Weak H -Spaces

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1. Introduction. When trying to represent an algebraic structure on a functor by a corresponding structure on its classifying space one runs into the notion of a weak H -space; see [5, Th. 8.1]. For example, in exposé 11, section 1, of [3] there is an explicit construction of a weak H -structure on B_U , the classifying space of the infinite unitary group, representing addition on $\mathbb{K}$. It is then natural to ask just what kinds of weak H -spaces exist. But before stating our answer, Theorem (1.1) below, we make precise the definitions involved.

We shall work exclusively in the category $\mathfrak{W}$ of spaces having the homotopy type of a CW complex. Each space has a base point, and all maps and homotopies preserve base points. It should be noted that $\mathfrak{W}$ is closed under the formation of cartesian products and loop spaces.

Let $[K, X]$ denote the set of homotopy classes $[h]$ of maps $h : K \rightarrow X$. A map $f : X \rightarrow Y$ induces a function $f_* : [K, X] \rightarrow [K, Y]$ by $f_*[h] = [fh]$. Two such maps $f, g : X \rightarrow Y$ are *weakly homotopic* if $f_* = g_*$ for every *finite* CW complex K . In particular, if $X \in \mathfrak{W}$ is compact then $f \simeq g$. To obtain the weak homotopy analogue of such familiar concepts as an H -space one simply replaces the word "homotopy" in the usual definition by the words "weak homotopy." In this way we get definitions of *weak H -structure*, *weak H -space*, *weakly homotopy-associative*, and *weakly homotopy-commutative*; these last two will be abbreviated by *w.h.a.* and *w.h.c.*, while *h.a.* and *h.c.* will refer to the usual notion. Furthermore, when speaking of a weak property we will mean it in the *strict* sense; thus " μ is w.h.a." means, in addition, that μ is not h.a.

A weak H -structure $\mu : X \times X \rightarrow X$ on X is either h.a., w.h.a., or neither, and similarly for commutativity. In this way μ falls into one of nine possible classes. We do not consider inversion here because, as will be shown in the Appendix, weak H -spaces always have both left and right homotopy inversions.

(1.1) Theorem. *There is no weak H -space whose multiplication is homotopy-associative, but each of the other six classes is non-empty.*

To provide examples let $n \geq 2$ and let $T_n = K(\mathbb{Q}, n)$ be an Eilenberg-MacLane space with a given cell structure, where $\mathbb{Q}$ is the additive group of rational

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numbers. Let W_n be constructed by attaching disjoint cones over the skeletons of T_n . Both spaces are $(n - 1)$ -connected and (W_n, T_n) may be taken to be a CW pair. Define

$$(1.2) \quad \begin{cases} X_n = \Omega^2(T_{n+2} \times W_{n+2}), \\ Y_n = \Omega^n(T_{n+2} \times W_{n+2}) \end{cases}, \quad n \geq 2.$$

Thus, X_n is $(n - 1)$ -connected, Y_n is a 1-connected n -fold loop space, and both are in $\mathfrak{W}$.

(1.3) Proposition. *On both X_n and Y_n there exists a countably infinite family of non-homotopic w.h.a. and h.c. weak H -structures, and a countably infinite family of non-homotopic w.h.a. and w.h.c. weak H -structures. Moreover, all of these structures are weakly homotopic to the usual loop space H -structure.*

The construction of these weak H -structures rests on the following result.

(1.4) Proposition. *For $n \geq 2$, the inclusion map $i : T_n \rightarrow W_n$ is weakly null-homotopic but not null-homotopic. Moreover, the map $\Omega^j i$ on the j -fold loop space is weakly null-homotopic, $j \geq 0$. If $n \geq 3$, $1 \leq j \leq n - 2$, and $r \geq 1$, then $r[\Omega^j i] \neq 0$ in the group $[\Omega^j T_n, \Omega^j W_n]$.*

Certainly W_n has been constructed so that i is weakly null-homotopic. To show that i is not null-homotopic we construct a map f from $\Sigma^{n-2} \mathbb{C}P^\infty$, the $(n - 2)$ -fold suspension of infinite complex projective space, to T_n whose composition with i is not null-homotopic. This procedure is patterned after, but distinct from, that given by Adams and Walker in [2]. (Their result can be used to construct weak H -spaces whose commutativity status is unclear.)

The paper is organized as follows. The main result (1.1) is deduced from (1.3) in §2. In §3 we give a general method of constructing weak H -structures given data analogous to that in (1.4). In particular, (1.3) follows as a consequence of (1.4). The proof of (1.4) occupies §4. The final section, §5, exhibits another family of weak H -spaces Z_n , $n \geq 2$, and examples of H -maps and weak H -maps of weak H -spaces from Z_n to Y_n .

2. Proof of Theorem (1.1). Let μ be a h.a. multiplication on space $X \in \mathfrak{W}$. We show that μ cannot be a strictly weak H -structure. To do this let $i_1 : X \rightarrow X \times X$ and $j_1 : X \rightarrow X \times X \times X$ be injections onto the first factor, and set $f = \mu i_1$. Since μ is h.a. we have that

$$ff = \mu i_1 f = \mu(\mu \times 1)j_1 \simeq \mu(1 \times \mu)j_1 = \mu i_1 = f.$$

But f is a homotopy equivalence if μ is a weak H -structure (it's weakly homotopic to 1). Hence we can cancel f in the above equation to get $\mu i_1 = f \simeq 1$. Similarly $\mu i_2 \simeq 1$ and μ is an H -structure. This establishes the first assertion of (1.1).

The following result will be useful for constructing new examples of weak H -spaces from old.

(2.1) **Lemma.** Let X, Y be spaces with multiplications μ, ν respectively, and define $\rho : (X \times Y) \times (X \times Y) \rightarrow X \times Y$ by $\rho(x, y, a, b) = (\mu(x, a), \nu(y, b))$. Then each of the following properties of ρ is equivalent to the corresponding property of both μ and ν : H -structure; h.a.; h.c.; and the weak analogues of these.

Thus, for example, ρ is a weak H -structure if and only if both μ and ν are.

Proof. Its completely straightforward to show that if a property holds for both μ and ν then it holds for ρ . We shall give two cases of the proof of the converse and leave the rest to the reader.

If ρ is an H -structure the following homotopy-commutative diagrams ($j = 1, 2$) shows that μ is an H -structure on X :

$$\begin{array}{ccccc}
 & & & & 1 \\
 & & & & \downarrow \\
 X \times Y & \xrightarrow{i_j} & (X \times Y) \times (X \times Y) & \xrightarrow{\rho} & X \times Y \\
 \uparrow i_1 & & \uparrow i_1 \times i_1 & & \downarrow p_1 \\
 X & \xrightarrow{i_j} & X \times X & \xrightarrow{\mu} & X
 \end{array}$$

Similarly, ν is an H -structure on Y .

If ρ is h.a., then

$$\begin{aligned}
 \mu(1 \times \mu) &= p_1 \rho(i_1 \times i_1)(1 \times \mu) = p_1 \rho(1 \times \rho)(i_1 \times i_1 \times i_1) \\
 &\simeq p_1 \rho(\rho \times 1)(i_1 \times i_1 \times i_1) = \mu(\mu \times 1)
 \end{aligned}$$

and μ is h.a. Similarly, ν is h.a.

We are now ready to produce examples of the six kinds of weak H -spaces. Two of these are provided by taking X to be the space X_2 of (1.2) and letting μ, μ' be w.h.a. weak H -structures which are h.c., w.h.c., respectively; these exist by (1.3).

The topological group S^3 of unit quaternions under multiplication ν is not h.c. by [4, Th. II], and hence is *not* w.h.c. (since the notions of homotopy and weak homotopy agree on finite complexes). It follows from (2.1) that the product multiplication on $X \times S^3$ induced by μ and ν is a w.h.a. weak H -structure which is *not* w.h.c.

To get the remaining three examples we appeal to the following result of Adams [1, p. 54].

(2.2) **Lemma.** There exists a space $Y \in \mathfrak{W}$ with a commutative H -structure ν which is not w.h.a.

Actually, Adams shows that ν is not h.a., but his proof establishes the stronger statement (2.2). Indeed, Y is constructed so as to have a certain cohomology

algebra over $\mathbf{Z}_3$, and his Lemma 2 shows that this algebra is not realizable from a h.a. H -structure on Y . Now if ν were w.h.a. the cohomology algebra of Y would be the same as if ν were h.a., because of the weak-homotopy theorem in cohomology [5, Th. 11.2]; consequently ν is not w.h.a.

Proceeding as above, μ and ν induce a h.c. weak H -structure on $X \times Y$ which is not w.h.a. If we use μ' instead of μ the structure is w.h.c. To get our final example take $X \times Y \times S^3$; its induced weak H -structure is neither w.h.a. nor w.h.c.

3. Construction of Weak H -Structures. Let X, Y be h.a. and h.c. H -spaces, and suppose that $f : X \rightarrow Y$ is weakly null-homotopic but that $r[f] \neq 0$ in the group $[X, Y]$, $r \geq 1$. We show how to modify the induced h.a., h.c. H -structure ρ on $Z = X \times Y$ to obtain weak H -structures. For $r \geq 0$ let $h_r : Z \rightarrow Z$ be a map such that

$$[h_r] = [1] + r[i_2 f p_1] \in [Z, Z]$$

and set

$$(3.1) \quad \mu_{r,s} = \rho(h_r \times h_s) : Z \times Z \rightarrow Z, \quad r, s \geq 0.$$

(3.2) Proposition. $\mu_{0,0} \simeq \rho$. If $(r, s) \neq (0, 0)$ then $\mu_{r,s}$ is a w.h.a. weak H -structure which is h.c. if $r = s$ and w.h.c. otherwise. Moreover, $\mu_{r,s}$ and $\mu_{a,b}$ are weakly homotopic but not homotopic if $(r, s) \neq (a, b)$.

Remark. Its natural to ask if we could get even more distinct weak H -structures on Z by using the s -fold composition $h^s : Z \rightarrow Z$ of $h = h_1$ as well as the maps h_r . However, it turns out that if $f : X \rightarrow Y$ is an H -map, as it is in our examples, then h_r is too and, as a consequence, $h^r \simeq h_r$.

Before giving the proof let's see how (1.3) follows from (1.4) and this construction. Let $n \geq 2$. Then $\Omega^2 i : \Omega^2 T_{n+2} \rightarrow \Omega^2 W_{n+2}$ is weakly null-homotopic, and $r[\Omega^2 i] \neq 0$ if $r \geq 1$, so (3.2) yields our weak H -structures on $X_n = \Omega^2 T_{n+2} \times \Omega^2 W_{n+2}$. To get weak H -structures on Y_n just apply our construction to the map $\Omega^n i : \Omega^n T_{n+2} \rightarrow \Omega^n W_{n+2}$.

The proof of (3.2) depends on the following lemma.

(3.3) Lemma. For $r, s \geq 0$, $r \neq s$, h_r and h_s are weakly homotopic but not homotopic. In particular, $h_r \not\simeq 1$ if $r > 0$.

Proof. Since $p_2 : X \times Y \rightarrow Y$ is an H -map, the composition

$$\varphi : [Z, Z] \xrightarrow{p_2*} [Z, Y] \xrightarrow{i_1*} [X, Y]$$

is a group homomorphism. Therefore

$$\varphi(r[g]) = r\varphi[g] = r[f] \neq 0, \quad r > 0,$$

where $g = i_2 f p_1$; hence $r[g] \neq 0$, $r > 0$. Now $[h_r] = [1] + r[g]$, so if $h_r \simeq h_s$ with $r > s$ then $(r - s)[g] = 0$ and we have a contradiction.

It remains to show that each h_r is weakly homotopic to 1. Let $f_n : X \rightarrow Y$ be defined inductively by $f_1 = f$, $f_n = \nu(f \times f_{n-1})\Delta$, where Δ is the diagonal map and ν is the H -structure on Y . Then $[f_r] = r[f]$. Since $\psi = p_1^*i_{2*} : [X, Y] \rightarrow [Z, Z]$ is a homomorphism, $r[g] = r\psi[f] = \psi[f_r] = [i_2f_r p_1]$. Set $g_r = i_2f_r p_1$. Now f is weakly null-homotopic, and hence so are f_r and g_r . Therefore $h_r \simeq \rho(1 \times g_r)\Delta$ is weakly homotopic to $\rho(1 \times 0)\Delta = \rho i_1 \simeq 1$.

Proof of (3.2). $\mu_{0,0} \simeq \rho$ since $h_0 \simeq 1$. Also observe that $\mu_{r,s}i_1 = \rho i_1 h_r \simeq h_r$, $\mu_{r,s}i_2 \simeq h_s$. It follows from (3.3) that $\mu_{r,s}$ is a weak H -structure for $(r, s) \neq (0, 0)$. In view of the first assertion of (1.1) proved at the beginning of §2, $\mu_{r,s}$ is w.h.a. Clearly each $\mu_{r,s}$ is weakly homotopic to ρ , and if $\mu_{r,s} \simeq \mu_{a,b}$, then $h_r \simeq h_a$, $h_s \simeq h_b$, and by (3.3), $r = a$, $s = b$. Finally, let $T : Z \times Z \rightarrow Z \times Z$ be the transposition map and observe that $\mu_{r,s}T \simeq \mu_{s,r}$ since $\rho T \simeq \rho$. It follows that $\mu_{r,s}$ is h.c. if and only if $\mu_{r,s} \simeq \mu_{s,r}$, i.e. $r = s$. If $r \neq s$, then $\mu_{r,s}$ is w.h.c. since its weakly homotopic to ρ .

4. Proof of Proposition (1.4). Let T be a CW complex and let $CT^k \supset T^k$ denote the (reduced) cone over the k -skeleton T^k of T . We construct a CW complex W containing T as a subcomplex by first forming the disjoint union of the CT^k , $k = 0, 1, \dots$, and then identifying points $x \in CT^n$ and $y \in CT^m$ if and only if $x \in T^n$, $y \in T^m$, and $x = y$. Notice that the CW complex W/T obtained by collapsing T to the base point of W is given by

$$(4.1) \quad W/T = \bigvee_{k=0}^{\infty} (\Sigma T^k),$$

where Σ is the (reduced) suspension functor. Also observe that the homotopy type of W depends on the cell structure of T .

(4.2) Lemma. *Let $i : T \rightarrow W$ be the inclusion map. Then the induced loop-space map $\Omega' i : \Omega' T \rightarrow \Omega' W$ is weakly null-homotopic for any $r \geq 0$.*

The result is obvious for $r = 0$ since $i(T^k) \subset CT^k \subset W$. The general case follows by induction from

(4.3) *If $f, g : X \rightarrow Y$ are weakly homotopic maps, then so are $\Omega f, \Omega g$.*

Proof. If K is a finite CW complex then $f_* = g_* : [\Sigma K, X] \rightarrow [\Sigma K, Y]$. By adjointness it follows that $(\Omega f)_* = (\Omega g)_* : [K, \Omega X] \rightarrow [K, \Omega Y]$ and the proof is complete.

For the remainder of this section we let $T = T_n$ be an Eilenberg-MacLane space $K(Q, n)$, $n > 0$. Without loss of generality we may assume that T^k is the base point $*$ if $k < n$, and that $T^n = \bigvee_{\alpha=1}^{\infty} S_{\alpha}^n$, a countable bouquet of n -spheres.

(4.4) Lemma. *The Steenrod operations Sq^i and ϕ^i are zero on $H^{n+1}(W/T; \mathbb{Z}_p)$, $p \geq 2$ a prime, for any $i > 0$.*

Proof. Because of (4.1) we have natural isomorphisms

$$H^{r+1}(W/T; \mathbb{Z}_p) \cong \prod_{k \geq n} H^{r+1}(\Sigma T^k; \mathbb{Z}_p) \cong \prod_{k \geq n} H^r(T^k; \mathbb{Z}_p),$$

where Π denotes complete direct sum. By naturality of the cohomology operations it therefore suffices to show that Sq^i and φ^i are zero on $H^n(T^k; \mathbb{Z}_p)$ for $k \geq n$. For $k = n$, T^k is a bouquet of spheres so our operations are zero (because of naturality). For $k > n$, the operations are zero simply because

$$H^n(T^k; \mathbb{Z}_p) \cong H^n(T; \mathbb{Z}_p) \cong \text{Hom}(\mathbb{Q}, \mathbb{Z}_p) = 0.$$

Now assume that $n \geq 2$ and set $X = \Sigma^{n-2} \mathbb{C}P^\infty$. Since $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$ -space we have that

$$[X, T] \cong [K(\mathbb{Z}, 2), K(\mathbb{Q}, 2)] \cong \text{Hom}(\mathbb{Z}, \mathbb{Q}) \cong \mathbb{Q}.$$

Let $f_r : X \rightarrow T$, for $r > 0$ an integer, be a cellular map such that $[f_r]$ corresponds to the group homomorphism $r : \mathbb{Z} \rightarrow \mathbb{Q}$ sending m to rm . Then $f_{r*} : H_n(X) = \mathbb{Z} \rightarrow H_n(T) = \mathbb{Q}$ is the homomorphism r .

As a preliminary to the proof of (1.4) we have the following analogue of the main result of [2].

(4.5) Proposition. *Using the above notation, the composition $if_r : X \rightarrow W$ is weakly null-homotopic but not null-homotopic for $n \geq 2$, $r > 0$.*

The proof will make use of two lemmas concerning f_r .

(4.6) Lemma. *There exists a prime p such that $(f_r | X^n)^* : H^n(T^n; \mathbb{Z}_p) \rightarrow H^n(X^n; \mathbb{Z}_p)$ is non-zero.*

Proof. Write $f = f_r$, $S_\alpha = S_\alpha^n$, and let $i : X^n \rightarrow X$, $i' : T^n \rightarrow T$ denote the inclusion maps. Since X^n is compact we may factor $f | X^n$ as follows:

$$f | X^n : X^n \xrightarrow{g} S_1 \vee \cdots \vee S_m \xrightarrow{j} T^n.$$

Identifying $H_n(S_1 \vee \cdots \vee S_m)$ with $\Sigma_k H_n(S_k)$ we have the commutative diagram

$$\begin{array}{ccc} H_n(X) & \xrightarrow{f_*} & H_n(T) \\ \uparrow i_* & & \uparrow i'_* \\ H_n(X^n) & \xrightarrow{g_*} \sum H_n(S_k) \xrightarrow{j_*} & H_n(T^n) \end{array}$$

where $g_*(u) = (f_*^1 u, \dots, f_*^m u)$ and $f^k : S_k \rightarrow X^n$ is the composition of g followed by projection p_k onto S_k . But i_* is an isomorphism and $f_* \neq 0$, so it follows that $f_*^k : H_n(X^n) = \mathbb{Z} \rightarrow H_n(S_k) = \mathbb{Z}$ is non-zero for some k . Choose prime p to be relatively prime to $f_*^k(1) \in \mathbb{Z}$.

We claim that $g^* : H^n(S_1 \vee \cdots \vee S_m; \mathbb{Z}_p) \rightarrow H^n(X^n; \mathbb{Z}_p)$ is non-zero. Since

$j^* : H^n(T^n; \mathbb{Z}_p) \rightarrow H^n(S_1 \vee \cdots \vee S_m; \mathbb{Z}_p)$ is surjective, the proof of (4.6) will be complete. By the universal coefficient theorem g^* corresponds to

$$\text{Hom}(g_*, 1) : \text{Hom}(H_n(S_1 \vee \cdots \vee S_m), \mathbb{Z}_p) \rightarrow \text{Hom}(H_n(X^n), \mathbb{Z}_p).$$

Let $\varphi = \rho p_{k*} : H_n(S_1 \vee \cdots \vee S_m) \rightarrow \mathbb{Z}_p$ where $\rho : \mathbb{Z} \rightarrow \mathbb{Z}_p$ is reduction mod p . Then $\text{Hom}(g_*, 1)\varphi = \varphi g_* = \rho f_*^k \neq 0$ implies that $g^* \neq 0$.

(4.7) Lemma. *Let $F : (CX, X) \rightarrow (W, T)$ be a map extending i_r , and let $G : \Sigma X \rightarrow W/T$ be the map induced on the quotients. Then $G^* : H^{n+1}(W/T; \mathbb{Z}_p) \rightarrow H^{n+1}(\Sigma X; \mathbb{Z}_p)$ is non-zero for the prime p of (4.6).*

Proof. Write $f = f_r$ and delete the coefficient group $\mathbb{Z}_p$ from the notation. Let $i : (W, T^n) \rightarrow (W, T)$ and $j : (CX^n, X^n) \rightarrow (CX, X)$ denote the inclusion maps. Now F may be assumed cellular and we let $F' : (CX^n, X^n) \rightarrow (W, T^n)$ be its restriction. Thus the diagram

$$\begin{array}{ccccccc} H^n(T^n) & \xrightarrow{\delta} & H^{n+1}(W, T^n) & \xleftarrow{i^*} & H^{n+1}(W, T) & \cong & H^{n+1}(W/T) \\ \downarrow (f|_{X^n})^* & & \downarrow F'^* & & \downarrow F^* & & \downarrow G^* \\ H^n(X^n) & \xrightarrow{\delta'} & H^{n+1}(CX^n, X^n) & \xleftarrow{j^*} & H^{n+1}(CX, X) & \cong & H^{n+1}(\Sigma X) \end{array}$$

is commutative. Moreover,

$$W/T^n = \Sigma T^n \vee A, \quad W/T = \Sigma T^n \vee B,$$

where

$$A = (W - \text{Int } CT^n)/T^n, \quad B = (W - \text{Int } CT^n)/T,$$

and it follows that i induces the map $1 \vee q : \Sigma T^n \vee A \rightarrow \Sigma T^n \vee B$ on the quotient spaces, q being an identification. Hence

$$\begin{array}{ccc} H^{n+1}(W, T) & \cong & H^{n+1}(\Sigma T^n) \oplus H^{n+1}(B) \\ \downarrow i^* & & \downarrow 1 \oplus q^* \\ H^{n+1}(W, T^n) & \xrightarrow{\rho} & H^{n+1}(\Sigma T^n) \oplus H^{n+1}(A) \end{array}$$

is a commutative diagram, the isomorphism ρ being

$$H^{n+1}(W, T^n) \cong H^{n+1}(W/T^n) \cong H^{n+1}(\Sigma T^n) \oplus H^{n+1}(A).$$

Let $C'T^n$ be a cone over T^n such that $W \cap C'T^n = T^n$. Let $p : W \cup C'T^n \rightarrow W \cup C'T^n/C'T^n = (\Sigma T^n) \vee A$ and $q : W \cup C'T^n \rightarrow W \cup C'T^n/W = \Sigma T^n$ be the identification maps, and let $p_1 : (\Sigma T^n) \vee A \rightarrow \Sigma T^n$ be the projection map. Its easy to see that $q \simeq p_1 p$. Passing to cohomology we obtain the following com-

$$\begin{array}{ccc}
 \rho : H^{n+1}(\Sigma T^n) \oplus H^{n+1}(A) \cong H^{n+1}(W/T^n) & \xrightarrow{p^*} & H^{n+1}(W \cup C' T^n) \cong H^{n+1}(W, T^n) \\
 \uparrow i_1 & \nearrow q^* & \uparrow \delta \\
 H^{n+1}(\Sigma T^n) & \xleftarrow{\Sigma} & H^n(T^n)
 \end{array}$$

mutative diagram where Σ is the canonical suspension isomorphism and i_1 is injection onto the first factor. Therefore δ maps $H^n(T^n)$ isomorphically onto $\rho H^{n+1}(\Sigma T^n)$. In view of this, the two earlier diagrams, and the fact that $(f \mid X_n)^* \neq 0$ by (4.6), it is clear that $G^* \neq 0$.

Proof of (4.5). Since i is weakly null-homotopic by (4.2), if_r is too. Suppose $if_r \simeq 0 : X \rightarrow W$. Then there is a map $F : (CX, X) \rightarrow (W, T)$ extending $f_r : X \rightarrow T$, and we let $G : \Sigma X \rightarrow W/T$ be the induced map. Let p be a prime in accordance with the above lemmas and consider the commutative diagram ($\mathbb{Z}_p$ coefficients)

$$\begin{array}{ccc}
 H^{n+1}(W/T) & \xrightarrow{G^*} & H^{n+1}(\Sigma X) \cong H^2(\mathbb{CP}^\infty) \\
 \downarrow \varphi^1 & & \cong \downarrow \varphi^1 \\
 H^r(W/T) & \xrightarrow{G^*} & H^r(\Sigma X) \cong H^{2p}(\mathbb{CP}^\infty)
 \end{array}$$

where $r = n + 1 + 2(p - 1)$. (If $p = 2$, we use Sq^2 instead.) But $G^* \neq 0$ in dimension $n + 1$ by (4.7) while $\varphi^1 = 0$ on space W/T by (4.4). This contradiction completes the proof.

Remark. Its instructive to examine the proof strategy to see just why $\mathbb{Q}$ was used instead of $\mathbb{Z}$ or a finite group $\mathbb{Z}_p$ in the definition of T . Its clear that $X = \Sigma^{n-2}\mathbb{CP}^\infty$ was selected because all of its Steenrod operations φ^1 and Sq^2 are isomorphisms. On the other hand, the space T is to be such that the corresponding operations on $H^{n+1}(W/T; \mathbb{Z}_p)$ are trivial; by (4.1), the same must be true of $H^n(T; \mathbb{Z}_p)$. Moreover, the easy way of getting control of maps $f : X \rightarrow T$ is to choose $T = K(G, n)$ since $[X, T] \cong G$ then. Suppose $[f]$ corresponds to $\alpha \neq 0 \in G$. Since $f^* : H^n(T; G) \rightarrow H^n(X; G) \cong G$ sends the fundamental class to α we have that $f^* \neq 0$. On the other hand, $f^* : H^n(T; \mathbb{Z}_p) \rightarrow H^n(X; \mathbb{Z}_p)$ must be zero for every prime p since the Steenrod operation φ^1 (or Sq^2) is an isomorphism for X and is trivial for T . In particular, if $G = \mathbb{Z}$ or $\mathbb{Z}_p$, p a prime, it follows that our method for showing $if \neq 0$ will fail. However, if we choose $G = \mathbb{Q}$ then $H^n(T; \mathbb{Z}_p) = 0$ for all p and this difficulty disappears.

Proof of (1.4). The statements concerning weak null-homotopy follow from (4.2). Moreover, (4.5) implies that i is not null-homotopic. It remains to show that if $n \geq 3$, $r \geq 1$, and $n - 2 \geq j \geq 1$, then $r[\Omega^j i] \neq 0$ in $[\Omega^j T, \Omega^j W]$.

Set $f = f_1$. Then $[f_r] = r[f] \in [X, T]$. Since the suspension structure on X also defines our group structures, $i_* : [X, T] \rightarrow [X, W]$ is a homomorphism and $[if_r] = r[if] \neq 0$ by (4.5).

Let $g : \Sigma^{n-2-i} \mathbb{C}P^\infty \rightarrow \Omega^i W$ be the j -fold adjoint of if , and $h : \Sigma^{n-2-i} \mathbb{C}P^\infty \rightarrow \Omega^i T$ the j -fold adjoint of f . Since $[X, W] \cong [\Sigma^{n-2-i} \mathbb{C}P^\infty, \Omega^i W]$ is an isomorphism of groups we see that $r[if]$ corresponds to

$$r[g] = r[(\Omega^i i)h] = rh^*[\Omega^i i] = h^*(r[\Omega^i i]),$$

which is therefore non-zero. In particular, $r[\Omega^i i] \neq 0$.

Remark. As a biproduct of the proof we have the following result which will be used in §5.

(4.8) $g : \mathbb{C}P^\infty \rightarrow \Omega^{n-2} W_n$ is weakly null-homotopic and $r[g] \neq 0 \in [\mathbb{C}P^\infty, \Omega^{n-2} W_n]$ if $r \geq 1, n \geq 3$.

That g is weakly null-homotopic is an immediate consequence of the corresponding fact about if and the following result.

(4.9) If $f, g : \Sigma Y \rightarrow Z$ are weakly homotopic maps then so are their adjoints $\hat{f}, \hat{g} : Y \rightarrow \Omega Z$.

Proof. Let K be a finite CW complex and consider the commutative diagram

$$\begin{array}{ccc} [K, Y] & \xrightarrow{\Sigma} & [\Sigma K, \Sigma Y] \\ \downarrow \hat{f}_* & & \downarrow f_* \\ [K, \Omega Z] & \cong & [\Sigma K, Z] \end{array}$$

Since $f_* = g_*$, we have $\hat{f}_* = \hat{g}_*$ and the proof is complete.

5. More Examples. As before, let W_{n+2} be constructed from $T_{n+2} = K(Q, n+2)$ and consider the space

$$Z_n = \mathbb{C}P^\infty \times \Omega^n W_{n+2}, \quad n \geq 2.$$

The usual h.a. and h.c. H -structures on $\mathbb{C}P^\infty = K(Z, 2)$ and $\Omega^n W_{n+2}$ induce the product H -structure ρ on Z_n which is h.a. and h.c. On the other hand we can use (4.8) and (3.1) to construct weak H -structures on Z_n . In particular we have

(5.1) **Proposition.** There is a countably infinite set of distinct w.h.a. weak H -structures on Z_n each of which is weakly homotopic to ρ .

We next give examples of an H -map and a weak H -map $F : Z_n \rightarrow Y_n$ relative to certain weak H -structures ν, μ on Z_n, Y_n respectively. F is an H -map if $\mu(F \times F) \simeq F\nu$, and a weak H -map if we have weak homotopy instead.

Let $f : \mathbb{C}P^\infty = K(\mathbb{Z}, 2) \rightarrow \Omega^n T_{n+2} \simeq K(\mathbb{Q}, 2)$ be an H -map induced by the group inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$. Then $F = f \times 1 : Z_n \rightarrow Y_n$ ($n \geq 2$) is an H -map relative to the usual H -structures ρ, σ . Let $\mu = \mu_{1,1}$ on Y_n and $\nu = \nu_{1,1}, \nu' = \nu_{1,0}$ on Z_n be the weak H -structures given by equation (3.1). We then have the following result:

(5.2) Proposition. *$F : Z_n \rightarrow Y_n$ is an H -map relative to ν, μ , and a weak H -map relative to ν', μ .*

Proof. We first recall the definitions of μ, ν , and ν' :

$$\mu = \sigma(h \times h), \quad \nu = \rho(k \times k), \quad \nu' = \rho(k \times 1),$$

where

$$\begin{cases} [h] = [1] + [i_2(\Omega^n i)p_1] \in [Y_n, Y_n], \\ [k] = [1] + [i_2 g p_1] \in [Z_n, Z_n], \end{cases}$$

and

$$\Omega^n W_{n+2} \xrightarrow{i_2} Y_n = \Omega^n T_{n+2} \times \Omega^n W_{n+2} \xrightarrow{p_1} \Omega^n T_{n+2}$$

for the space Y_n . Here, σ and ρ are used to define the group structures on the sets of homotopy classes.

We claim that $hF \simeq Fk$. First observe that $g \simeq (\Omega^n i)f$; this follows from the definition of g given in §4. Next, $F_* : [Z_n, Z_n] \rightarrow [Z_n, Y_n]$ and $F^* : [Y_n, Y_n] \rightarrow [Z_n, Y_n]$ are homomorphisms since F is an H -map relative to ρ, σ . Hence

$$\begin{aligned} [Fk] &= F_*[k] = F_*[1] + F_*[i_2 g p_1] \\ &= [F] + [i_2 g p_1] = [F] + [i_2(\Omega^n i)p_2(f \times 1)] \\ &= F^*([1] + [i_2(\Omega^n i)p_2]) = [hF]. \end{aligned}$$

We are now ready to prove (5.2). Since $\sigma(F \times F) \simeq F\rho$ and $hF \simeq Fk$, we have that $\mu(F \times F) \simeq F\nu$. Also, ν' is weakly homotopic to ν , so $\mu(F \times F)$ and $F\nu'$ are weakly homotopic too; it remains to show that they are not homotopic. Suppose otherwise. By composing this homotopy with the injection $i_2 : Z_n \rightarrow Z_n \times Z_n$, we would have $hF \simeq F$. The above computation would then imply that $i_2 g p_1$, and hence g itself, is null-homotopic contrary to (4.8).

Appendix. Although a weak H -structure is never h.a. the answer to the corresponding question concerning homotopy inversion is just the opposite.

Proposition. *A weak H -structure on a space having the homotopy type of a connected CW complex has both left and right homotopy inversion.*

The proof is the same as that for the corresponding fact about H -spaces but we shall give the details for the reader's convenience.

Let μ be a weak H -structure on X as in the hypothesis. First observe that $\mu_* : \pi_n(X \times X) = \pi_n(X) \times \pi_n(X) \rightarrow \pi_n(X)$ is the usual group operation (here

written additively). To see this let $\nu : S^n \rightarrow S^n \vee S^n$ be the coproduct collapsing $S^{n-1} \subset S^n$ to a point. Then $i\nu : S^n \rightarrow S^n \vee S^n \rightarrow S^n \times S^n$ is homotopic to the diagonal map Δ . Moreover, $\mu i : X \vee X \rightarrow X \times X \rightarrow X$ is weakly homotopic to the folding map ∇ . If $f, g : S^n \rightarrow X$ then $\mu(f \times g)\Delta \simeq \mu i(f \vee g)\nu$ is weakly homotopic, and hence homotopic, to $\nabla(f \vee g)\nu$. Therefore

$$\mu_*([f], [g]) = [\mu(f \times g)\Delta] = [\nabla(f \vee g)\nu] = [f] + [g].$$

If $g, h : Y \rightarrow Z$ let $(g, h) : Y \rightarrow Z \times Z$ denote the map $(g \times h)\Delta$. Now set $f = (p_1, \mu) : X \times X \rightarrow X \times X$. By the above observation, $f_* : \pi_n(X) \times \pi_n(X) \rightarrow \pi_n(X) \times \pi_n(X)$ is the isomorphism taking (α, β) to $(\alpha, \alpha + \beta)$. Therefore f is a homotopy equivalence and we let $g = (g_1, g_2)$ be a homotopy inverse. Then $p_1 \simeq gfp_1 = gp_1 = g_1$ and $p_2 \simeq gfp_2 = g\mu$. It follows that

$$0 = p_2 i_1 \simeq \mu(g_1, g_2) i_1 \simeq \mu(p_1, g_2) i_1 = \mu(1, g_2 i_1);$$

that is, $g_2 i_1$ is a right homotopy inverse for μ . To obtain a left homotopy inverse simply carry through the argument using $f = (\mu, p_2)$.

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