A Limitation of Fourier Analysis

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1. Statement of conjecture. In his classic paper [7] on generalized harmonic analysis, Wiener extended the notion of *energy spectrum* to a wide class of *random functions*. Typical physical examples of these are the outputs from filters having “white noise” (Brownian motion) inputs. Wiener defined the *autocorrelation function* of a sample function $f(x) = f(x, \omega)$ as the function

$$(1) \quad R(h, f) = R^*(-h, f) = \lim_{X \rightarrow \infty} \frac{1}{2X} \int_{-X}^X f(x)f^*(x+h) dx.$$

If this function exists and is in $L_1 \cap L_2$, then the Fourier transform

$$(2) \quad \hat{R}(q, f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iqh} R(h, f) dh,$$

is nonnegative by a well-known result of Bochner. This fact and consideration of the sample function $f(x) = ce^{iax}$ led Wiener to suggest that $\hat{R}(q, f)$ should be interpreted as the mean squared amplitude of the Fourier component with wave-number q . That is, one should think of $\hat{R}(q, f)$ as the mean *power density* of f (per unit wave-number) at the wave-number q .

Wiener justified these ideas rigorously by showing, for a given continuous power density $E(q)$, how to construct *normal homogeneous random functions* (N.H.R.F.) $f(x, \omega)$ and an associated probability measure μ_E on an abstract sample space, such that $\hat{R}(q, f)$ as defined by (1)-(2) exists and equals $E(q)$ with μ_E -probability one. In physical language, “time averages of sample functions are ensemble averages with probability one”: metric transitivity.

Wiener went further: he showed how to write almost every sample function $f(x, \omega)$ as a symbolic Fourier-Stieltjes transform

$$(3) \quad f(x, \omega) = \int_{-\infty}^{\infty} e^{iax} dF(q, \omega).$$

Namely, he defined $F(q, \omega)$ up to an additive constant for almost every sample function by the formula

$$(3') \quad F(q'', \omega) - F(q', \omega) = \frac{i}{2\pi} \text{l.i.m.}_{X \rightarrow \infty} \int_{-X}^X f(x, \omega) [e^{-iq''x} - e^{-iq'x}] \frac{dx}{x}.$$

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Moreover he showed that $F(q, \omega)$ was continuous (though generally of unbounded variation, so that it is not clear how to reconstruct f from F ; c.f. [6, p. 78]).

Since $F(q, \omega)$ is continuous, it is a Schwartz distribution, and so $F'(q, \omega)$ also exists as a Schwartz distribution (because all distributions are differentiable (as distributions; see [5, p. 20]), and as a tempered distribution [3] as well. This leads one to write

$$(4) \quad f(x, \omega) = \int_{-\infty}^{\infty} e^{iax} F'(q, \omega) dq,$$

where the improper integral on the right is to be interpreted heuristically. Symbolically, one can thus think of the generalized function $F'(q, \omega)$ as the *Fourier transform* $\hat{f}(q, \omega)$ of $f(q, \omega)$.

It is therefore attractive to speculate that one can write, at least heuristically,

$$(5) \quad \hat{f}(q, \omega) = F'(q, \omega) = A(q, \omega) e^{i\theta(q, \omega)},$$

where the *amplitude* $A(q, \omega)$ might be, for example, proportional to the square root of the mean power density per unit wave-length, $E(q)$. This speculation is especially attractive because, in the past, Fourier analysis has often been applied successfully as a heuristic guide long before rigorous theorems were available.

2. Disproof of conjecture. It will be proved below that the preceding conjecture is false: it is impossible to assign a *phase* $\theta(q, f) = \theta(q, \omega)$ to the N.H.R.F. defined by Wiener so as to satisfy (5), even heuristically. For, to be acceptable even heuristically, the "Fourier transform" $\hat{f}$ in (5). should have (with probability one) the property that

$$(6) \quad \text{If } f(x, \omega') = f(x + h, \omega), \text{ then } \hat{f}(q, \omega') = e^{iah} \hat{f}(q, \omega).$$

But for N.H.R.F., there exists for each h a measure-preserving transformation $T_h: \Omega \rightarrow \Omega$ of the sample space $\Omega = \Lambda$, such that $f(x + h, \omega) = f(x, T_h \omega)$. Therefore, (5) would have to define a *group* of measure-preserving transformations T_h acting on the probability measure space $\Omega = \Lambda$, so that

$$(7) \quad \hat{f}(q, T_h \omega) = e^{iah} \hat{f}(q, \omega).$$

It will now be shown that this is impossible, at least if $\hat{f}(q, \omega)$ is a Borel function on $\mathbf{R} \times \Omega$ (with probability one). Accordingly, suppose that $\hat{f}(q, \omega)$ were a Borel function on $\mathbf{R} \times \Omega$ satisfying (7). Then $\theta(q, f)$ would be defined except where $\hat{f}(q, \omega) = 0$. For each ω , there would thus be a Borel set $I = I(\omega)$ of positive length $m(I)$, invariant under the action of T_h , on which $\theta(q, \omega)$ in (5) would be a Borel function from I to the unit circle Θ .

We now apply the Lebesgue Density Theorem: for any $\epsilon > 0$, $N = N(\epsilon, \omega)$ exists so large that on each of N non-overlapping subintervals I_i of I , whose total length $\sum m(I_i) > m(I) - \epsilon$, the values of $\theta(q, \omega)$ lie in an interval $(a_i - \epsilon$,

$a_i + \epsilon$) of width 2ϵ . Hence there is a least such $N = N_{\min}(\epsilon, \omega)$; moreover for fixed ϵ , I the function $N_{\min}(\epsilon, \omega)$ is a Borel function on Ω , (or, equivalently, on Λ or S' as defined below). We let $J = J(\omega)$ be the difference set $I - \cup I_i$; its measure is at most ϵ .

For example, consider the N.H.R.F. with power density given by (metrically transitive case):

$$E(q) = E(q, f) = \begin{cases} 1 & \text{on } (0, 1) \\ 0 & \text{elsewhere.} \end{cases}$$

In this example, sample functions should satisfy, with μ_E probability one,

$$(8) \quad f(q, \omega) = \begin{cases} e^{i\theta(q, \omega)} & |q| < 1 \\ 0 & |q| > 1. \end{cases}$$

Now fix $\epsilon < 1/10$, and let Σ_N denote the set of all ω such that $N_{\min}(\epsilon, \omega) = N$. For any regular measure, Σ_N is a Borel set and so $\mu(\Sigma_N)$ exists. Moreover almost every ω belongs to one and only one Σ_N , by (8) and the Lebesgue Density Theorem; hence $\cup \Sigma_N = \Omega$, $\Sigma_i \cap \Sigma_j = \emptyset$ if $i \neq j$, and

$$(9) \quad \mu(\Sigma_1) + \mu(\Sigma_2) + \mu(\Sigma_3) + \cdots = \mu(\Omega) = 1.$$

We now apply Poincaré's Recurrence Theorem, which asserts that if $\mu(\Sigma_N) > 0$, then

$$(10) \quad \liminf_{h \rightarrow \infty} N_{\min}(f, T_h \omega) \leq N$$

on set of f of positive μ -measure. This contradicts the following lemma.

Lemma. *If $\epsilon < 1/10$, then for any fixed Borel function $\theta(q, f)$,*

$$(11) \quad \lim_{h \rightarrow \infty} N_{\min}(\epsilon, f(x + h)) = \infty \quad \text{on } (0, 1).$$

Proof. The idea is that, as $h \uparrow \infty$, the semigroup (7) mixes or "smears out" phase values uniformly over Θ . Specifically, let $K_i = I_i \rightarrow J' \subset I$; we will ignore $q \in J(\omega)$ systematically. Since for all $q \in K_i$, $\theta(q, f) \in (a_i - \epsilon, a_i + \epsilon)$, by (5),

$$(12) \quad \theta(q, T_h \omega) \in (a_i - \epsilon + qh, a_i + \epsilon + qh).$$

Hence if $\epsilon < 1/10$ and $h > 20\pi/m(I_i)$, regardless of $\theta(q, \omega)$ for $q \in J(\omega)$, the inverse image of any interval of width π/m on Θ can have relative measure at most $2/m$. This implies (11), and even that $N_{\min}(\epsilon, f(x + h)) = N_{\min}(\epsilon, T_h \omega) = O(h)$ as $h \rightarrow \infty$, completing the proof.

Remark. It seems hard to reconcile the preceding result with the following two known facts: (i) for the N.H.R.F. considered, $f(x, \omega)$ is in the space S'

of tempered distributions with probability one [3], and (ii) for all $f(x) \in S'$, $\hat{f}(q) \in S'$ also exists as a tempered distribution [5, p. 20]. Perhaps the answer is that the measure μ_E is not Borel in S' .

3. Implications for homogeneous turbulence. Wiener's construction, as described in §1, was extended to normal homogeneous random vector fields $\mathbf{u}(\mathbf{x}, \omega)$ (N.H.R.V.F.) in an incompressible fluid by Prof. Kampé de Fériet and the author [2], following related work by Maruyama [4]. Specifically, it was shown in [2] that the sample space Ω could be taken as the Fréchet space Λ of vector fields which are square-integrable on compact sets; $E(q)$ is the mean turbulent energy per unit volume in physical and wave-vector space. Moreover the measure μ_E is "regular": the Lebesgue completion of a measure defined on all Borel subsets of Ω . Alternatively [3], one can take the sample space as the space S' of all velocity-fields whose components are tempered distributions in Euclidean space. Finally one still has metric transitivity.

The mathematical model constructed in [2] simulates the kinematics of homogeneous turbulence remarkably closely. Experimental observations indicate homogeneous turbulence has a continuous spectrum, is nearly normal (Gaussian), and is often nearly isotropic. If it were strictly normal and isotropic, then the model of [2] would describe it exactly, and homogeneous incompressible turbulent motion could be regarded as a superposition of a large number of statistically independent shear flows having appropriately distributed wave-lengths and independently random phases. This is not strictly true: components of different wave-lengths interact weakly in real homogeneous turbulence.

Nevertheless, the simulation is close enough so as to make the following conclusion virtually certain. Though Wiener's definition of (3) has the analog

$$(13) \quad \mathbf{u}(\mathbf{x}) = \iiint e^{i\mathbf{q} \cdot \mathbf{x}} d\mathbf{F}(\mathbf{q}) \quad [1, (2.5.3)]$$

constructed by Batchelor, one *cannot* reconstruct $\mathbf{u}(\mathbf{x})$ from $\mathbf{F}(\mathbf{q})$. Specifically, one cannot specify the *phases* of the different "Fourier components" of homogeneous turbulence: they are undefined.

This is in sharp contrast with *hypothetical periodic turbulent velocity-fields*, even of large period. The difference is nevertheless perhaps "physically trivial", as stated by Batchelor in the first footnote of [1, p. 30], because the *phases* of adjacent wave-lengths in the periodic case are uncorrelated and so fluctuate wildly, as the wave-length changes by small steps. Moreover they cannot be averaged mod 2π . The *amplitudes*, on the other hand, have Gaussian distributions having well-defined averages even over narrow bands of wave-lengths.

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