The Three-Dimensional Wave Equation in a Characteristic Quarter-Space

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Introduction and Summary. In this paper we study the wave equation,

$$\square u \equiv u_{tt} - u_{xx} - \sum_{i=1}^2 u_{y_i y_i} = 0,$$

in a characteristic quadrant of space-time, $|x| \leq |t|$. This problem, so well known in the one-dimensional case, has not been studied much in more than one spatial dimension.

In one dimension, there is no difference between t and x in the equation $u_{tt} = u_{xx}$. Using the method of characteristics one easily constructs a unique solution u for arbitrary given boundary values, either in the quadrant $|x| < t$ or in $|t| < x$.

In higher dimensions, these two problems become essentially different. The “interior problem,” $|x| < t$, is still solvable for general boundary values. This is perhaps well known, but we include a simple proof for the sake of completeness. (See Hadamard, [2], para. 119, for a proof using finite parts of divergent integrals; see Riesz, [6], for a proof using Riemann–Liouville operators and analytic continuation.)

The “exterior problem,” $|t| < x$, is no longer well posed in higher dimensions. We show that a solution exists for analytic boundary values, but not for C^∞ boundary values with compact support. (Pucci [5] gives similar results by a different method. For further references on problems in exterior domains, see Lax and Phillips, [4]. Friedlander [7] has an interesting example of a physical problem leading to an incorrectly posed Goursat problem.)

Having proved that data cannot be prescribed arbitrarily on the whole boundary, we then show that the boundary values on half the boundary ($t = x$, say) already determine the solution uniquely—but only if in addition some restriction is placed on the growth of u and u_t at ∞ . This leads to the conclusion (Corollary 2) that if u exists in all space-time, it is uniquely determined by the data needed to determine it in one interior quadrant!

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A more restricted version of Theorem 2 was announced in [1]. There a proof was sketched which depended on a fairly elaborate theory of iterated paraboloidal means. It turns out that transform methods yield a shorter proof of a much stronger theorem, which is presented here. On the other hand, the representation of the solution in terms of the paraboloidal means in [1] leads directly to a necessary and sufficient condition for compatible data on $t = x$ and $t = -x$.

A theorem for a whole class of partial differential equations, suggested by the facts summarized in Theorem 1, is stated in [3]. See also Friedlander [7], where Fritz John's extension of Holmgren's method is used to show that the values in the pair of strips $|y| \leq a$, $|t| = x$, determine the solution of the exterior problem uniquely.

Although we state our results only for $y = (y_1, y_2)$, the arguments and results of Theorem 1 are valid for $y = (y_1, \dots, y_n)$, any n . The restriction to $n = 2$ is used in Theorem 2, where we use the representation of solutions of the wave equation in terms of spherical means.

Notations (see Figure 1) and statements of results. We will let Q_E denote the "exterior" quadrant $\{x > |t|, y \text{ unbounded}\}$; Q_I , the "interior" quadrant $\{0 < t < |x|, y \text{ unbounded}\}$. C_+ is the half-plane $x = t, x > 0$; C_- , the half-plane $x = -t, x > 0$; C^- , the half-plane $x = -t, t > 0$; and C_0 the subspace $\{x = t = 0\}$. Thus the boundary of Q_E is $C_+ \cup C_0 \cup C_-$; the boundary of Q_I is $C_+ \cup C_0 \cup C^-$. The projections of these sets into the $x - t$ plane will be written $\bar{Q}_E, \bar{Q}_I, \bar{C}_+, \bar{C}_-, \bar{C}^-, \bar{C}_0$. We let R_+ denote the positive half-space $\{t = 0, x > 0, y \text{ unbounded}\}$ and $\bar{R}_+$ the positive x -axis. We let $-Q_E, -Q_I$ and C^+ denote the reflections in the origin of Q_E, Q_I and C_+ .

We let $u(t, x, y_i)$ denote a solution of the wave equation, and $U(t, x, \eta_i)$ its Fourier transform. We understand the Fourier transformation in the sense

of the theory of (tempered) distributions, so that we may apply it to functions having polynomial ("tempered") growth in y . Fourier transformation will always be applied to the y -variables only. We say that a function, $u(t, x, y)$, is "tempered" if u , $u_{tt} - u_{xx}$, $u_{y,y}$ and $u_{y,y,y}$ all can be bounded in absolute value by some polynomial in y with coefficients depending on t and x .

We set $a = \frac{1}{2}(x + t)$, $b = \frac{1}{2}(x - t)$, so that C_- is the set $\{a = 0, b > 0\}$; C_- , the set $\{a = 0, b < 0\}$; C_+ , the set $\{b = 0, a > 0\}$. We let u_+ , u_- , u_- , and U_+ , U_- , U_- denote the values assumed respectively on C_+ , C_- , C_- by u and on $\hat{C}_+$, $\hat{C}_-$, $\hat{C}_-$ by U . The letters f, g will be reserved for the values assumed on R_+ by u_t and u respectively. $F(x, \eta)$ and $G(x, \eta)$ will be their transforms.

If u_- and u_+ are given, and if $\square u \equiv 0$ in Q_t , $u = u_-$ on C_- , $u = u_+$ on C_+ , we call u a "solution to the interior Goursat problem." If u_- and u_+ are given, $\square u = 0$ in Q_E , $u = u_-$ on C_- , $u = u_+$ on C_+ , we call u a "solution to the exterior Goursat problem."

We always assume that the data u_- and u_+ are consistent; i.e., $\lim_{b \rightarrow 0} u_-(b, y) = \lim_{a \rightarrow 0} u_+(a, y)$ a.e. with respect to y .

Theorem 1a. *For arbitrary data $u_-(b, y)$ and $u_+(a, y)$ which are locally square-integrable in y , there exists a unique solution to the interior Goursat problem.*

Theorem 1b. *If $u_- \equiv 0$ and u_+ is C^∞ with compact support, not $\equiv 0$, there exists no tempered solution of the exterior Goursat problem.*

Theorem 1c. *If u_+ and u_- are tempered and entire of exponential type in y , there exists a unique tempered solution of the exterior Goursat problem, which is moreover entire of exponential type in y .*

Our next theorem gives conditions under which a solution of the exterior problem is determined by its values on C_- alone. To understand what sort of theorem can be expected, one should first notice that if $g(a)$ is an arbitrary function of $a = (x + t)/2$ satisfying $g(0) = 0$, and if $h(y_1, y_2)$ is an arbitrary harmonic function, then $u = gh$ is a solution of the exterior problem vanishing on C_- . (This remark is due to P. Lax by oral communication to one of the authors. We want to thank him.) Thus the best uniqueness theorem one could hope for would be one that says " $u \equiv 0$ on $C_- \Rightarrow u = \sum g_i h_i$," where the g are functions of a vanishing at the origin, and the h are harmonic functions of y_1, y_2 . It turns out that in order to obtain such a conclusion, one needs to restrict *a priori* the growth at infinity of both f and g , the Cauchy data for u on R_+ .

Theorem 2. *If two tempered solutions of the exterior Goursat problem are identical on C_- , and if on R_+ their difference and the t -derivative of their difference grow at ∞ no faster than a polynomial, then in Q_E their difference is of the form*

$$\sum_0^n g_i(a) h_i(y_1, y_2),$$

where the h_i are harmonic polynomials, and $g_i(0) = 0$ for all j .

The proof of both theorems follows below, but notice that Theorem 2 immediately gives the obvious

Corollary 1. *If two tempered solutions of the exterior Goursat problem are identical on C_- , and if on R_+ their difference and the t -derivative of their difference grow at infinity no faster than a polynomial, and if, moreover, their difference vanishes as $|y| \rightarrow \infty$ for all x , then they are identically equal in Q_E .*

It will be clear in the proof of Theorem 2 that *a priori* growth conditions are needed only for two reasons: to justify formal use of the Fourier transform in y , and to guarantee that the Laplace transforms with respect to x of the Cauchy data f, g are holomorphic in the right half-plane. Now, square-integrability in x, y serves these needs even more readily than polynomial growth. Moreover, once we have $u(t, x, y)$ as a polynomial in y , square-integrability in y means $u \equiv 0$. Recall that for a function u satisfying the wave equation in all space-time, the energy is defined as $\frac{1}{2} \int (u_t^2 + u_x^2 + u_{y_1}^2 + u_{y_2}^2) dx dy_1 dy_2$. Then we have

Corollary 2. *If $\square u \equiv 0$ for all t, x, y_i , if $u \equiv 0$ on C_+ and on C^- , and if at $t = 0$ u has finite energy, then $u \equiv 0$ for all t, x, y_i .*

Proof. By Theorem 1a, $u \equiv 0$ in Q_I . By Theorem 2, $u \equiv 0$ in Q_E . The hypotheses of Corollary 2 are invariant if we replace x by $-x$, so Theorem 2 also gives $u \equiv 0$ in $-Q_E$. Now $u \equiv 0$ in Q_E and in $-Q_E$ means in particular $u \equiv 0$ on C_- and C^+ ; thus for u in $-Q_I$ we have a well-posed interior Goursat problem whose solution is $u \equiv 0$.

It remains to prove Theorems 1a, b, c and 2.

Remark. Following the proofs, we give a counter-example to show that Theorem 2 would become false if for $u_i(0, x, y)$ we weakened the condition of polynomial growth to growth of order less than $(e^{\epsilon x})$ for any given $\epsilon > 0$.

Proof of Theorem 1a. First of all, we can suppose without loss of generality that the data u^-, u_0, u_+ have compact support. (Here $u_0(y)$ means the given common value assumed on C_0 by u^- and u_+ .) For the property of finite speed of propagation possessed by solutions of the wave equation means that for a fixed but arbitrary point $P = \{t, x, y\}$ in Q_I , the domain of dependence of P intersects $\{C^- \cup C_0 \cup C_+\}$ in a bounded set. To determine the solution at P , it is irrelevant what values the data take on outside this bounded set, and we may as well suppose they vanish there.

Next, we observe that the solution, if it exists, can be written as the sum of three solutions: one with boundary values depending only on y ; a second which vanishes on C^- ; and a third which vanishes on C_+ . In fact, we set $h_0(y) = u_0$, $h_+(a, y) = u_+$, $h^-(b, y) = u^-$. If u^1 is a solution of the exterior Goursat problem with boundary values $u_+^1(a, y) \equiv u^-^1(b, y) \equiv h_0(y)$; if u^2 is a solution vanishing on C^- , with boundary value $h_+(a, y) - h_0(y)$ on C_+ ; and if u^3 is a solution vanishing on C_+ , with boundary value $h^-(b, y) - h_0(y)$ on C^- —then $u = u^1 + u^2 + u^3$ is the desired solution of the general problem.

Since construction of u^2 is clearly the same problem as construction of u^3 , we need construct only u^1 and u^2 . For these purposes we will relabel the data, $h_0 = u_0$, $h_+ - h_0 = u_+$.

Since the data and solutions (if they exist) have compact support in y for fixed x, t , we can consider the Fourier transforms, which we regard as functions of a, b, η , and which satisfy

$$U_{xx}^i - U_{tt}^i \equiv U_{ab}^i = |\eta|^2 U^i \quad \text{in } \hat{Q}_t,$$

where $|\eta|^2 = \sum \eta_i^2$ and $i = 1, 2$. The boundary conditions are $U^1 \equiv U_0(\eta)$ on $\hat{C}^-$ and $\hat{C}_+$ and $U^2 \equiv 0$ on $\hat{C}^-$, $U^2 = U_+(a, \eta)$ on $\hat{C}_+$, with $U_+(0, \eta) = 0$ a.e. The solutions to these two simple problems in two independent variables are well known and may be obtained, for example, by iterated indefinite integration with respect to a and b .

We find

$$\begin{aligned} (1) \quad U^2 &= U_+(a, \eta) + \int_0^a U_+(\tau, \eta) \sum_1^\infty \frac{|\eta|^2|^n b^n (a - \tau)^{n-1}}{n!(n-1)!} d\tau \\ &= U_+(a, \eta) + |\eta| b^{\frac{1}{2}} \int_0^a U_+(\tau, \eta) \frac{I_1(2|\eta| b^{\frac{1}{2}}(a - \tau)^{\frac{1}{2}}) d\tau}{(a - \tau)^{\frac{1}{2}}} \end{aligned}$$

and $U^1 = U_0(\eta)I_0(2|\eta| a^{\frac{1}{2}}b^{\frac{1}{2}})$, where I_0 and I_1 are the modified Bessel functions. Now, since we have $a > 0, b < 0$, in Q_t , we replace $b^{\frac{1}{2}}$ by $i|b|^{\frac{1}{2}}$, and, using $I_0(it) = J_0(t)$, $I_1(it) = iJ_1(t)$, we get

$$(2) \quad U^2 = U_+(a, \eta) - |\eta| |b|^{\frac{1}{2}} \int_0^a \frac{U_+(\tau, \eta) J_1(2|\eta| |b|^{\frac{1}{2}}(a - \tau)^{\frac{1}{2}}) d\tau}{(a - \tau)^{\frac{1}{2}}}$$

and $U^1 = U_0(\eta)J_0(2|\eta| |ab|^{\frac{1}{2}})$.

If u_+ and U_+ are differentiable with respect to a , equation (2) can be reduced, by integration by parts, to

$$(3) \quad U^2 = \int_0^a \frac{d}{d\tau} U_+(\tau, \eta) J_0(2|\eta| |b|^{\frac{1}{2}}(a - \tau)^{\frac{1}{2}}) d\tau.$$

All that remains is to check that $U^1(\eta)$ and $U^2(\eta)$ are the transforms of some functions $u^1(y)$ and $u^2(y)$ with compact support; these will be the desired solutions. Since u_+ and u_0 had compact support, U_+ and U_0 are in L_2 with respect to η , and moreover are entire of exponential type. Since J_0 and J_1 are bounded, decay at ∞ like a $-\frac{1}{2}$ power, and are entire of exponential type, we see that $U^1(\eta)$ is in L_2 and entire of exponential type; U^2 is entire of exponential type in η (for any fixed t, x) and can be estimated by a constant multiple of $|\eta|^{\frac{1}{2}} U_+(\eta)$. By the Paley-Weiner-Schwartz theorem, it follows that $u^1(y)$ is an L_2 function with compact support for each fixed t, x ; $u^2(y)$ is a tempered distribution with compact support in y for each fixed x, t . Equation (3) shows that $u^2(y)$ is in L_2 if $\partial u_+ / \partial a$ exists and is in L_2 (with respect to y).

Proof of Theorem 1b. Now we have the same problem as before, except that we need only look at the component u^2 of the solution, and except also that in Q_E both a and b are positive. If a tempered solution u exists, its transform U will satisfy equation (1) above. Now $I_1(|\eta|)$ is asymptotic to $e^{|\eta|}/(2\pi|\eta|)^{\frac{1}{2}}$, so we see from (1) that u can exist as a tempered distribution (i.e., U can have tempered growth in η) only if $U_+(\eta)$ decays exponentially at ∞ . But if U_+ is $O(e^{-a|\eta|})$ for any $a > 0$, its inverse transform u_+ is an analytic function of y . Such a function can have compact support only if it vanishes identically.

Proof of Theorem 1c. As in Theorem 1a, we can write $u = u^1 + u^2 + u^3$. Now, by the Paley-Weiner theorem, our hypothesis implies that for any fixed x, t U_- , U_+ and U_0 will likewise have compact support in η , and the conclusion follows by the converse of the Paley-Weiner theorem.

Remark. A more refined theorem could be stated here, if we observe that for U to be in $L_2(\eta)$ formula (1) does not really require U_+ or U_0 to have compact support, but merely to decay exponentially. Such decay obtains if u_+ and u_0 are analytic in a sufficiently wide strip in the complex y_1, y_2 -planes. If we ask that u exists only for a, b bounded, not necessarily in all of Q_E , then the data need not be entire. On the other hand, by Theorem 1b, it is not enough if they are C^∞ . Pucci [5] has proved for two-space variables x and y by a method of analytic continuation that a solution exists locally for data analytic in y . He also gives an example to show that a solution need not exist for data with an arbitrarily high finite number of smooth derivatives.

Now we turn to Theorem 2.

Proof of Theorem 2. Here we restrict ourselves to the three-dimensional wave equation. We will let u be the difference of the two solutions described in the theorem; we want to prove $u \equiv 0$. We will do this by proving that the Cauchy data for u — u and u_t , evaluated on R_+ —both vanish identically.

Our principal tool is Poisson's solution of the wave equation in terms of its Cauchy data:

$$u = t \text{SM}[f; X, t] + \frac{\partial}{\partial t} t \text{SM}[g; X, t].$$

Here $g = u$ and $f = u_t$ on R_+ . SM, the spherical mean, is the operator that assigns to a function of $X = (x, y_1, y_2)$ a new function of X, t whose value is the average of the values of the original function, taken over the surface of the t -sphere with center X .

When $t = x$, the first coordinate of X , we write $\text{SM}[f; X, x] = \text{SM}^*f$, so that

$$(4) \quad \text{SM}^*f = \frac{1}{4\pi} \iint f(x + x\lambda_0, y_1 + x\lambda_1, y_2 + x\lambda_2) d\omega.$$

Here $d\omega$ is the surface element on the unit sphere $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$.

If u is a solution to the exterior Goursat problem with $u_+ = h$, $u_- \equiv 0$, we define $\bar{u} = u(t) - u(-t)$. This $\bar{u}$ is an odd function of t , a solution to the exterior Goursat problem with $\bar{u}_+ = h$, $\bar{u}_- = -h$. If on R_+ , $u = g$, $u_t = f$, then on R_+ , $\bar{u} \equiv 0$, $\bar{u}_t = 2f$.

Now the idea of the proof can be summarized. We apply Poisson's formula to $\bar{u}$ and then take Fourier transforms to express H in terms of F . We use formula (1), with $t = 0$, to express G in terms of H . Together, these two formulas give a relation between G and F —a necessary condition on the (transforms of) the Cauchy data for a tempered solution of the exterior Goursat problem which vanishes on C_- . Finally, we show that if f and g both have polynomial growth, this necessary condition implies that they both are actually polynomials in y . Then u also is a polynomial in y , and the theorem follows.

First, we return to equation (1) to express G in terms of H . Equation (1) gives a solution vanishing on $\hat{C}_-$ or $\hat{C}^-$; to get one vanishing on $\hat{C}_+$, interchange a and b in the formula. In any case, when we set $t = 0$ to get U on $\hat{R}_+$, a and b both become $x/2$, and we get

$$U^2 \equiv U = G(x, \eta) = H(x/2, \eta) + |\eta| (x/2)^{\frac{1}{2}} \int_0^{x/2} \frac{H(\tau, \eta) I_1 \left(2 |\eta| \left(\frac{x}{2} - \tau \right) \right)^{\frac{1}{2}} d\tau}{(x/2 - \tau)^{\frac{1}{2}}}$$

or

$$(5) \quad G(2x, \eta) = H(x, \eta) + |\eta| x^{\frac{1}{2}} \int_0^x \frac{H(\tau, \eta) I_1 (2 |\eta| x^{\frac{1}{2}} (x - \tau)^{\frac{1}{2}}) d\tau}{(x - \tau)^{\frac{1}{2}}}.$$

Now to express H in terms of F , use Poisson's formula for $\bar{u}$, evaluated on C_+ , i.e., by setting $t = x$, to get

$$h = 2xSM^*f.$$

Therefore, applying the Fourier transformation to both sides, and using (4),

$$H = \frac{2x}{4\pi(2\pi)^2} \iint_{-\infty}^{\infty} e^{-iy\eta} dy \iint_{|\lambda|=1} f(x + x\lambda_0, y_1 + x\lambda_1, y_2 + x\lambda_2) d\omega,$$

where $y\eta = y_1\eta_1 + y_2\eta_2$. We reduce this four-fold integral to a single integral by interchanging the order of integration. Setting $y'_i = y_i + x\lambda_i$, ($i = 1, 2$) and $\lambda\eta = \lambda_1\eta_1 + \lambda_2\eta_2$,

$$\begin{aligned} H &= \frac{x}{8\pi^3} \iint_{|\lambda|=1} d\omega \iint_{-\infty}^{\infty} f(x + x\lambda_0, y') e^{-iy'\eta} e^{ix\lambda\eta} dy' \\ &= \frac{x}{2\pi} \iint e^{ix\lambda\eta} F(x + x\lambda_0, \eta) d\omega. \end{aligned}$$

On the unit λ -sphere we use polar coordinates: $\lambda_0 = \cos \varphi$, $\lambda_1 = \sin \varphi \cos \theta$, $\lambda_2 = \sin \varphi \sin \theta$, $d\omega = \sin \varphi d\varphi d\theta = -d\lambda_0 d\theta$. Then

$$H = \frac{x}{2\pi} \int_{-1}^1 d\lambda_0 F(x + x\lambda_0, \eta) \int_0^{2\pi} e^{ix\lambda\eta} d\theta$$

$$= \frac{x}{2\pi} \int_{-1}^1 d\lambda_0 F(x + x\lambda_0, \eta) \int_0^{2\pi} e^{ix\lambda \sin \varphi (\eta_1 \cos \theta + \eta_2 \sin \theta)} d\theta.$$

Let $\eta' = \eta/|\eta|$ and define θ' by $\eta'_1 \cos \theta + \eta'_2 \sin \theta = \sin(\theta + \theta')$. Then

$$\begin{aligned} H &= \frac{x}{2\pi} \int_{-1}^1 d\lambda_0 F(x + x\lambda_0, \eta) \int_0^{2\pi} e^{ix|\eta|(1-\lambda_0^2)^{\frac{1}{2}} \sin(\theta + \theta')} d(\theta + \theta') \\ &= x \int_{-1}^1 F(x + x\lambda_0, \eta) J_0(x|\eta|(1-\lambda_0^2)^{\frac{1}{2}}) d\lambda_0. \end{aligned}$$

If we define $\sigma = \frac{1}{2}x(1 + \lambda_0)$ and $z = x|\eta|(1 - \lambda_0^2)^{\frac{1}{2}} = 2|\eta|\sigma^{\frac{1}{2}}(x - \sigma)^{\frac{1}{2}}$, we obtain finally

$$(6) \quad H(x, \eta) = 2 \int_0^x J_0(z) F(2\sigma, \eta) d\sigma.$$

The above formula has been derived under the assumption that $f(x, y_1, y_2)$ is integrable with respect to y . In theorem 2, however, we allow f to have polynomial growth in y , hence the argument has to be modified. We apply the same argument to $e^{-\epsilon|y|}f$ instead of f with $\epsilon > 0$. When $\epsilon \rightarrow 0$, one obtains (6) as the limit in the weak topology of the space of tempered distributions.

Now we substitute (6) into (5):

$$\begin{aligned} G(2x, \eta) &= 2 \int_0^x J_0(z) F(2\sigma, \eta) d\sigma \\ &\quad + 2|\eta|x^{\frac{1}{2}} \int_0^x d\tau \frac{I_1(2|\eta|x^{\frac{1}{2}}(x - \tau)^{\frac{1}{2}})}{\sqrt{x - \tau}} \int_0^\tau J_0(2|\eta|\sigma^{\frac{1}{2}}(\tau - \sigma)^{\frac{1}{2}}) F(2\sigma, \eta) d\sigma \end{aligned}$$

which becomes, on changing the order of integration,

$$\begin{aligned} (7) \quad G(2x, \eta) &= 2 \int_0^x J_0(z) F(2\sigma, \eta) d\sigma \\ &\quad + 4\eta^2 x \int_0^x F(2\sigma, \eta) d\sigma \int_\sigma^x \frac{I_1(2|\eta|x^{\frac{1}{2}}(x - \tau)^{\frac{1}{2}}) J_0(2|\eta|\sigma^{\frac{1}{2}}(\tau - \sigma)^{\frac{1}{2}})}{2|\eta|x^{\frac{1}{2}}(x - \tau)^{\frac{1}{2}}} d\tau. \end{aligned}$$

At this point we state an identity which may be known, but for which we have found no reference in the literature:

$$\begin{aligned} (8) \quad 4x|\eta|^2 \int_\sigma^x \frac{I_1(2|\eta|x^{\frac{1}{2}}(x - \tau)^{\frac{1}{2}}) J_0(2|\eta|\sigma^{\frac{1}{2}}(\tau - \sigma)^{\frac{1}{2}})}{2|\eta|x^{\frac{1}{2}}(x - \tau)^{\frac{1}{2}}} d\tau \\ = 2I_0(2|\eta|(x - \sigma)) - 2J_0(2|\eta|\sigma^{\frac{1}{2}}(x - \sigma)^{\frac{1}{2}}) \\ = 2I_0(2|\eta|(x - \sigma)) - 2J_0(z). \end{aligned}$$

A proof is given in an appendix below. In view of (8), equation (7) reduces to

$$G(2x, \eta) = 2 \int_0^x F(2\sigma, \eta) I_0(2|\eta|(x - \sigma)) d\sigma$$

which, by the change of variables $s = 2\sigma$ followed by the substitution of x for $2x$, takes on the even simpler form

$$(9) \quad G(x, \eta) = \int_0^x F(s, \eta) I_0(|\eta| (x - s)) ds.$$

Now, the growth assumptions on f and g guarantee that they, and therefore also F and G , possess Laplace transforms with respect to x , which are holomorphic in the right half-plane. Noting that the right side of (9) is a convolution, we apply the Laplace transformation to both sides, obtaining

$$(10) \quad \hat{G}(p, \eta) = \hat{F}(p, \eta)(p^2 - |\eta|^2)^{-\frac{1}{2}}.$$

Now, for every real η , $\hat{G}$ and $\hat{F}$ are both holomorphic, and therefore in particular single-valued, for all $\text{Re } p > 0$. But by (10) this is impossible for $\eta \neq 0$, in any neighborhood of $p = |\eta|$, unless $\hat{F}$ (and therefore also $\hat{G}$) vanishes identically there. Then $\hat{F}$ and $\hat{G}$ must vanish identically in p for all $\eta \neq 0$. By the uniqueness of the Laplace transform, $F(x, \eta)$ and $G(x, \eta)$ must also vanish for $\eta \neq 0$; that is, their η -support is concentrated at the origin, which implies that f and g as functions of y are polynomials. Now, for $\eta = 0$, (10) implies $u_x(0, x, y) = g_x = f$. On the other hand, by definition $f = u_t(0, x, y)$. Writing u as a polynomial in y with coefficients depending on t and x , one now finds that the coefficients c_i satisfy $(\partial/\partial t - \partial/\partial x)c_i = 0$, so that c_i is a function of x ; then we can conclude from $\square u = 0$ that u is harmonic in y_1, y_2 . Q.E.D.

Comment and counterexample. It is clear that the hypothesis of polynomial growth for f, g could be weakened; all that we need is the analyticity of $\hat{F}$ and $\hat{G}$ in $\text{Re } p > 0$ for all real non-zero η , and for this polynomial growth in x, y is sufficient but not necessary. On the other hand, we can show by an example that a somewhat weaker growth condition is already too weak to give the theorem.

For this purpose, define G to be $\equiv 0$ outside the sphere of radius ϵ with center at $x = 1, \eta_1 = \eta_2 = 0$. Inside the sphere, $G = (\epsilon^2 - |\eta|^2 - (x - 1)^2)^\mu$, with large μ .

This G is in C^μ and has compact support, so it corresponds to a g which has compact support in x , decays in y like a $-\mu$ power, and is C^μ in x , entire analytic in y . For large μ , then, it is as "regular" as we please.

We can insert this G into equation (5), regarding H as an unknown. We recognize that we have for H a Volterra equation of the second kind, with a unique solution H for arbitrary G . Then equation (1) with $H = H_+$ determines U . Since G has compact support in η , so do H and U , and therefore u , the inverse transform of U , certainly exists and is a non-zero solution of the exterior Goursat problem, vanishing on C_- . But what about the growth at ∞ of f , the initial value on R_+ of u_t ?

To study this, we can invert (10), after multiplying by $(p^2 - |\eta|^2)^{1/2}$, to obtain

$$(11) \quad F(x, \eta) = G_x(x, \eta) - |\eta| \int_0^x \frac{G(\sigma, \eta) I_1(|\eta| (x - \sigma)) d\sigma}{x - \sigma}.$$

For fixed x , G and F depend only on $|\eta|$. This enables us to carry out the inverse transform integral in polar coordinates in the η_1, η_2 plane, obtaining

$$(12) \quad f(x, y) = 2\pi \int_0^\infty J_0(|\eta| |y|) F(x, |\eta|) |y| d|\eta|,$$

where $|y| = (y_1^2 + y_2^2)^{1/2}$.

Using (11) and (12), it is not hard to estimate $f(x, 0, 0)$ for large x ; we find $f(x) \sim \text{const. } e^{\epsilon x} x^{-\nu}$, $\nu > 0$. Since ϵ is arbitrary, we see that Theorem 2 becomes false if we allow u_i to have exponential growth in x of order ϵ for any $\epsilon > 0$; that is, if we let the Laplace transform have a singularity in the right half-plane.

Appendix. To prove (8), we let Q stand for the integrand on the left side. Then, using the power series expansions of I_1 and J_0 ,

$$Q = \frac{1}{2} \left(\sum_0^\infty \frac{1}{p!} \frac{1}{(p+1)!} |\eta|^{2p} x^p (x - \tau)^p \right) \left(\sum_0^\infty \frac{1}{k!} \frac{1}{k!} (-1)^k |\eta|^{2k} \sigma^k (\tau - \sigma)^k \right).$$

In forming $\int_\sigma^x Q d\tau$, use the elementary identity

$$\int_\sigma^x (x - \tau)^p (\tau - \sigma)^k d\tau = \frac{p!k!}{(p+k+1)!} (x - \sigma)^{p+k+1}.$$

Then termwise integration gives

$$\begin{aligned} 4 |\eta|^2 x \int_\sigma^x Q d\tau &= 2x \sum_{\substack{p=0 \\ k=0}}^\infty \frac{(-1)^k |\eta|^{2p+2k+2} x^p \sigma^k (x - \sigma)^{p+k+1}}{(p+1)!k!(p+k+1)!} \\ &= 2 \sum_{p+k=0}^\infty \frac{|\eta|^{2(p+k+1)} (x - \sigma)^{p+k+1}}{(p+k+1)!(p+k+1)!} \\ &\quad \cdot \left(\sum \frac{(p+k+1)!x^{p+1}(-\sigma)^k}{(p+1)!k!} \right). \end{aligned}$$

In the inner sum above, p and k must both vary in such a way that $p+k$ takes on an arbitrary fixed value, say $m-1$. Then we can evaluate the inner sum:

$$\sum_{\substack{m-1+p+k \\ p=0 \text{ to } m-1 \\ k=0 \text{ to } m-1}} \frac{m!x^{p+1}(-\sigma)^k}{(p+1)!k!} = (x - \sigma)^m - (-\sigma)^m.$$

Hence

$$4 |\eta|^2 x \int_\sigma^x Q d\tau = 2 \sum_{m=1}^\infty \frac{|\eta|^{2m} (x - \sigma)^{2m}}{m!m!} - 2 \sum_{m=1}^\infty \frac{|\eta|^{2m} (-1)^m (x - \sigma)^m \sigma^m}{m!m!}.$$

We can let m start with $m = 0$ in both sums, since this amounts to adding and subtracting $2 \cdot 1$ from the right side of the equation. We then recognize both series as Bessel functions, and so we have

$$4 |\eta|^2 x \int_\sigma^x Q d\tau = 2I_0(2|\eta|(x - \sigma)) - 2J_0(2|\eta|\sigma^{\frac{1}{2}}(x - \sigma)^{\frac{1}{2}})$$

which is the desired identity.

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