the coordinates $x_1, \dots, x_{n-1}$ (and $\xi_1, \dots, \xi_{n-1}$) enter into K and $\tilde{K}$ only through Γ so that, e.g., $W = W(t, \tau, r)$, where

$$r = \left[\sum_{i=1}^{n-1} (x_i - \xi_i)^2 \right]^{1/2} = |x - \xi|.$$

To prove the theorem, we first show that

$$(3.7) \quad l_\tau W(\tau, t, r) - l_\tau^* \tilde{W}(\tau, t, r) = 0$$

for each point (τ, ξ) in $\mathfrak{D} = I' \times R^{n-1}$ and all (t, x) in $\mathfrak{D}$ lying in the interior of the backwards characteristic cone $(t - \tau)^2 - r^2 = 0$. In (3.7) the subscripts on l and l^* indicate the variables in which the operators are acting, e.g. $l_\tau = \partial/\partial\tau - \mu_\tau(\tau)/\mu(\tau)$.

(The same identity holds for the corresponding elementary solutions in the sense of L. Schwartz. Furthermore we have the same identity for both Riesz-kernels (see [13]) $K_\alpha(x, y)$ and $\tilde{K}_\alpha(x, y)$, i.e. $l_{x_1} K_\alpha(x, y) - l_{x_1}^* \tilde{K}_\alpha(x, y) = 0$ for every complex α for which the lefthand side is defined. (3.7) can be obtained as a consequence of this identity. M. Riesz's theory offers some advantages in connection with the problem treated here, in particular for the performance of the extended method of descent (and ascent) which we use, at least implicitly, so often in this paper. However Hadamard's theory is still better adapted to the present problem.)

Write (t_0, t_1) for I' and fix a point (τ, ξ) in $\mathfrak{D} = I' \times R^{n-1}$. Let R be the region bounded by the backwards characteristic cone with vertex at (τ, ξ) and the plane $t = t_0$, and $f(t, x)$ a function infinitely differentiable with compact support in R . We consider the initial value problem $\mathfrak{L}_n u = f$ in $\mathfrak{D}$, $u = \partial u/\partial t = 0$

on $t = t_0$, whose solution at the point (τ, ξ) is given by Hadamard's theory and (2.2) as

$$u(\tau, \xi) = - \int_R W(\tau, t, |x - \xi|) f(t, x) dx dt.$$

Therefore

$$(3.8) \quad l_\tau u(\tau, \xi) = - \int_R l_\tau W(\tau, t, |x - \xi|) f(t, x) dx dt.$$

On the other hand, $l_\tau u(t, x)$ is a solution of $\tilde{\mathfrak{L}}_n v = l_\tau f(t, x)$ in $\mathfrak{D}$, $v = \partial v / \partial t = 0$ on $t = t_0$ so that

$$(3.9) \quad \begin{aligned} l_\tau u(\tau, \xi) &= - \int_R \tilde{W}(\tau, t, |x - \xi|) l_\tau f(t, x) dx dt \\ &= - \int_R l_\tau^* \tilde{W}(\tau, t, |x - \xi|) f(t, x) dx dt. \end{aligned}$$

(3.7) follows immediately from (3.8) and (3.9).

If now $W \equiv 0$, then $l_\tau^* \tilde{W}(\tau, t, r) \equiv 0$. From $\tilde{\mathfrak{L}}_n \tilde{W} = (\Delta_{n-1} + l_\tau l_\tau^*) \tilde{W} = 0$ follows therefore that $\Delta_{n-1} \tilde{W} = 0$. But $\tilde{W}$ depends on the variables appearing in Δ_{n-1} only implicitly in terms of r . As $\tilde{W}$ must be regular as $r \rightarrow 0$ it follows that $\tilde{W}$ is independent of r and therefore the unique power series development $\tilde{W} = \sum_{\nu=(n-2)/2}^{\infty} k_\nu \tilde{V}_\nu \Gamma^{\nu-(n-2)/2}$ must terminate after the first term. This implies the theorem.

It may be mentioned that similar considerations, together with the symmetry relation $\tilde{W}(t, \tau, r) = \tilde{W}(\tau, t, r)$, give explicitly

$$\tilde{W} = \frac{k}{\mu(t)\mu(\tau)}, \quad k = \text{constant}.$$

If $k \neq 0$, the operators $\mathfrak{L}_n$ and $\tilde{\mathfrak{L}}_n$ cannot be equivalent in the sense of Section 1 since the trivial transformations preserve the Huygens' character of an operator, but $\tilde{W} \neq 0$ and $W \equiv 0$. (compare (2.3a)).

4. In order to construct classes of H -operators with the aid of Theorem 1, we start with the wave operator $\mathfrak{L}_4^0 = \square_4$ and take $\mu(t) = k_1 t + k_2$ ($k_1 \neq 0$), so that the operators in (3.3) become

$$l_t^0 = \frac{\partial}{\partial t} - \frac{1}{t+k}, \quad l_t^{0*} = -\frac{\partial}{\partial t} - \frac{1}{t+k}, \quad k = \frac{k_2}{k_1}.$$

The operators $\Delta_5 + l_t^0 l_t^{0*} = \Delta_5 - \partial^2 / \partial t^2 + 2/(t+k)^2$, which are obviously equivalent to

$$(4.1) \quad \mathfrak{L}_5^1 = \Delta_5 - \left(\frac{\partial^2}{\partial t^2} - \frac{2}{t^2} \right),$$

are H -operators according to Theorem 1. This coincides with a result in [5].

The ordinary differential equation $\mu_{tt} - (2/t^2)\mu = 0$ associated with (4.1) has for its general solution $\mu(t) = k_1 t^2 + k_2 t^{-1}$ so that

$$l_i^1 = \frac{\partial}{\partial t} - \frac{\mu_i}{\mu} = \frac{\partial}{\partial t} - \frac{2t^3 - k}{t(t^3 + k)}, \quad l_i^{1*} = -\frac{\partial}{\partial t} - \frac{2t^3 - k}{t(t^3 + k)}, \quad k = \frac{k_2}{k_1}.$$

Therefore

$$(4.2) \quad \Delta_7 + l_i^1 l_i^{1*} = \Delta_7 - \left(\frac{\partial^2}{\partial t^2} - \frac{6t(t^3 - 2k)}{(t^3 + k)^2} \right)$$

is a family of H -operators, in agreement with [7]. The parameter k in (4.2) may be eliminated through the trivial transformations

$$(4.3) \quad \alpha t + \beta_0 \rightarrow t, \quad \alpha x_i + \beta_i \rightarrow x_i, \quad \frac{1}{\alpha^2} \mathfrak{L} \rightarrow \mathfrak{L}$$

with a suitable choice of the constants α and β_i , e.g. $\alpha = (-k)^{-1/3}$, $\beta_i = 0$, $i = 0, \dots, 7$. (These coordinate transforms form the subgroup of the general conformal group (*i.e.* the group of coordinate transforms which leaves the principle part of (3.1) unchanged up to a scalar factor) which leaves (3.1) self-adjoint and preserves the dependence of c on t alone.) Thus the operators in (4.2) are all equivalent to

$$(4.4) \quad \mathfrak{L}_3^2 = \Delta_7 - \left(\frac{\partial^2}{\partial t^2} - \frac{6t(2 + t^3)}{(1 - t^3)^2} \right).$$

Continuing in this fashion we obtain a sequence of operators $\mathfrak{L}_n^p = \Delta_{n-1} - (\partial^2/\partial t^2 + c^p(t))$ defined recursively by

$$(4.5) \quad \mathfrak{L}_n^0 = \square_n, \quad l_i^{p-1} \dots l_i^0 \square_n = \mathfrak{L}_n^{p l_i^{p-1} \dots l_i^0},$$

where

$$(4.6) \quad l_i^\nu = \frac{\partial}{\partial t} - \frac{\mu_i^\nu}{\mu^\nu}, \quad \mu_{it}^\nu + c^\nu(t)\mu^\nu = 0, \quad (\nu = 0, 1, \dots).$$

$\mathfrak{L}_n^p$ will be Huygens' for $n \geq 2(p+2)$, n even. In each l_i^p one arbitrary integration constant can be chosen so that $\mathfrak{L}_n^p$ will in general depend on p arbitrary parameters, two of which may be eliminated through the transformations (4.3). What remains is a $(p-2)$ -parameter family of H -operators, no two of which will be, in general, equivalent. For example, if we apply our transformation to (4.4), choosing for the solution of the associated ordinary differential equation the function

$$\mu(t) = \frac{t}{1 - t^3} + k \frac{5 + 5t^3 - t^6}{1 - t^3},$$

we obtain the one parameter family of H -operators

$$\mathfrak{L}_{10}^3 = \Delta_9 - \left(\frac{\partial^2}{\partial t^2} - \frac{2[1 + 18kt - 3k^2t(50 - 75t^3 - 2t^9)]}{[t + k(5 + 5t^3 - t^6)]^2} \right).$$

The so-called E.P.D. (Euler-Poisson-Darboux) operators in self-adjoint form

$$\Delta_{n-1} = \left(\frac{\partial}{\partial t^2} - \frac{p(p+1)}{t^2} \right), \quad (p = 0, 1, \dots)$$

which are H -operators for n even and $n \geq 2(p+2)$ (see [5]), may be obtained by choosing the sequence $\mu^p = t^{p+1}$, $l_t^p = \partial/\partial t - \mu_t^p/\mu^p = \partial/\partial t - (p+1)/t$, ($p = 0, 1, \dots$). In this case our transformation reduces essentially to the "correspondence principle" which has been used successfully by A. Weinstein and others.

However it should be emphasized that there is little connection between the present paper and the theory of E.P.D. operators as it is developed by Weinstein [9, 10], Diaz and Weinberger [11] and others, because these authors have been interested mainly in the singular Cauchy problem for the nonself-adjoint E.P.D. operator. (One of the authors would like to use this occasion to complete a bibliographical note made in [5]. Due to an unfortunate oversight, the well-known and important papers of A. Weinstein dealing with the singular Cauchy problem for the E.P.D. equation were not mentioned at that time. For this literature, see A. Weinstein ([9, 10] and references contained therein).) In that case the preservation of the Huygens' character of the solution under our transformation is just a simple application of the correspondence principle. This, however, is true only for the singular problem. (See in particular [11], p. 712.)

5. In order to achieve some generalization of Theorem 1, it seems close at hand to start with the interchange relation $l\mathcal{L} = \tilde{\mathcal{L}}l$, or somewhat more generally

$$(5.1) \quad l\mathcal{L} = \tilde{\mathcal{L}}l,$$

where l and $\mathcal{L}$ are given linear differential operators of order one and two respectively. (5.1) can be considered as the integrability condition for the overdetermined system of equations $\mathcal{L}u = 0$, $lu = 0$. This system is completely integrable if there is a scalar α and a first order operator b such that $l\mathcal{L} - \mathcal{L}l = \alpha\mathcal{L} + bl$, which is the same as (5.1) after writing $\tilde{l} - \alpha = l$, $\mathcal{L} + b = \tilde{\mathcal{L}}$. It is well known that (5.1) can be satisfied in many ways. However (5.1) alone will not in general lead to a simple generalization of Theorem 1. By way of example, choose $\mathcal{L} = \square_4$ and instead of using the infinitesimal translation $\partial/\partial t$, consider the infinitesimal rotation $x_2 \partial/\partial x_1 - x_1 \partial/\partial x_2$ which commutes with $\square_4$. We introduce cylindrical coordinates $x_1 = r \sin \phi$, $x_2 = r \cos \phi$ so that $x_2 \partial/\partial x_1 - x_1 \partial/\partial x_2 = \partial/\partial \phi$ and

$$\square_4 = \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial x_3^2} + \frac{1}{r} \frac{\partial}{\partial r} \right).$$

Define the operators

$$l = \frac{\partial}{\partial \phi} - \frac{1}{\phi}, \quad l^* = -\frac{\partial}{\partial \phi} - \frac{1}{\phi},$$

and write $\partial^2/\partial\phi^2 = -l^*l$. In analogy with (3.6) we have $l\Box_4 = \tilde{\mathfrak{L}}_4 l$, where

$$(5.2) \quad \tilde{\mathfrak{L}}_4 = \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left[\frac{\partial^2}{\partial \phi^2} - \frac{2}{\phi^2} \right] + \frac{\partial^2}{\partial x_3^2} \right),$$

or in Cartesian coordinates

$$\tilde{\mathfrak{L}}_4 = \Box_4 + \frac{2}{(x_1^2 + x_2^2) \left[\arctan \frac{x_1}{x_2} \right]^2}.$$

But none of the operators $\tilde{\mathfrak{L}}_{2n} = \tilde{\mathfrak{L}}_4 + \Delta_{2n-4}$ ($n \geq 3$) can be an H -operator. For $\tilde{\mathfrak{L}}_4$ this follows from [4] where all H -operators of the form $\Box_s + c(t, x_1, \dots, x_s)$ are given explicitly. To prove our statement for all $n \geq 3$, let

$$\tilde{u} = \frac{1}{\Gamma} + \tilde{W} \log \Gamma + R, \quad \tilde{W} = \sum_{\nu=0}^{\infty} \tilde{W}_\nu \Gamma^\nu,$$

be the fundamental solution of (5.2). Here R is a regular function and

$$\Gamma = \bar{t}^2 - \bar{x}_3^2 - r^2 - \rho^2 + 2r\rho \cos \bar{\phi},$$

where $\bar{t} = t - \tau$, $\bar{x}_i = x_i - \xi_i$, $\bar{\phi} = \phi - \phi_0$ and $\xi_1 = \rho \sin \phi_0$, $\xi_2 = \rho \cos \phi_0$. Since the $\tilde{W}_\nu$ are independent of x_3 it suffices to show that the series development for $\tilde{W}$ does not terminate. Beyond that we will get the fundamental solution in explicit form.

Since $1/\Gamma$ is the fundamental solution of $\Box_4$ we have $\tilde{\mathfrak{L}}_4[l(1/\Gamma)] = 0$ for $\Gamma \neq 0$. In addition $\tilde{\mathfrak{L}}_4(\partial \tilde{u}/\partial \phi_0) = 0$ so that $v = l(1/\Gamma) + \partial \tilde{u}/\partial \phi_0$ is again a singular solution of $\tilde{\mathfrak{L}}_4 v = 0$. But v has precisely the singularities as $\tilde{u}$, and because of the uniqueness of the irregular part of $\tilde{u}$ we must have

$$(5.3) \quad l\left(\frac{1}{\Gamma}\right) + \frac{\partial}{\partial \phi_0} \left(\frac{1}{\Gamma}\right) + \frac{\partial \tilde{W}}{\partial \phi_0} \log \Gamma + \frac{1}{\Gamma} \frac{\partial \Gamma}{\partial \phi_0} \tilde{W}_0 = \left(\frac{1}{\Gamma} + \tilde{W} \log \Gamma\right)g,$$

where g depends only on $(\tau, \xi_3, \rho, \phi_0)$. This function is easily found to be $-1/\phi_0$ so that (5.3) can be written

$$(5.4) \quad \left(\frac{\partial \tilde{W}}{\partial \phi_0} + \frac{\tilde{W}}{\phi_0}\right) \log \Gamma = \frac{1}{\Gamma} \left(\tilde{W}_0 \Gamma_\phi - \frac{\bar{\phi}}{\phi \phi_0}\right), \quad \Gamma_\phi = \frac{\partial \Gamma}{\partial \phi}.$$

Since the last relation must hold identically in Γ , we get

$$\frac{\partial \tilde{W}}{\partial \phi_0} + \frac{\tilde{W}}{\phi_0} = 0, \quad \tilde{W} = -\frac{V}{\phi \phi_0}, \quad \frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \phi_0} = 0.$$

Furthermore the coefficient of $1/\Gamma$ in (5.4) gives us

$$(5.5) \quad \tilde{W}_0 = \bar{\phi}/\phi \phi_0 \Gamma_\phi.$$

We write $V = -\sum_{r=0}^{\infty} V_r \Gamma^r$ where $V_r = \phi \phi_0 \tilde{W}_r$. From $\partial V / \partial \phi = 0$ we obtain the recursion formulas

$$(5.6) \quad \frac{\partial V_r}{\partial \phi} + (\nu + 1) \Gamma_{\phi} V_{r+1} = 0, \quad (\nu = 0, 1, \dots).$$

If we introduce the operator $D_{\phi} = -1/\Gamma_{\phi} \partial/\partial \phi$, (5.6) can be expressed as

$$(5.7) \quad V_r = \frac{1}{\nu!} (D_{\phi})^r (V_0), \quad (\nu = 0, 1, \dots).$$

According to (5.5)

$$V_0 = \phi \phi_0 \tilde{W}_0 = \frac{\bar{\phi}}{\Gamma_{\phi}} = -(\frac{1}{2}) D_{\phi} (\bar{\phi})^2.$$

Introducing $\psi = 2r\rho \cos \bar{\phi}$ for $0 < |\bar{\phi}| < \pi$ we have $D_{\phi} = -\partial/\partial \psi$, $\bar{\phi} = \arccos(\psi/2r\rho)$. Therefore (5.7) can be written

$$V_r = \frac{1}{2} \frac{(-1)^r}{\nu!} \left(\frac{\partial}{\partial \psi} \right)^{r+1} \left[\arccos \left(\frac{\psi}{2r\rho} \right) \right]^2, \quad (\nu = 0, 1, \dots),$$

from which it follows that no V_r can vanish identically.

In order to get V explicitly we write

$$V_r^* = \frac{1}{2} \frac{(-1)^r}{\nu!} \left(\frac{\partial}{\partial \psi} \right)^r \left[\arccos \left(\frac{\psi}{2r\rho} \right) \right]^2.$$

Then

$$V^* = \sum_{r=0}^{\infty} V_r^* \Gamma^r = \frac{1}{2} \left[\arccos \left(\frac{\psi - \Gamma}{2r\rho} \right) \right]^2, \quad V = \frac{\partial V^*}{\partial \Gamma}.$$

All these formulas are valid for $r, \rho, \phi, \phi_0 \neq 0$, $0 < |\phi - \phi_0| < \pi$, and can be extended in the limit to $\bar{\phi} = 0$ and $\bar{\phi} = \pi$.

6. There is, however, a generalization in another direction that can be established. We have seen that there is a sequence of operators $\mathfrak{L}_n^2 = \Delta_{n-1} - (\partial^2/\partial t^2 + c^p(t))$, defined by (4.5), (4.6), which are Huygens' for $n \geq 2(p+2)$, n even. The same construction can of course be performed with any space variable, say $x_1 = x$. If we define a sequence of operators $L_n^q = (\partial^2/\partial x^2 + d^q(x)) - \partial^2/\partial t^2 + \Delta_{n-2}$ by the recursion formulas

$$L_n^0 = \square_n, \quad l_x^{q-1} \dots l_x^0 \square_n = L_n^q l_x^{q-1} \dots l_x^0,$$

$$l_x^r = \frac{\partial}{\partial x} - \frac{\lambda_x^r}{\lambda^r}, \quad \lambda_{xx}^r + d^r(x) \lambda^r = 0,$$

we see in the same way as before that L_n^q is an H -operator for n even and $n \geq 2(q+2)$.

Each of the operators l_x^r commutes with every l_t^r . We now define a new set of operators $\mathfrak{L}_n^{pq}$ by $\mathfrak{L}_n^{00} = \square_n$, $\mathfrak{L}_n^{p0} = \mathfrak{L}_n^p$, $\mathfrak{L}_n^{0q} = \mathfrak{L}_n^q$ and the recursion formulas

$$l_x^{q-1} \dots l_x^0 l_t^{p-1} \dots l_t^0 \square_n = \mathfrak{L}_n^{pq} l_x^{q-1} \dots l_x^0 l_t^{p-1} \dots l_t^0.$$

Explicitly we have

$$(6.1) \quad \begin{aligned} \mathfrak{L}_n^{pq} &= \left(\frac{\partial^2}{\partial x^2} + d^q(x) \right) - \left(\frac{\partial^2}{\partial t^2} + c^p(t) \right) + \Delta_{n-2} \\ &= l_t^{p-1} l_t^{p-1*} - l_x^{q-1} l_x^{q-1*} + \Delta_{n-2} \end{aligned}$$

and we claim the following

Theorem 2. *Each $\mathfrak{L}_n^{pq}$ is an H -operator for $n \geq 2(p + q + 2)$, n even.*

We shall omit reference in the text to the domain of validity of Huygens' principle for the operator $\mathfrak{L}_n^{pq}$, $n \geq 2(p + q + 2)$. This domain is that portion of R^n in which the coefficients of $\mathfrak{L}_n^{pq}$ are defined and analytic.

We remark that Theorem 2 does not imply Theorem 1, as the latter has been proved under the only hypothesis that the predecessor $\mathfrak{L}_n$ is an H -operator. Here both predecessors have to be chosen out of a special sequence of H -operators.

Lemma. *If $\mathfrak{L}_{2m}^{p-1,q}$ and $\mathfrak{L}_{2m}^{p,q-1}$ are both H -operators, so is $\mathfrak{L}_{2(m+1)}^{pq}$.*

Proof. We evidently have

$$l_t^{p-1} \mathfrak{L}_{2m}^{p-1,q} = \mathfrak{L}_{2m}^{pq} l_t^{p-1}, \quad l_x^{q-1} \mathfrak{L}_{2m}^{p,q-1} = \mathfrak{L}_{2m}^{pq} l_x^{q-1}.$$

Let $W^{p-1,q}$, $W^{p,q-1}$ and W^{pq} be the coefficients of $\log \Gamma$ in the fundamental solution of the corresponding operators. These functions depend on $x_2, \dots, x_{n-1}$ and $\xi_2, \dots, \xi_{n-1}$ only through Γ so that, e.g., $W^{pq} = W^{pq}(t, x; \tau, \xi; r)$ where $r^2 = \sum_{i=2}^{n-1} (x_i - \xi_i)^2$. As in Section 3 we conclude that

$$(6.2) \quad l_\tau^{p-1} W^{p-1,q} = l_t^{p-1*} W^{pq}, \quad l_\xi^{q-1} W^{p,q-1} = l_x^{q-1*} W^{pq}.$$

By hypothesis $W^{p-1,q} = W^{p,q-1} \equiv 0$ so from (6.2)

$$(6.3) \quad l_t^{p-1*} W^{pq} = l_x^{q-1*} W^{pq} = 0.$$

From (6.1) and (6.3) we conclude that $\Delta_{2(m-1)} W^{pq} = 0$. Since W^{pq} involves variables appearing in $\Delta_{2(m-1)}$ only implicitly in terms of r and is regular as $r \rightarrow 0$, it follows that W^{pq} is independent of r and its series development in powers of Γ terminates after the first term, as was to be proved.

As in Section 3, W^{pq} may be determined explicitly and we obtain

$$W^{pq} = \frac{k}{\mu^{p-1}(t) \mu^{p-1}(\tau) \lambda^{q-1}(x) \lambda^{q-1}(\xi)}, \quad k = \text{constant}.$$

To prove Theorem 2, we show that for any given positive integer r , all $\mathfrak{L}_n^{pq}$ with $p + q = r$ are H -operators for $n = 2(r + 2)$. By Theorem 1, $\mathfrak{L}_n^{p0}$ and $\mathfrak{L}_n^{0q}$ are all H -operators for $n = 2(p + 2)$ and $n = 2(q + 2)$ respectively. In par-

ticular, $\mathfrak{L}_0^{1,0}$ and $\mathfrak{L}_0^{0,1}$ are H -operators, so our lemma implies Theorem 2 for $p + q = 1$. But also the lemma shows that if $\mathfrak{L}_n^{p,q}$ are H -operators for $p + q = r$ fixed and $n = 2(r + 2)$, then $\mathfrak{L}_n^{p,q}$ are all H -operators for $p + q = (r + 1)$ and $n = 2(r + 3)$. Theorem 2 therefore follows by complete induction.

The procedure of this section may be generalized in an obvious way to include the variables $x_2, \dots, x_{n-1}$ as well, and one can establish a generalization of Theorem 2 in an analogous way. The result can be expressed as follows.

Theorem 3. *If the operators*

$$\mathfrak{L}_{n_0} = \Delta_{n_0-1} - \left(\frac{\partial^2}{\partial x_0^2} + c_0(x_0) \right),$$

$$\mathfrak{L}_{n_i} = \left(\frac{\partial^2}{\partial x_i^2} + c_i(x_i) \right) - \frac{\partial^2}{\partial x_0^2} + \sum_{\substack{j=1 \\ j \neq i}}^{n_i-1} \frac{\partial^2}{\partial x_j^2}$$

are H -operators for $n_i = 2(m_i + 2)$, ($i = 0, 1, \dots, k - 1$), then

$$\sum_{i=1}^{k-1} \left(\frac{\partial^2}{\partial x_i^2} + c_i(x_i) \right) - \left(\frac{\partial^2}{\partial x_0^2} + c_0(x_0) \right) + \Delta_{n-k}$$

is an H -operator for $n = 2[\sum_{i=0}^{k-1} m_i + 2]$ provided $\mathfrak{L}_{n_i}$ is derived from $\square_{n_i}$ through a sequence of l_{x_i} -transforms ($i = 0, 1, \dots, k - 1$).

If the $c_i(x_i)$ are of the form $c_i(x_i) = k_i/x_i^2$ we obtain in particular the results of [5].

The class of H -operators which have been found in Sections 3 and 4 give rise to other applications of the method developed in [12]. In particular we get immediately an independent proof of Theorem 3 of the present paper by direct application of Theorem 2 of [12]. That way we get even a slightly more general result than above, because the hypothesis that $\mathfrak{L}_{n_i}$ is derived from $\square_{n_i}$ through a sequence of l_{x_i} -transforms may be omitted. Nevertheless all H -operators which are known and can be constructed by one of the methods can be constructed by the other as well.

In a forthcoming paper the question whether there are H -operators of the form $\mathfrak{L}_n = \square_n + c(x_0)$ other than those generated through a sequence of l_{x_0} -transforms from $\square_n$ will be answered in the negative.

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