Representability of Milnor's Functor k_{PL} .

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1. Introduction. For the introduction we shall assume that the spaces under consideration are connected and have base points, and that all maps and homotopies preserve base points. $[X, B]$ denotes the set of homotopy classes $[f]$ of maps $f: X \rightarrow B$; the associated set-valued functor is denoted by $[_, B]$. Let $\mathfrak{W}$ be the category of finite CW complexes and maps.

Let f and g be maps from a space X to Y . Recall that f and g are said to be *weakly homotopic* if $f_* = g_*: [K, X] \rightarrow [K, Y]$ for every $K \in \mathfrak{W}$. (Here, $f_*[\varphi] = [f\varphi]$, $[\varphi] \in [K, X]$.) If in the definition of " H -space" we replace the term "homotopy" by "weak homotopy", we then get the definition of a *weak H -space*. One similarly defines the notion of *weak homotopy associativity*, etc., of a multiplication $\mu: X \times X \rightarrow X$ on X . If X and Y are spaces with multiplications μ and ν respectively, a map $f: X \rightarrow Y$ is called a *weak homomorphism* if $f\mu$ is weakly homotopic to $\nu(f \times f)$.

A multiplication μ on B induces a multiplication (written additively) on each $[X, B]$ by $[f] + [g] = [\mu(f, g)]$ where $(f, g): X \rightarrow B \times B$ is defined by $(f, g)(x) = (f(x), g(x))$. If $f: B \rightarrow B'$ is a weak homomorphism, then $f_*: [X, B] \rightarrow [X, B']$, $X \in \mathfrak{W}$, is a homomorphism.

Milnor has developed in [5] the theory of PL -microbundles. In particular, he has defined $k_{PL}(X)$ as the set of stable equivalence classes of PL -microbundles over X . Whitney sum immediately makes $k_{PL}(X)$ an abelian monoid (it's really a group but this requires proof).

Our main result is the following.

(1.1) Theorem. *The functor k_{PL} is representable on $\mathfrak{W}$ in the sense that there exists a countable CW complex B_{PL} such that k_{PL} is naturally equivalent to $[_, B_{PL}]$. Addition on k_{PL} is represented by a weakly homotopy commutative and weakly homotopy associative H -space structure μ_{PL} on B_{PL} . Moreover, (B_{PL}, μ_{PL}) is unique in the sense that if (B, μ) is another H -space representing k_{PL} on $\mathfrak{W}$ then there exists a weak homomorphism $B_{PL} \rightarrow B$ which is a weak homotopy equivalence.*

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The existence of B_{PL} is deduced in [5] by much different methods.

By using orientable PL -microbundles one obtains a functor k_{PL0} and (1.1) holds equally well for this.

The proof of (1.1) is carried out by verifying the axioms of Brown [1] to obtain the classifying space and by generalizing known results and methods of Eckmann and Hilton to represent the multiplication on the functor. The former requires a geometric construction in which two PL -microbundles are glued together; it is summarized as follows (base points are not used here):

(1.2) Theorem. *Let $X = X_1 \cup X_2$ be a simplicial complex such that $A = X_1 \cap X_2$ is a finite subcomplex. Let ξ_i be an n -dimensional PL -microbundle over X_i , $j = 1, 2$, such that $\xi_1|_A \cong \xi_2|_A$. Then there exists an n -dimensional PL -microbundle ξ over X such that $\xi|_{X_i} \cong \xi_i$, $j = 1, 2$.*

The paper is divided into two parts, the first of which consists of the geometric items needed for proving (1.1). In §2 we give some preliminary technical facts about PL -maps. Section 3 contains equivalent formulations of the PL -microbundle notions of local triviality and isomorphism; our motivation here is simply that finite subcomplexes are much easier to work with than infinitely triangulated open subsets. The gluing construction and the proof of (1.2), except for a technical lemma proven in §5, are given in §4. In §6 we examine uniqueness of the construction. The orientable case is treated in §7.

Part II is devoted to representability. Representation of multiplication on functors of the form $[, B]$ is considered in sections 8 and 9. Theorem (1.1) is proved in §10. The final section, §11, establishes a weak-homotopy theorem in singular homology. As a consequence, we obtain (11.3) which states that the Hopf algebras $H_*(B_{PL}; k)$ and $H^*(B_{PL}; k)$, k a field, are independent (up to isomorphism) of the choice of classifying H -space. We also show, in (11.4), that the natural transformation $\tilde{K}O \rightarrow k_{PL}$ of Milnor induces unique Hopf algebra homomorphisms of the homology and cohomology of their classifying spaces.

PART I: GLUING PL -MICROBUNDLES TOGETHER

2. PL -maps. By a complex K we mean a simplicial complex K contained in euclidean space R^n such that each point of the underlying space $|K|$ has a neighborhood intersecting only finitely many simplices of K . Thus the usual topology of $|K|$ is the same as the subspace topology. A subdivision K' of K will be assumed to be simplicial unless otherwise stated.

Let K and L be complexes. Recall that a map $f: K \rightarrow R^n$ is said to be *linear* if f maps each simplex of K linearly into R^n . A map $f: K \rightarrow L$ is linear if it maps each simplex of K linearly into a simplex of L . More generally, $f: K \rightarrow L$ is *piecewise linear*, or f is a *PL -map*, if there exists a subdivision K' of K such that the induced map $f: K' \rightarrow L$ is linear. A *PL -homeomorphism* is a PL -map which is also a homeomorphism.

Many of the following results are well-known.

(2.1) *The collection of complexes and PL-maps forms a category.*

(2.2) *Every open subset of a complex can be triangulated such that inclusion into the complex is a PL-map.*

(2.3) *Let K and L be complexes and let $f: K \rightarrow R^n$ be a linear map. If $f(|K|) \subset |L|$, then $f: K \rightarrow L$ is a PL-map.*

Proof. The sets $\sigma \cap f^{-1}\tau$ form a cell subdivision of K as σ ranges over K and τ ranges over L . If K' is a simplicial subdivision of this cell complex, then $f: K' \rightarrow L$ is linear.

(2.4) Corollary. *If L and L_1 are complexes with $|L_1| \subset |L|$, then the inclusion map $i: L_1 \rightarrow L$ is piecewise linear.*

From the above results we obtain the following.

(2.5) Corollary. *Let $f: K \rightarrow L$ be a PL-map, and let K_1 and L_1 be complexes with $|K_1| \subset |K|$ and $|L_1| \subset |L|$. If $f(|K_1|) \subset |L_1|$, then the induced map $f_1: K_1 \rightarrow L_1$ is piecewise linear.*

(2.6) Corollary. *If the map $f: X \rightarrow Y$ is piecewise linear for specific triangulations of spaces X and Y , then it is also piecewise linear for any other triangulations.*

(2.7) Proposition. *Let $f: K \rightarrow L$ be a PL-map with K a finite complex. Then there exist subdivisions K' of K and L' of L such that the induced map $f: K' \rightarrow L'$ is simplicial.*

This follows from Lemma 9 of [5]. Note that finiteness of K is essential. (For example, let L be a closed interval of reals, let K be its interior, and let f be the inclusion map.) However, we do have the following easily proved generalization for PL-homeomorphisms.

(2.8) *If $f: K \rightarrow L$ is a PL-homeomorphism, then there exist subdivisions K' of K and L' of L such that the induced map $f: K' \rightarrow L'$ is simplicial. In particular, the inverse of a PL-homeomorphism is again a PL-homeomorphism.*

Let K_1 and K_2 be complexes and let A be a full finite subcomplex of K_1 . If $h: A \rightarrow K_2$ is an injective simplicial map, the identification space $K = K_1 \cup_h K_2$ obtained from the disjoint union $K_1 + K_2$ by identifying each $a \in A$ with its image $h(a) \in K_2$ is a complex containing K_1 and K_2 as subcomplexes. Let L_1, L_2, B , and $k: B \rightarrow L_2$ be analogous, and form $L = L_1 \cup_k L_2$. Using (2.7) one easily proves the following result.

(2.9) *Let $f_1: (K_1, A) \rightarrow (L_1, B)$ and $f_2: K_2 \rightarrow L_2$ be PL-maps such that $f_2 h = k(f_1|_A)$. Then the induced map $f: K \rightarrow L$ on the quotient complexes is piecewise linear.*

3. Equivalent conditions for local triviality and isomorphism. Let X and E be complexes and let $i : X \rightarrow E$ and $p : E \rightarrow X$ be PL -maps with $pi : X \rightarrow X$ the identity map 1_X . Recall that the set $\xi = \{X, E, i, p\}$ is *locally n -trivial* at $x \in X$ if there exist open neighborhoods X_0 of x , E_0 of $i(x)$, and a PL -homeomorphism $k : E_0 \rightarrow X_0 \times \mathbb{R}^n$ such that the following diagram is commutative.

$$(3.1) \quad \begin{array}{ccccc} & & E_0 & & \\ & i|_{X_0} \nearrow & \downarrow k & \nwarrow p|_{E_0} & \\ X_0 & & & & X_0 \\ & \iota \searrow & & \nearrow \pi_1 & \\ & & X_0 \times \mathbb{R}^n & & \end{array}$$

Here, π_1 is projection onto the first factor and $\iota(x) = (x, 0)$, $x \in X_0$. We assume that X_0 and E_0 are triangulated in accordance with (2.2); because of (2.6), the choice of triangulation is immaterial. If this condition holds for every $x \in X$, then ξ is a PL -microbundle of dimension n .

If K and H are finite complexes, it will be convenient to assume that their product space $K \times L$ is triangulated so that the two projection maps are simplicial.

The following notation will be needed. J^n consists of those points $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that $|x_j| \leq 1$ for each j . Triangulate J^n so that 0 is a vertex. If X is a complex, the closed and open stars of $x \in X$ are respectively denoted by $St(x, X)$ and $st(x, X)$.

(3.2) Proposition. $\{X, E, i, p\}$ is locally n -trivial at $x \in X$ if and only if the following condition holds:

there exist i) a subdivision X' of X with x as a vertex,

ii) a subdivision E' of E and a subcomplex B' of E' contained in $p^{-1}(|St(x, X')|)$, and

iii) a PL -homeomorphism $k : B' \rightarrow St(x, X') \times J^n$

such that $k^{-1}(|St(x, X')| \times \text{Int } J^n)$ is an open subset of $p^{-1}(|St(x, X')|)$ and the diagram

$$(3.3) \quad \begin{array}{ccccc} & & B' & & \\ & i|_{St(x, X')} \nearrow & \downarrow k & \nwarrow p|_{B'} & \\ St(x, X') & & & & St(x, X') \\ & \iota \searrow & & \nearrow \pi_1 & \\ & & St(x, X') \times J^n & & \end{array}$$

is commutative.

Proof. Suppose the condition holds. Let $X_0 = |st(x, X')|$ and $E_0 = k^{-1}(X_0 \times$

$\text{Int } J^n$). Since there is a PL -homeomorphism of $\text{Int } J^n$ with R^n , local n -triviality at x follows from (2.5) and diagram (3.3).

Conversely, suppose we have local n -triviality at x . Using the notation of diagram (3.1), let X' be a subdivision of X such that i) holds and $|St(x, X')| \subset X_0$. Then $B = k^{-1}(|St(x, X') \times J^n|)$ is a finitely triangulable space. By (2.4) and (2.7) there exists a subdivision E' of E such that B becomes a subcomplex B' . Since the restriction $k: B' \rightarrow St(x, X') \times J^n$ of the PL -homeomorphism $k: E_0 \rightarrow X_0 \times R^n$ is a PL -homeomorphism by (2.5), the proof is complete.

Let $\xi_i = \{X, E_i, i_i, p_i\}$ be an n -dimensional PL -microbundle over X , $j = 1, 2$. Recall that ξ_1 and ξ_2 are *isomorphic* if there exist open neighborhoods U_i of $i_i(X)$ and a PL -homeomorphism $h: U_1 \rightarrow U_2$ such that the diagram

$$(3.4) \quad \begin{array}{ccccc} & & U_1 & & \\ & \nearrow i_1 & \downarrow h & \nwarrow p_1|_{U_1} & \\ X & & & & X \\ & \searrow i_2 & & \nearrow p_2|_{U_2} & \\ & & U_2 & & \end{array}$$

is commutative. We write $h: \xi_1 \cong \xi_2$.

(3.5) Proposition. *Suppose the base space X is a finite complex. Then ξ_1 and ξ_2 are isomorphic if and only if for $j = 1, 2$ there exist a subdivision E'_j of E_j , a full finite subcomplex K_j of E'_j containing an open (in E_j) neighborhood of $i_j(X)$, and a simplicial isomorphism $h: K_1 \rightarrow K_2$ such that the following diagram is commutative.*

$$(3.6) \quad \begin{array}{ccccc} & & K_1 & & \\ & \nearrow i_1 & \downarrow h & \nwarrow p_1|_{K_1} & \\ X & & & & X \\ & \searrow i_2 & & \nearrow p_2|_{K_2} & \\ & & K_2 & & \end{array}$$

Proof. Suppose the condition holds. By hypothesis there are open neighborhoods V_i of $i_i(X)$ in E_i with $V_i \subset |K_i|$, $j = 1, 2$. Put $U_1 = V_1 \cap h^{-1}(V_2)$ and $U_2 = V_2 \cap h(V_1)$. Since $h(U_1) = U_2$, (2.5) and diagram (3.6) imply that $\xi_1 \cong \xi_2$.

Conversely, suppose that $\xi_1 \cong \xi_2$. Using the notation of (3.4) we may, by (2.8), assume that U_1 and U_2 are triangulated so that h is simplicial. By local compactness of E_1 and compactness of $i_1(X)$ there exists an open neighborhood V of $i_1(X)$ in E_1 such that $\bar{V}$ is compact and is contained in U_1 . Hence there

exists a finite subcomplex K_1 of U_1 containing $\bar{V}$. Setting $K_2 = h(K_1)$, the restriction $h : K_1 \rightarrow K_2$ is then a simplicial isomorphism and diagram (3.6) is commutative. Since inclusion $K_i \subset E_i$ is a PL -map (2.7) allows us to subdivide E_i in the desired manner.

4. The gluing construction. Let $X = X_1 \cup X_2$ be a complex with $A = X_1 \cap X_2$ a finite subcomplex. Let $\xi_j = \{X_j, E_j, i_j, p_j\}$ be an n -dimensional PL -microbundle, $j = 1, 2$, and let $h : \xi_1 \mid A \cong \xi_2 \mid A$. Using (3.5) we thus have a finite full subcomplex K_j of E_j containing an open neighborhood of $i_j(A)$ in $p_j^{-1}(A)$ and a simplicial isomorphism $h : K_1 \rightarrow K_2$ such that diagram (3.6) (with X replaced by A) is commutative.

We can now describe the PL -microbundle $\xi(h, K_1, K_2)$ obtained by *gluing* ξ_1 and ξ_2 together by h . The total space E is the complex $E_1 \cup_h E_2$. Let $q : E_1 + E_2 \rightarrow E$ be the identification map and define $i : X \rightarrow E$ by $i(x) = qi(x)$ if $x \in X_j$. Then i is a well-defined PL -map by (2.9). The PL -map $p : E \rightarrow X$ is obtained by factoring the map $P : E_1 + E_2 \rightarrow X$, defined by $P \mid E_j = p_j$, $j = 1, 2$, through q . Clearly $pi = 1_X$ and $\xi(h, K_1, K_2) \mid X_j \cong \xi_j$, $j = 1, 2$. Therefore Theorem (1.2) will be proved once we establish local triviality. Because of (3.2) it suffices to prove the following.

(4.1) Proposition. $\xi(h, K_1, K_2)$ satisfies the condition of (3.2) for each $x \in X$.

Remark. $h : \xi_1 \mid A \cong \xi_2 \mid A$ is initially given as a PL -homeomorphism $h : U_1 \rightarrow U_2$, as in (3.4), while $h : K_1 \rightarrow K_2$ is a restriction. It is straightforward to prove that the isomorphism class of $\xi(h, K_1, K_2)$ is actually independent of the choice of K_1 and K_2 .

The proof of (4.1) is based on two technical lemmas, the first of which allows us to shrink local product neighborhoods in the fiber direction.

(4.2) Lemma. Let $\{X, E, i, p\}$ satisfy the condition of (3.2) at $x \in X$. Let A be a finite subcomplex of X containing x . Let K be a subcomplex of E contained in $p^{-1}(A)$ such that $|K|$ contains an open neighborhood U of $i(|A|)$ in $p^{-1}(|A|)$. Then there exist a subdivision E'' of E' , a subcomplex B_1 of the induced subdivision B'' of B' , and a PL -homeomorphism $k_1 : B_1 \rightarrow St(x, X') \times J^n$ with $|B_1| \cap p^{-1}(|A|) \subset |K|$ such that the analogue of diagram (3.3) is commutative.

Proof. Let C_ϵ^n , $0 < \epsilon < 1$, denote the closed n -cube with center 0 and edge length 2ϵ . Now $|St(x, A')| \times 0 \subset k(U \cap |B|)$ and $k(U \cap |B|)$ is an open subset of $|St(x, A') \times J^n|$. Choose ϵ such that $|St(x, A')| \times C_\epsilon^n \subset k(U \cap |B|)$. Setting $|B_1| = k^{-1}(|St(x, X')| \times C_\epsilon^n)$, we have that $|B_1| \cap p^{-1}(|A|) \subset |K|$. Finally, let $g : C_\epsilon^n \rightarrow J^n$ be the obvious homeomorphism and put $k_1 = (1 \times g)(k \mid B_1) : B_1 \rightarrow St(x, X') \times J^n$. We leave it to the reader to construct the required subdivisions.

Our other preliminary result is a local extension lemma. Let M be a finite star complex, that is, $M = St(x, M)$ for some vertex $x \in M$. Let $Lk(x, M) =$

$St(x, M) - st(x, M)$ be the link of x in M . Suppose N is a subcomplex of M containing x such that if $\tau \in Lk(x, M) \cap N$, then the join $x * \tau$ of x with τ is in N . Thus N is contractible to x .

(4.3) Lemma. *Let $f : N \times J^n \rightarrow M \times J^n$ be an injective PL-map such that the diagram*

$$\begin{array}{ccc} & N \times J^n & \\ \iota \nearrow & \downarrow f & \searrow \pi_1 \\ N & & M \\ \iota \searrow & & \nearrow \pi_1 \\ & M \times J^n & \end{array}$$

is commutative. Then there exist a subdivision M' of M and an injective PL-map $\bar{f} : St(x, M') \times J^n \rightarrow M \times J^n$ such that the restrictions of $\bar{f}$ and f to $(St(x, M') \cap N) \times J^n$ agree and the diagram

$$(4.4) \quad \begin{array}{ccc} & St(x, M') \times J^n & \\ \iota \nearrow & \downarrow \bar{f} & \searrow \pi_1 \\ St(x, M') & & M \\ \iota \searrow & & \nearrow \pi_1 \\ & St(x, M') \times J^n & \end{array}$$

is commutative.

The proof of this result will be given in the next section.

Proof of (4.1). We can assume that $x \in Bd X_1 \cap Bd X_2$ since otherwise the result follows from hypothesis. Using (4.2) we can assume, without loss of generality, the following:

- i) x is a vertex of X ;
- ii) for $j=1, 2$, there exists subcomplex B_j of E_j such that $|B_j| \cap p_j^{-1}(|A|) \subset |K_j|$ and $h(|B_1| \cap p_1^{-1}(|A|)) \subset |B_2|$;
- iii) for $j=1, 2$, there exists a PL-homeomorphism $k_j : B_j \rightarrow St(x, X_j) \times J^n$ such that $k_j^{-1}(|St(x, X_j)| \times \text{Int } J^n)$ is open in $p_j^{-1}(|St(x, X_j)|)$ and the corresponding diagram (3.3) is commutative; and
- iv) if $\tau \in Lk(x, X) \cap A$, then $x * \tau \in A$.

Let f be the injective PL-map defined by the composition

$$St(x, A) \times J^n \xrightarrow{k_1^{-1}} B_1 \cap p_1^{-1}A \xrightarrow{h} B_2 \xrightarrow{k_2} St(x, X_2) \times J^n.$$

By (4.3) there exists a subdivision X' of X and an injective PL-map $\bar{f} : St(x, X'_2) \times J^n \rightarrow St(x, X'_2) \times J^n$ extending $f|_{St(x, A') \times J^n}$ such that the corresponding diagram (4.4) is commutative.

Set $|B_3| = k_1^{-1}(|St(x, X'_1) \times J^n|) \subset |B_1|$ and $|B_4| = k_2^{-1}f(|St(x, X'_2) \times J^n|) \subset |B_2|$, and let E'_j be a subdivision of E_j such that $p_i^{-1}(|A|)$ and $|B_{j+2}|$ become subcomplexes, $j = 1, 2$. By (2.8) we may assume that $h : K'_1 \rightarrow K'_2$ is simplicial.

Let $k_3 : B_3 \rightarrow St(x, X'_1) \times J^n$ be the restriction of k_1 and define $k_4 : B_4 \rightarrow St(x, X'_2) \times J^n$ by $k_4(b) = \tilde{f}^{-1}k_2(b)$, $b \in B_4$. Both are *PL*-homeomorphisms and the analogues of diagrams (3.3) are commutative because of (4.4).

Now $h(B_3 \cap p_1^{-1}A') = k_2^{-1}k_2hk_1^{-1}(St(x, A') \times J^n) = k_2^{-1}f(St(x, A') \times J^n) = B_4 \cap p_2^{-1}A'$ and the diagram

$$\begin{array}{ccc}
 B_3 \cap p_1^{-1}A' & & \\
 \downarrow h & \searrow k_3 & \\
 B_4 & \xrightarrow{k_4} & St(x, X'_2) \times J^n
 \end{array}$$

is commutative. Then $B = B_3 \cup B_4$ is a subcomplex of the subdivision $E' = E'_1 \cup E'_2$ of E and the map $k : B \rightarrow St(x, X') \times J^n$ induced by k_3 and k_4 is a *PL*-homeomorphism by (2.9). Since $k^{-1}(|St(x, X')| \times \text{Int } J^n)$ is open in $p^{-1}(|St(x, X')|)$, the proof is complete.

5. Proof of Lemma (4.3). The map $f : N \times J^n \rightarrow M \times J^n$ is piecewise linear; here, $M \times J^n$ is assumed to be a specific triangulation of $|M \times J^n|$. There exists a subdivision M' of M and another triangulation K of $|M \times J^n|$ such that $\pi_1 : K \rightarrow M'$ is simplicial and $f : L \rightarrow M \times J^n$ is linear, where L is the induced triangulation of $|N \times J^n|$. Put $M_1 = St(x, M')$, $N_1 = N' \cap M_1$, $K_1 = \pi_1^{-1}(M_1) \subset K$, and $L_1 = \pi_1^{-1}(N_1) \subset K_1$. Then $\pi_1 : K_1 \rightarrow M_1$ is simplicial and $f : L_1 \rightarrow M \times J^n$ is linear.

We next construct a special subdivision M'_1 of M_1 containing N_1 as a subcomplex. First we subdivide $Lk(x, M_1)$, to get $Lk'(x, M_1)$, as follows. Let $\zeta \in Lk(x, M_1)$. If $\zeta \in N_1$, ζ is not subdivided. If $\zeta \cap N_1 = \emptyset$, subdivide ζ barycentrically. If $\zeta \cap N_1 = \tau \neq \zeta$, let b be the barycenter of ζ and subdivide ζ by taking as simplexes the joins $b * \tau$ and $b * \rho$ where ρ is a simplex of the barycentric subdivision of $Bd \zeta - \text{Int } \tau$. By taking the join of x with the simplexes of $Lk'(x, M_1)$ we obtain M'_1 .

Let K'_1 be a subdivision of K_1 containing L_1 as a subcomplex such that the map $\pi_1 : K'_1 \rightarrow M'_1$ is simplicial.

Consider f as the linear map $f : L_1 \rightarrow M \times J^n \subset R^n$. Define $\tilde{f} : K'_1 \rightarrow R^n$ as the linear extension of the following map of the vertex set of K'_1 : Let $v \in K'_1$ be a vertex. If $v \in L_1$, let $\tilde{f}(v) = f(v)$. If $v \notin L_1$, let $\tilde{f}(v) = (\pi_1(v), \pi_2 f(\pi_2(v))) \in |M \times J^n|$. Clearly $\tilde{f}|_{L_1} = f$, $\tilde{f}(|K'_1|) \subset |M \times J^n|$, and diagram (4.4) is commutative (since it is commutative for the vertices of K'_1). Once we show that $\tilde{f}$ is injective, it will follow from §2 that $\tilde{f}$ is piecewise linear since $|\tilde{f}(K'_1)|$ is triangulable and $|\tilde{f}(K'_1)| \subset |M \times J^n|$.

To show injectivity of $\tilde{f}$, it suffices (by (4.4)) to show that if $\sigma = x * \rho \in M'_1$,

where $\rho \in Lk'(x, M_1)$, then $\bar{f} : \sigma \times J^n \rightarrow R^n$ is injective. If $\rho \cap N_1 = \phi$, this is clear since $\bar{f} : y \times J^n \rightarrow R^n$, $y \in \sigma$, is essentially the injective map $\bar{f} : x \times J^n \rightarrow R^n$. Injectivity is also clear if $\rho \in N_1$. The final case is where $\rho \cap N_1 = \tau$, a proper face of ρ . Thus $\rho = \tau * b_1 * \cdots * b_k$ where b_i is a barycenter of a simplex of $Lk(x, M_1)$. Since $\bar{f}|_{(x * \tau) \times J^n}$ is injective by hypothesis, an induction on k allows us to assume that $k = 1$. But if $y \in x * b$, then $\bar{f}|_{(y * \tau) \times J^n}$ is essentially $\bar{f}|_{(x * \tau) \times J^n}$, by definition of $\bar{f}$, and hence $\bar{f} : (y * \tau) \times J^n \rightarrow R^n$ is injective and the proof is complete.

6. Uniqueness of the gluing construction. Let ξ and ξ' be PL -microbundles over X with total spaces E and E' respectively, and let $f_j : \xi \cong \xi'$, $j = 0, 1$. Thus we have open sets $U_i \subset E$, $U'_i \subset E'$, and a PL -homeomorphism $f_i : U_i \rightarrow U'_i$ so that diagram (3.4) is commutative. Recall [5] that f_0 and f_1 define the same (PL) -isomorphism germ F from ξ to ξ' if they coincide on some sufficiently small open neighborhood of $i(X)$ in E . $F : \xi \rightarrow \xi'$ is then an equivalence class of such isomorphisms $\xi \cong \xi'$.

Let $\xi \times I$ be the PL -microbundle $\{X \times I, E \times I, i \times 1, p \times 1\}$. We say that isomorphism germs $F_j : \xi \rightarrow \xi'$ are (PL) -germ isotopic if there exists an isomorphism germ $F : \xi \times I \rightarrow \xi' \times I$ such that the diagrams

$$\begin{array}{ccc} \xi \times I & \xrightarrow{F} & \xi' \times I \\ i_j \uparrow & & \uparrow i_j \\ \xi & \xrightarrow{F_j} & \xi' \end{array} \quad (j = 0, 1)$$

are commutative. Here $i_j : \xi \rightarrow \xi \times I$ denotes the injection of ξ onto $\xi \times \{j\}$. Germ isotopy is clearly an equivalence relation.

(6.1) Proposition. *The isomorphism class of $\xi(h, K_1, K_2)$ depends only on the germ isotopy class of the isomorphism germ of h .*

Proof. By the remark following (4.1), the isomorphism class of $\xi(h) = \xi(h, K_1, K_2)$ depends only on the isomorphism germ of $h : \xi_1|A \cong \xi_2|A$.

Suppose that the isomorphism germ of h is germ isotopic to the isomorphism germ of $h' : \xi_1|A \cong \xi_2|A$. Let $H : (\xi_1 \times I)|A \times I \cong (\xi_2 \times I)|A \times I$ be a representative of the germ isotopy and form $\xi(H)$ over $X \times I$. By definition of germ isotopy and the above paragraph, we have that $\xi(H)|X \times 0 \cong \xi(h)$ and $\xi(H)|X \times 1 \cong \xi(h')$. Moreover, $1 : X \times I \rightarrow X \times I$ is a homotopy $i_0 \cong i_1$; the homotopy theorem ([5], Thm. 1) implies

$$\xi(H)|X \times 0 = i_0^* \xi(H) \cong i_1^* \xi(H) = \xi(H)|X \times 1$$

and completes the proof.

7. Orientable PL-microbundles. A PL -microbundle $\xi = \{X, E, i, p\}$ is said to be *orientable* if there exists a cover of $i(X)$ by local product neighborhoods

$\{(X_{0\alpha}, k_\alpha, E_{0\alpha})\}$, as in (3.1), such that all transition functions $g_{\alpha\beta x} : (U_{\alpha\beta x}, 0) \rightarrow (R^n, 0)$ defined by

$$g_{\alpha\beta x}(y) = k_\beta k_\alpha^{-1}(x, y), x \in X_{0\alpha} \cap X_{0\beta}, y \in U_{\alpha\beta x} = \pi_2 k_\alpha(E_{0\alpha} \cap E_{0\beta} \cap p^{-1}x) \subset R^n,$$

are orientation preserving. A maximal such cover defines an *orientation on ξ* .

It is straightforward to verify that if in the gluing construction ξ_1 and ξ_2 are oriented and $h : \xi_1|_A \cong \xi_2|_A$ is orientation preserving (with obvious definition), then $\xi(h)$ is oriented in such a way that the isomorphisms $\xi(h)|_{X_i} \cong \xi_i$ are orientation preserving, $j = 1, 2$.

We will need the following facts later.

(7.1) Lemma. *If ξ_1 and ξ_2 are oriented PL-microbundles over X , there is an induced orientation on their Whitney sum $\xi_1 \oplus \xi_2$.*

(7.2) Lemma. *If ξ is a non-orientable PL-microbundle, then $\xi \oplus \xi$ is orientable.*

(7.3) Lemma. *Let ξ be an oriented PL-microbundle over X and let ξ' be ξ with the reverse orientation. Let $0^1 = \{X, X \times R, \iota, \pi_1\}$ be the trivial oriented 1-dimensional microbundle over X . Then there exists an orientation preserving isomorphism $\xi \oplus 0^1 \cong \xi' \oplus 0^1$.*

The proofs of these facts are straightforward.

Let $X = X_1 \vee X_2$, i.e., $A = X_1 \cap X_2$ is a point, and let ξ_j be an oriented PL-microbundle of dimension n over X_j , $j = 1, 2$. Since $\xi_j|_A$ is the trivial microbundle 0^n , there exists an orientation preserving isomorphism $h : \xi_1|_A \cong \xi_2|_A$. By ([3], Thm. 3), any other choice of h belongs to the germ isotopy class of h . Applying (6.1) we conclude that the isomorphism class of $\xi_1 \vee \xi_2 = \xi(h, K_1, K_2)$ is independent of the choice of h . Since

$$(\xi_1 \oplus 0^1) \vee (\xi_2 \oplus 0^1) \cong (\xi_1 \vee \xi_2) \oplus 0^1$$

we have the following corollary of (7.3).

(7.4) Corollary. *Let $X = X_1 \vee X_2$ and let ξ_j be an oriented n -dimensional PL-microbundle over X_j , $j = 1, 2$. Then*

$$(\xi_1 \vee \xi_2) \oplus 0^1 \cong (\xi_1 \vee \xi_2') \oplus 0^1.$$

PART II: REPRESENTABILITY

We shall assume throughout that all spaces under consideration are connected and have base points, and that all maps and homotopies preserve these base points. In many places, however, analogous results hold without these restrictions.

8. Representing multiplication on contravariant functors. Recall that $\mathcal{W}$ denotes the category of finite CW complexes. Let $\mathcal{S}$ be the category of based sets and functions which preserve the distinguished element. Let $\mathcal{M}$ be the

category of sets with multiplication. Precisely, the objects of $\mathfrak{M}$ are triples (S, ν_S, s_0) where $(S, s_0) \in \mathfrak{S}$ and $\nu_S : (S \times S, (s_0, s_0)) \rightarrow (S, s_0)$ is a function, called a *multiplication* on S . The morphisms of $\mathfrak{M}$ are those functions $f : (S, s_0) \rightarrow (T, t_0)$ such that $f\nu_S = \nu_T(f \times f)$. Let $\mathfrak{M}_1$ [respectively, $\mathfrak{M}_2$; $\mathfrak{M}_3$] be the full subcategory consisting of those objects whose multiplication has the base point as a two-sided identity [respectively, is associative; is commutative].

(8.1) Theorem. *Let B be a countable CW complex and suppose that the functor $[_, B] : \mathfrak{W} \rightarrow \mathfrak{S}$ lifts to a functor $F : \mathfrak{W} \rightarrow \mathfrak{M}$. Then the multiplication on F is represented by a multiplication μ on B ; μ is unique only up to weak homotopy. Moreover, if $F(\mathfrak{W}) \subset \mathfrak{M}_1$, then μ can be taken to be an H -structure. If $F(\mathfrak{W}) \subset \mathfrak{M}_2$ [respectively, $\mathfrak{M}_3$], μ is weakly homotopy associative [respectively, weakly homotopy commutative].*

This result is very similar to standard results due to Eckmann and Hilton, the basic difference being that here the classifying space B does not belong to the domain category, but is a limit of objects of it. Thus (8.1) will be proved by using limiting techniques together with standard methods. Details will be given in the next section; there we will also establish the following related result.

(8.2) Proposition. *Let B_j be a space with multiplication μ_j , $j = 1, 2$, and suppose B_1 is a CW complex. If $f : B_1 \rightarrow B_2$ is a map such that $f_* : [_, B_1] \rightarrow [_, B_2]$ is a natural transformation of functors from $\mathfrak{W}$ to $\mathfrak{M}$, then f is a weak homomorphism.*

9. Proofs of (8.1) and (8.2). The proof of (8.1) is based on three lemmas.

(9.1) Lemma. *Let B be a CW complex. If ν is a weak H -structure on B , there exists an H -structure μ on B weakly homotopic to ν .*

Proof. Let $i_k : B \rightarrow B \times B$ denote the canonical injection, $k = 1, 2$. Then $(\nu i_k)_*$ is the identity map on each homotopy group $\pi_i(B)$. Since B is a CW complex, νi_k is a homotopy self-equivalence of B ; let $g_k : B \rightarrow B$ be a homotopy inverse.

Define $\mu : B \times B \rightarrow B$ by $\mu = \nu(g_1 \times g_2)$. Then

$$\mu i_k = \nu(g_1 \times g_2) i_k = \nu i_k g_k \simeq 1_B, \quad k = 1, 2,$$

and μ is an H -structure on B .

It remains to show that μ and ν are weakly homotopic. For this it suffices to show that if A is a finite subcomplex of B (containing the base point), then $\mu_A \simeq \nu_A$ where $\mu_A = \mu|_{A \times A}$. Let $g_{kA} = g_k|_A$ and let $j : A \subset B$ be the inclusion map. Then

$$g_{kA} = g_k 1_B j \simeq g_k \nu i_k j \simeq j, \quad k = 1, 2.$$

Hence

$$\mu_A = \nu(g_1 \times g_2)(j \times j) = \nu(g_{1A} \times g_{2A}) \simeq \nu(j \times j) = \nu_A$$

and the proof is complete.

(9.2) Lemma. *Weakly homotopic multiplications on a space B induce the same multiplication on the functor $[_, B] : \mathfrak{W} \rightarrow \mathfrak{S}$.*

The proof is immediate from the definitions.

Let K be a CW complex and let $\mathfrak{C}$ be the directed set of finite subcomplexes K_α of K where $K_\beta \leq K_\alpha$ if $K_\beta \subset K_\alpha$. If B is any space, the inclusion map $i_{\beta\alpha}$ induces the function $i_{\beta\alpha}^* : [K_\alpha, B] \rightarrow [K_\beta, B]$. Thus $\{[K_\alpha, B], i_{\beta\alpha}^*\}$ is an inverse system over $\mathfrak{C}$. Define $\eta : [K, B] \rightarrow \text{inv lim}_{\mathfrak{C}} \{[K_\alpha, B]\}$ by $\eta[f] = \{[f|_{K_\alpha}]\}$. By using the homotopy extension theorem on a cofinal subsystem $K_1 \subset K_2 \subset \dots$ of $\mathfrak{C}$, the following well-known result can easily be shown.

(9.3) Lemma. *If K is countable, then η is surjective.*

(9.4) Corollary. *Let B be a countable CW complex, let $K = B \times B$, and let $\mathfrak{D}$ be the cofinal subsystem of $\mathfrak{C}$ consisting of subcomplexes of the form $A \times A$, where A is a finite subcomplex of B . Then $\eta : [B \times B, B] \rightarrow \text{inv lim}_{\mathfrak{D}} \{[A \times A, B]\}$ is surjective.*

Proof of (8.1). We denote the product of elements $[f]$ and $[g]$ in $F(X) = [X, B]$ by $[f] + [g]$, $X \in \mathfrak{W}$. If $A \times A \in \mathfrak{D}$, let $p_{iA} : A \times A \rightarrow B$ denote the composition of the projection onto the i^{th} factor A followed by the inclusion map $j : A \subset B$, $i = 1, 2$. Thus there are maps $\mu_A : A \times A \rightarrow B$ such that

$$(9.5) \quad [\mu_A] = [p_{1A}] + [p_{2A}].$$

We assert that $\{[\mu_A]\} \in \text{inv lim}_{\mathfrak{D}} \{[A \times A, B]\}$. Indeed, if $i : A \times A \subset C \times C$ is the inclusion map, $C \times C \in \mathfrak{D}$, then

$$i^*[\mu_C] = i^*[p_{1C}] + i^*[p_{2C}] = [p_{1A}] + [p_{2A}] = [\mu_A].$$

By (9.4), there exists a map $\mu : B \times B \rightarrow B$ such that $\mu|_{A \times A} \simeq \mu_A$ for each finite subcomplex A of B .

We show that μ represents the given multiplication on each $F(X)$, $X \in \mathfrak{W}$. Let $[f], [g] \in [X, B]$. Let A be a finite subcomplex of B containing the compact set $f(X) \cup g(X)$, and regard f and g as maps from X to A as well. Since $(f, g)^* : E[A \times A, B] \rightarrow [X, B]$ preserves multiplication it follows that

$$[\mu(f, g)] = [\mu_A(f, g)] = (f, g)^*[\mu_A] = (f, g)^*[p_{1A}] + (f, g)^*[p_{2A}] = [f] + [g]$$

as we were to show.

Suppose ν were another multiplication on B inducing the given multiplication on F . If A is a finite subcomplex of B , then, by definition,

$$[p_{1A}] + [p_{2A}] = [\nu(p_{1A}, p_{2A})] = [\nu(j \times j)] = [\nu_A].$$

It follows from (9.5) that $\nu_A \simeq \mu_A$. Thus μ is unique up to weak homotopy.

Suppose $F(\mathfrak{W}) \subset \mathfrak{M}_1$. By modifying standard techniques as we did above, one shows that μ is a weak H -structure. Because of (9.1) and (9.2), μ may be assumed to be an H -structure. The remaining cases in which $F(\mathfrak{W})$ is contained in $\mathfrak{M}_2$ or $\mathfrak{M}_3$ are treated similarly.

Proof of (8.2). It suffices to show that if A is a finite subcomplex of B , then $\mu_2(f_A \times f_A) \simeq f_{\mu_{1A}} : A \times A \rightarrow B_2$, where $f_A = f|_A$. Since $f_* : [A \times A, B_1] \rightarrow [A \times A, B_2]$ is a homomorphism, we have, by equation (9.5),

$$[f_{\mu_{1A}}] = f_*[p_{1A}] + f_*[p_{2A}] = [\mu_2(fp_{1A}, fp_{2A})] = [\mu_2(f_A \times f_A)]$$

and the proof is complete.

10. Proof of (1.1). We use the geometric results of Part I to verify the axioms of Brown [1]; this establishes existence of the classifying space. Then (8.1) applies to give the representing H -structure.

(10.1) Lemma. *Let F be either k_{PL} or k_{PL0} . Then F satisfies the following axioms of Brown:*

Axiom h: $f \simeq g$ implies $F(f) = F(g)$.

Axiom c: The sets $F(S^n)$ are countable for $n > 0$; S^n denotes the n -sphere.

Axiom (e1): $F(\text{point space})$ consists of a single element.

Proof. Axiom h follows from the homotopy theorem [5, p. 5]. Axiom c follows from the fact [5, p. 28] that there are only countably many isomorphism classes of PL -microbundles over a finite complex. Axiom (e1) is obvious.

(10.2) Lemma. *$F = k_{PL0}$ satisfies axiom (e2) of Brown: Let $X = X_1 \cup X_2$, $A = X_1 \cap X_2$, where X, X_1, X_2 , and A are in $\mathbb{W}$ and all have the same base point. Let $i_k : X_k \subset X$ and $j_k : A \subset X_k$ be the inclusion maps, $k = 1, 2$. If $u_k \in F(X_k)$ are such that $F(j_1)u_1 = F(j_2)u_2$, then there exists $v \in F(X)$ such that $F(i_k)v = u_k$, $k = 1, 2$. Furthermore, if A is a point, then v is unique.*

Proof. Suppose $\{\xi_k\} \in k_{PL0}(X_k)$ and $\{\xi_1|_A\} = \{\xi_2|_A\}$ in $k_{PL0}(A)$. Since the class $\{\xi_k\}$ has $\xi_k \oplus 0^r$ as representative for any $r \geq 0$, we can assume that ξ_1 and ξ_2 have the same dimension and that $\xi_1|_A \cong \xi_2|_A$. By our gluing construction, there exists an orientable PL -microbundle ξ over X such that $\xi|_{X_k} \cong \xi_k$, $k = 1, 2$. This proves the first part.

Now suppose that A is a point. Choosing orientations for ξ_1 and ξ_2 , construct the oriented PL -microbundle $\xi_1 \vee \xi_2$ over $X = X_1 \vee X_2$ such that $\xi_1 \vee \xi_2|_{X_k} \cong \xi_k$ by an orientation preserving isomorphism, $k = 1, 2$. We must show that if ξ is any other orientable PL -microbundle over X for which there are isomorphisms $f_k : \xi|_{X_k} \cong \xi_k$, $k = 1, 2$, then $\{\xi\} = \{\xi_1 \vee \xi_2\}$. Choose the orientation of ξ so that f_1 is orientation preserving. Then $\xi \cong \xi_1 \vee \xi_2$ or $\xi_1 \vee \xi'_2$ depending on whether f_2 is orientation preserving or reversing. (ξ'_2 is ξ_2 with the reverse orientation.) In the latter case, $\xi \oplus 0^1 \cong (\xi_1 \vee \xi_2) \oplus 0^1$ by (7.3). Thus $\{\xi\} = \{\xi_1 \vee \xi_2\}$ in either case.

The k_{PL} analogue of (10.2) is true, but the uniqueness requirement when A is a point seems difficult to establish directly. We avoid this by first proving (1.1) in the orientable case, from which we conclude that each $k_{PL0}(X)$ is an abelian group. Using (7.2), we deduce the same for $k_{PL}(X)$, a fact also known

directly from Milnor [5]. From this we easily prove Axiom (e2) for k_{PL} . The proof of (1.1) then follows exactly as it does in the orientable case.

Proof of (1.1) in the orientable case. Using Lemmas (10.1) and (10.2) with Brown's representability theorem, we get a countable CW complex classifying space B_{PL0} . Representation of the multiplication follows from (8.1), while uniqueness follows from Brown's theorem and (8.2).

(10.3) Corollary. $k_{PL0}(X)$ is an abelian group, $X \in \mathfrak{W}$.

Proof. This follows from James' result [4] which states that $[X, B]$ is a loop if X is a CW complex and B is an H -space.

From this we have the following.

(10.4) Corollary. $k_{PL}(X)$ is an abelian group, $X \in \mathfrak{W}$.

Proof. If $\{\xi\} \in k_{PL}(X)$, then $\{\xi \oplus \xi\} \in k_{PL0}(X)$ by (7.2). Thus there exists an (orientable) PL -microbundle η over X such that $\{\xi \oplus \xi\} + \{\eta\} = 0$. Hence $\{\xi\} + \{\xi \oplus \eta\} = 0$ in $k_{PL}(X)$.

Remark. A rewording of (10.3) and (10.4) gives the well-known result about the existence of Whitney sum inverses; this is proved directly in [5]. Our proof has been indirect, the result being a consequence of representability by an H -space. For the topological microbundle analogue, see [8].

We can now prove

(10.5) Lemma. k_{PL} satisfies Brown's axiom (e2).

Proof. We must verify the uniqueness condition when A is a point. To do this we observe first that the abelian group valued functor k_{PL} is *half exact* in the sense of Dold. Precisely, this means that if (X, A) is a pair in $\mathfrak{W}$ and $i : A \subset X, j : X \subset X \cup CA$ are inclusion maps, where CA is the reduced cone over A , then the sequence

$$k_{PL}(X \cup CA) \xrightarrow{i^*} k_{PL}(X) \xrightarrow{j^*} k_{PL}(A)$$

is exact. But this is immediate from our gluing construction and the fact that ji is null-homotopic. Since this property is easily shown (see [7] for example) to imply that $k_{PL}(X_1 \vee X_2) \cong k_{PL}(X_1) \oplus k_{PL}(X_2)$, the proof is complete.

Proof of (1.1). This is the same as for the orientable case given earlier.

11. Some further results. We first prove a weak-homotopy theorem and then give some applications. Singular theory is used throughout.

(11.1) Theorem. Let f and g be weakly homotopic maps from a space X to Y . Then

- i) $f_* = g_* : H_*(X; G) \rightarrow H_*(Y; G)$ for any G ;
- ii) $f^* = g^* : H^*(Y; G) \rightarrow H^*(X; G)$ if G is a compact topological group or a vector space over a field.

(11.2) Corollary. *If $f : X \rightarrow Y$ is a weak homomorphism of spaces with multiplication, then $f_* : H_*(X; k) \rightarrow H_*(Y; k)$ preserves the induced Pontrjagin product and $f^* : H^*(Y; k) \rightarrow H^*(X; k)$ preserves the induced coproduct, k being a field.*

To prove (11.1), suppose first that X is a CW complex. Let $\mathfrak{C}$ be the directed system of finite subcomplexes X_α of X ; see §8. Inclusion $X_\alpha \subset X$ induces canonical isomorphisms ϕ_α (see [2, p. 255] for example) such that the diagram

$$\begin{array}{ccc} \text{dir lim}\{H_q(X_\alpha; G)\} & \xrightarrow{\phi_q} & H_q(X; G) \\ & \searrow & \downarrow f_* \\ & \text{dir lim}\{(f|_{X_\alpha})_*\} & H_q(Y; G) \end{array}$$

is commutative. Since $f|_{X_\alpha} \simeq g|_{X_\alpha}$ for each α , part i) is established.

To prove i) for the general case, let X be a space and let $|SX|$ be the geometric realization of the singular complex of X . This construction is functorial and there is a natural transformation j such that each $j_X : |SX| \rightarrow X$ is a weak homotopy equivalence. Hence the induced function $j_* : [K, |SX|] \rightarrow [K, X]$, $K \in \mathfrak{W}$, is bijective by [6, Appendix]. It follows that if f and g are maps from X to Y , then: f and g are weakly homotopic

- iff $f_* = g_* : [K, X] \rightarrow [K, Y]$ for each $K \in \mathfrak{W}$,
- iff $|Sf|_* = |Sg|_* : [K, |SX|] \rightarrow [K, |SY|]$ for each $K \in \mathfrak{W}$,
- iff $|Sf|$ and $|Sg|$ are weakly homotopic maps from $|SX|$ to $|SY|$.

Since $|SX|$ is a CW complex and $j_* : H_*(|SX|; G) \cong H_*(X; G)$, the general case follows from the special case already established.

Similar arguments prove ii) to complete the proof.

As an immediate consequence of (1.1) and (11.2) we have the following result.

(11.3) Corollary. *Let k be a field and let B be the H -space B_{PL} or B_{PL0} . The Hopf algebras $H_*(B; k)$ and $H^*(B; k)$ are then independent of the choice of classifying space or H -structure (up to canonical Hopf algebra isomorphism).*

Let B_v and B_0 be the usual classifying H -spaces for the functors $\tilde{K}$ and $\tilde{KO}$ respectively. Analogues of (1.1), (1.2), and (11.3) of course hold for these functors, and may be proved by arguments similar, but technically simpler, to those we have used.

Let $\tilde{K} \rightarrow \tilde{KO}$ and $k_{PL0} \rightarrow k_{PL}$ be the obvious natural transformations, and let $\tilde{KO} \rightarrow k_{PL}$ be the natural transformation constructed by Milnor [5, Lemma 6].

(11.4) Proposition. *These natural transformations are represented by weak*

homomorphisms $B_U \rightarrow B_0$, $B_{PL0} \rightarrow B_{PL}$, and $B_0 \rightarrow B_{PL}$, respectively. They are unique up to weak homotopy and induce unique Hopf algebra homomorphisms in homology and cohomology (coefficients in a field).

Proof. The maps are obtained by applying Lemma 2.1 of Brown [1]. They are weak homomorphisms because of (8.2). The final statement follows from (11.1) and (11.2).

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