*Support Functions of Plurisubharmonic Functions**

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Introduction. When $f(x)$ is a distribution on the real line, let $K = K(f)$ be the convex hull of its support. Clearly, K is an interval, finite or infinite. In a somewhat modernized form, Titchmarsh's convolution theorem [10] (p. 327) states that if f_1, f_2 have non-empty compact supports and $(f_1 * f_2)(x) = \int f_1(x-y)f_2(y) dy$ is the convolution of f_1 and f_2 , then $K(f_1 * f_2) = K(f_1) + K(f_2)$ where the sum stands for the set of all $x_1 + x_2$ with $x_1 \in K(f_1), x_2 \in K(f_2)$. If the supports are bounded on one side, *e.g.* from below, the equality holds for the left endpoints of the supports. If, in particular, $f_1 * f_2 = 0$, *i.e.* $K(f_1 * f_2)$ is empty, and the supports of f_1 and f_2 are bounded from one side, the theorem shows that either f_1 or f_2 has to vanish. (The first chapter of [12] contains an elementary proof of this corollary.) Lions [6] has extended Titchmarsh's theorem to several variables. All proofs published so far (see *e.g.* [2], [4], [5], [6], [7], [10]) use the Fourier-Laplace transform $\hat{f}(\xi) = \int e^{-ix\xi} f(x) dx$, which is an entire function when f has compact support. The best organized proofs are, I believe, due to Hörmander [4], whose main interest, however, is the question of singular supports. The aim of this article is to give an exposition of the subject with simple proofs. The gist of the paper is Lemma 2.2 which states that taking the least harmonic majorant of a subharmonic function is a positive linear operation and characterizes the mass of this majorant at a point on the boundary in a way reminiscent of the Phragmén–Lindelöf principle. Lemma 2.1 dealing with boundary measures is taken from [3].

Anyone interested only in a proof of Titchmarsh's theorem for one variable can limit his reading to section 2, which, apart from a proof of this result, also gives a rather broad exposition of the properties of subharmonic functions. The Titchmarsh–Lions theorem is a corollary of our Theorem 4.1, which establishes a positive linear map from a certain class of plurisubharmonic function to the class of support functions. This theorem is first proved for one complex variable (Theorem 3.1). The classical correspondence between support functions

* Dedicated to Marcel Riesz on his eightieth birthday, Nov. 16th, 1966.

and compact convex sets is established in section 1. Finally, it should be remarked that Theorem 4.1 also implies the Lions-Titchmarsh theorem for Beurling's generalized distributions (see [1]).

1. Compact convex sets and support functions. Let $\mathcal{K}$ be the family of non-void compact convex sets K in R^n and let $\mathcal{H}$ be the family of support functions H , i.e. continuous maps $R^n \ni \eta \rightarrow H(\eta) \in R$ with the properties

$$(1) \quad a \geq 0 \Rightarrow H(a\eta) = aH(\eta); \quad H(\eta + \xi) \leq H(\eta) + H(\xi).$$

Any K defines a support function

$$(2) \quad K^*(\eta) = \max_{x \in K} x\eta; \quad x\eta = x_1\eta_1 + \cdots + x_n\eta_n.$$

If, for instance, K is a point, K^* is linear. If $n = 1$ and K is an interval $a \leq x \leq b$, then $K^*(\eta) = a\eta$ and $b\eta$ when $\eta \leq 0$ and $\eta \geq 0$ respectively. Conversely, every support function H defines a compact convex set

$$H^* = \{x : x\eta \leq H(\eta), \forall \eta\}.$$

A simple geometrical argument shows that any x outside K belongs to a closed half-space $y\eta \leq x\eta$ which contains K , i.e. K is the intersection of all closed half-spaces containing K . Since this intersection is given by $x\eta \leq K^*(\eta)$, $\forall \eta$, we have $K^{**} = K$. Similarly, if H is a support function, the set $\eta_0 \geq H(\eta)$ in R^{n+1} is a closed convex cone C , say, with its vertex at the origin. Its boundary is given by $\eta_0 = H(\eta)$. Since C is convex, given η there exists a half-space $\xi_0 - \xi x \geq 0$ containing C such that the plane $\xi_0 - \xi x = 0$ contains the point $(H(\eta), \eta)$. In particular, $H(\eta) = \eta x$ while $H(\xi) \geq \xi x$ for all ξ . In other words, $H(\eta)$ is the maximum of all linear functions $\eta \rightarrow x\eta$ that it majorizes, so that $H^{**} = H$.

We have seen that the star operation defines one-to-one maps $\mathcal{K} \rightarrow \mathcal{H}$ and $\mathcal{H} \rightarrow \mathcal{K}$ whose products are the identities on $\mathcal{K}$ and $\mathcal{H}$ respectively. These maps are also positive linear, i.e. if H_1, H_2, H_3 are the support functions of K_1, K_2, K_3 , then

$$(3) \quad H_1 = aH_2 \Leftrightarrow K_2 = aK_1; \quad a \geq 0,$$

$$(4) \quad H_3 = H_1 + H_2 \Leftrightarrow K_3 = K_1 + K_2,$$

where aK_1 and $K_1 + K_2$ are the sets of all $ax, x \in K_1$, and $x + y, x \in K_1, y \in K_2$, respectively. In fact, (2) shows that $(aK_1)^* = aK_1^*$ and that $(K_1 + K_2)^* = K_1^* + K_2^*$. Hence $(aH_1)^* = aH_1$ and $(H_1^* + H_2^*)^* = H_1 + H_2$ so that, applying the star operation once more, we get $(aH_1)^* = aH_1^*$ and $(H_1 + H_2)^* = H_1^* + H_2^*$.

We note that if $K(S)$ denotes the convex hull of a compact set S , then $S_3 = S_1 + S_2$ implies $K(S_3) = K(S_1) + K(S_2)$. The simple proof is left to the reader.

Let $C \subset R^n$ be a closed convex cone which is proper in the sense that there exists an η such that the intersections of C with the half-spaces $\eta x \leq \text{constant}$

are compact. Let S be a closed set and S_ν , $\nu = 1, 2, \dots$, a sequence of closed sets. We say that $S_\nu \rightarrow S$ if, for every compact set L , $S_\nu \cap L = S \cap L$ when ν is large enough. If all S_ν are contained in some fixed compact set then, trivially, $K(S_\nu) = K(S)$ when ν is large enough. Taking $S_\nu = S \cup x^{(\nu)}$ where $x^{(\nu)} \rightarrow \infty$, we see that this need not be the case when S_ν has parts tending to infinity. In the last section of this paper, we shall need the following lemma: If, for some fixed compact set L ,

$$(5) \quad L + C \supset A_\nu \rightarrow A \subset L + C; \quad \nu \uparrow \Rightarrow K(A_\nu) \uparrow,$$

then

$$(6) \quad K(A_\nu) \subset K(A) + C.$$

To prove this, let A^λ , A_ν^λ , B^λ be the parts of A , A_ν , $B = L + C$ where $x\eta \leq \lambda$ and B_λ the part of B where $x\eta > \lambda$. Taking L convex, which is no restriction, we have that B_λ is convex. By (5), $A_\nu \subset A_\nu^\lambda \cup B_\lambda$ and, since A^λ is contained in B^λ , which is compact, we have $A_\nu^\lambda = A^\lambda$ when ν is large enough. Hence, by (5), $K(A_\nu) \subset K(A^\lambda \cup B_\lambda)$ for all ν and λ . Hence for every $x \in K(A_\nu)$ there is a $y_\lambda \in K(A^\lambda)$ and a $z_\lambda \in B_\lambda$ and a number a_λ between 0 and 1 such that $x = a_\lambda y_\lambda + (1 - a_\lambda)z_\lambda$. Put $\mu = x\eta$ and assume that $\lambda > \mu$. Then $z_\lambda\eta > \lambda > \mu$ so that $y_\lambda\eta \leq \mu$, i.e. y_λ belongs to the compact set $K(A)^\mu$. Hence $(1 - a_\lambda)z_\lambda\eta \geq (1 - a_\lambda)\lambda$ is bounded as $\lambda \rightarrow \infty$, so that $a_\lambda \rightarrow 1$. Choosing a sequence of λ 's such that y_λ converges to a limit y , we have $x = y + z$ where $y \in K(A)^\mu$ and $z \in C$. In fact, $z_\lambda = z'_\lambda + z''_\lambda$ where $z'_\lambda \in L$ and $z''_\lambda \in C$, so that $\lim (1 - a_\lambda)z_\lambda = \lim (1 - a_\lambda)z''_\lambda \in C$.

2. Boundary measures of subharmonic functions. We shall review some basic properties of subharmonic functions. A function $v(z)$ defined in an open connected part Ω of the complex plane $\mathbb{C}$ taking real values and the value $-\infty$ is said to be subharmonic in Ω if it is continuous from above, in particular locally bounded from above, and satisfies the maximum principle with respect to harmonic functions in every open part ω of Ω : if u is harmonic in ω and $\lim (v - u) \leq 0$ on $\partial\omega$, then $v \leq u$ in ω . Conformal maps do not change these properties. It is clear that if v, w are subharmonic in Ω so is $\max(v, w)$. The subharmonic function which equals $-\infty$ at all points of Ω will be called trivial.

Now let $P(z, e^{i\theta}) = (2\pi)^{-1}(1 - |z|)^2 |e^{i\theta} - z|^{-2}$ be the Poisson kernel of the unit disk $|z| < 1$. Then $P(z/r, e^{i\theta})$ is the Poisson kernel of the disk $|z| < r$. If v is subharmonic in Ω and Ω contains the closed disk $|z| \leq r$, then

$$(1) \quad |z| < r \Rightarrow v(z) \leq u_r(z) \equiv \int_0^{2\pi} P(z/r, e^{i\theta}) v(re^{i\theta}) d\theta.$$

In fact, replacing $v(re^{i\theta})$ in the integral by a continuous majorant $f(\theta)$, by the maximum principle we get a harmonic majorant of v and taking a sequence of majorants $f(\theta)$ decreasing to $v(re^{i\theta})$, we get a sequence of harmonic majorants decreasing to u_r , which is identically $-\infty$ or harmonic in the disk $|z| < r$.

In particular, if v is not trivial in the disk $|z| < r$, v is integrable on $|z| = r$ and integrable on a neighborhood of this circle. A translation of the origin shows that if v is subharmonic and not trivial, then v is locally integrable and integrable on every circle. Putting $z = 0$ in (1) and translating the origin to a point $z_0 \in \Omega$, we get

$$(1') \quad v(z_0) \leq \int_0^{2\pi} v(z_0 + \rho e^{ia}) da/2\pi,$$

where it is supposed that Ω contains the disk $|z - z_0| \leq \rho$. A classical argument shows that if v is continuous from above in Ω and satisfies (1') for every $z_0 \in \Omega$ and all sufficiently small ρ , then v is subharmonic, so that subharmonicity is a local property. If $f(z)$ is holomorphic in Ω , then $v(z) = \log |f(z)|$ satisfies the requirements just stated and hence is subharmonic. We notice also that if v_1, v_2 are subharmonic in Ω so is $a_1 v_1 + a_2 v_2$ provided $a_1, a_2 \geq 0$.

Although we shall not need it here, we quote the F. Riesz representation theorem [8]: v is subharmonic and non trivial in a region Ω if and only if there exists a measure $d\sigma \geq 0$ in Ω such that v is locally of the form

$$h(z) - (2\pi)^{-1} \int_{\omega} \log |z - z'|^{-1} d\sigma(z').$$

Here $z \in \omega$, $\omega \subset \Omega$ is open, $\bar{\omega} \subset \Omega$ is compact, and $h = h_{\omega}$ is a harmonic function depending on ω . L. Schwartz [9] has observed that $d\sigma(z)$ is nothing but the distribution $\Delta v(z)$, (Δ the Laplace operator), and this remark together with the fact that weak solutions of Laplace's equation $\Delta u = 0$ are harmonic gives a simple proof of F. Riesz's theorem and a new characterization of subharmonic non trivial functions: distributions v such that $\Delta v \geq 0$. In fact, an easy argument shows that any such distribution is represented by a unique subharmonic function.

Returning to (1), we observe that the functions $r \rightarrow u_r(z)$ are increasing. In fact, if $r' > r = |z|$, then $v(z) \leq u_{r'}(z)$ while, by the properties of the Poisson integral, $\lim_{r' \rightarrow r} u_{r'}(z) \leq v(z)$. Hence, by the maximum principle for harmonic functions, $u_r(z) \leq u_{r'}(z)$ when $|z| < r < r'$. We shall see next that if v is non trivial, then u_r is the least harmonic majorant of v in the disk $|z| < r$. In fact, this least harmonic majorant obviously equals $U_r(z) = \lim_{\epsilon \downarrow 0} u_{r-\epsilon}(z)$, and since $u_r(0) = U_r(0)$ implies $u_r(z) = U_r(z)$ for all z such that $|z| < r$, it suffices to show that the function $r \rightarrow g(r) = u_r(0)$ is continuous. Now putting $z = 0$ in (1) and applying (1') with $z_0 = re^{i\theta}$, using Fubini's theorem and a change of variables, we get

$$(1'') \quad g(r) \leq \int g(|r + \rho e^{ia}|) da/2\pi,$$

where ρ is small enough. The function $\log |z|$ is harmonic outside the origin and hence satisfies (1'') with equality sign. Hence, by a classical argument, g satisfies the maximum principle with respect to linear polynomials in $\log r$, i.e.,

g is a convex function of $\log r$, in particular continuous. We can also prove that if v is subharmonic and bounded from above in the whole plane, then v is a constant. In fact, in that case $g(r)$ is constant and its limit as $r \rightarrow 0$, lying between $v(0)$ and $\lim v(0)$, equals $v(0)$ since v is continuous from above. Hence $v(0) = \int v(re^{i\theta}) d\theta/2\pi = u_r(0)$ for all r so that, by the maximum principle $v(z) = u_r(z)$ is harmonic for $|z| < r$. This means that v is harmonic and hence a constant. We remark in passing that if $d\sigma \geq 0$ is the measure associated with v , then

$$(2) \quad v(z) = u_r(z) - \int_{|z'| < r} G_r(z, z') d\sigma(z'),$$

where G_r is Green's function for the disk $|z| < r$ (F. Riesz [8]).

Next, we shall study the least harmonic majorant of v in the unit disk $|z| < 1$, supposing that v is non trivial and subharmonic there but perhaps in no larger disk. In the first place, v need not have a harmonic majorant, but if it has one, w , say, then $u_r(0) \leq w(0)$ is bounded from above so that, by Harnack's theorem, $u(z) = \lim u_r(z)$, $r \uparrow 1$, is the least harmonic majorant. In particular, $u_r(0) = \int v(re^{i\theta}) d\theta/2\pi$ increases to $u(0) = \int u(re^{i\theta}) d\theta/2\pi$ so that

$$(3) \quad r \uparrow 1 \Rightarrow \int (u(re^{i\theta}) - v(re^{i\theta})) d\theta \downarrow 0.$$

As a consequence (2) holds for $r = 1$, $u = \lim u_r$, $r \uparrow 1$, if v has a harmonic majorant in the unit disk.

Now let V be the class of all non trivial subharmonic functions in the unit disk having *positive* harmonic majorants there. If $v \in V$, we can, in the arguments above, choose $w > 0$ and then we can say more. In fact, then also $v_+ = \max(0, v)$ belongs to V and $\int v_+(re^{i\theta}) d\theta/2\pi \leq w(0)$ is bounded from above so that, by (3), $\int |v(re^{i\theta})| d\theta$ is bounded. Hence by a classical property of bounded measures it follows that there exists a measure ν such that, for any continuous real function $f(\theta)$,

$$(4) \quad r \rightarrow 1 \Rightarrow \int v(re^{i\theta})f(\theta) d\theta \rightarrow \int f(\theta) d\nu(e^{i\theta})$$

at least when r tends to 1 on some sequence. This is true also when f depends continuously on r . Hence, passing to the limit in (1) we get

$$(5) \quad u(z) = \int P(z, e^{i\theta}) d\nu(e^{i\theta}).$$

Multiplying this by a continuous function $f(a)$ and integrating for $|z| = \rho$ constant < 1 , since $P(\rho e^{ia}, e^{i\theta})$ is symmetric in a and θ , we get

$$\int u(\rho e^{ia})f(a) da = \int h(\rho e^{ia}) d\nu(e^{ia}),$$

where h is the Poisson integral of f . Letting ρ tend to 1, we see that (4) holds

with v replaced by u without any restriction to a sequence. Hence the measure ν is unique and (4) holds. We shall express (4) by saying that ν is the *boundary measure* of v . Notice that if v is subharmonic in some open set intersecting the unit circle in an open set E and if $f(\theta) \geq 0$ has its support in E , then by Fatou's lemma, the limit in (4) is majorized by $\int f(\theta)v(e^{i\theta})d\theta$ so that $d\nu(e^{i\theta}) \leq v(e^{i\theta})d\theta$ in E . Actually, using the F. Riesz representation theorem, it is easy to see that there is equality, but we shall not need that fact. Notice that, combining (2) with $r = 1$ and (5), we get

$$(2') \quad v(z) = \int P(z, e^{i\theta}) d\nu(e^{i\theta}) - \int_{|z'| < 1} G(z, z') d\sigma(z')$$

when $v \in V$. Conversely, if $d\nu$ and $d\sigma \geq 0$ are measures and $\int (1 - |z'|) d\sigma(z') < \infty$, this defines a function $v \in V$. However, we shall not need this characterization of the class V . Let us only state what we have proved so far, namely,

Lemma 2.1. *If $v \in V$, then v has a unique boundary measure ν , common to v and its least harmonic majorant.*

It is clear that the map $V \ni v \rightarrow \nu$ is positive linear, i.e. linear under addition and multiplication by constants ≥ 0 and that if $u, v \in V$ and u is harmonic, then $u \geq v$ if and only if their boundary measures satisfy the same inequality.

Let us now consider the point mass Mv of ν at $z = 1$, i.e., $Mv = \lim_{\epsilon \rightarrow 0} \int_{-}^{+} d\nu(e^{i\theta})$, $\epsilon \rightarrow 0$. We shall call Mv the mass of ν at the point $z = 1$. It is again clear that the map $V \ni v \rightarrow Mv$ is positive linear and the following statement holds: if $u, v \in V$, $Mu = 0$, u is harmonic and there exist numbers A such that $v(z) \leq u(z) + AP(z, 1)$, then $A = Mv$ is the least number with this property. In fact, in terms of the boundary measures ν, μ of v, u , respectively the inequality reads $d\nu(e^{i\theta}) \leq d\mu(e^{i\theta}) + Ad\beta(e^{i\theta})$ where β has mass 1 at $z = 1$ and vanishes elsewhere. Hence $M\nu \leq A$ and, putting $\nu_1 = \nu - (M\nu)\beta$, we have $M\nu_1 = M\mu = 0$ and $d\nu_1 \leq d\mu$ outside $z = 1$ so that $d\nu_1 \leq d\mu$ on the whole circle $|z| = 1$.

Now map the unit disk conformally onto the upper half-plane $y > 0$ and let V refer to the upper half-plane so that V consists of all functions which are subharmonic and have positive harmonic majorants there. Further, let $Mv = M_z v(z)$ be the mass of ν at ∞ normalized in such a way that $Mv = 1$ when $v = y$. Transported from the unit disk, the formula (5) for the least harmonic majorant u of v reads

$$(5') \quad u(z) = \int_{-\infty}^{+\infty} Q(z - t) d\nu(t) + yMv, \quad z = x + iy,$$

where $Q(z) = y/\pi |z|^2$ is the Poisson kernel for the upper half-plane and

$$\int (1 + t^2)^{-1} |d\nu(t)| < \infty.$$

A simple computation, left to the reader, shows that, for all x ,

$$(6) \quad Mv = \lim_{y \rightarrow \infty} u(z)y^{-1}, \quad y \rightarrow \infty.$$

Rephrasing two of our previous results, we get

Lemma 2.2. *The map $V \ni v \rightarrow Mv$ is positive linear. If $u, v \in V$, $Mu = 0$ and u is harmonic and if there are numbers A such that $v \leq u + Ay$, then $A = Mv$ is the least number with this property.*

We can now prove Titchmarsh's theorem as it is presented in the introduction. Let $K(f)$ be the interval $a \leq x \leq b$ and put $v(\zeta) = \log |\hat{f}(\zeta)|$. According to a theorem by Paley and Wiener in the form given in Schwartz [9] II, p. 128, b is the least number B with the following properties: there exist numbers c and N such that $v(\zeta) \leq u(\zeta) + B\eta$ where $\zeta = \xi + i\eta$, $\eta > 0$, $u(\zeta) = c + N \log |\zeta + i|$. It is clear that u is harmonic and belongs to V in the upper half-plane and (6) shows that u has mass 0 at ∞ , i.e., $Mu = 0$. Hence the lemma applies so that $b = Mv$ is the mass of v at ∞ . Now, putting $f_3 = f_1 * f_2$, we have $\hat{f}_3 = \hat{f}_1 \hat{f}_2$ so that $v_3 = v_1 + v_2$ and, consequently, $b_3 = b_1 + b_2$. The same reasoning for the lower half-plane shows that $a_3 = a_1 + a_2$ and this finishes the proof when f_1 and f_2 have compact supports. If their supports are bounded from below, $f_3 = f_1 * f_2$ is still defined and, if a_1, a_2, a_3 have their previous meaning, a look at the convolution shows that $a_3 \geq a_1 + a_2$. Now, close to $a_1 + a_2$, f_3 is independent of the values of f_1 and f_2 for large arguments and if g_1, g_2 have compact supports and equal f_1, f_2 when $a_1 \leq x \leq a_1 + 1$ and $a_2 \leq x \leq a_2 + 1$ respectively, then a_3 is the left endpoint of the support of $g_3 = g_1 * g_2$, so that $a_3 = a_1 + a_2$. As the example $f_1(x) = f'(x)$, $f_2 = (1 + \operatorname{sgn} x)/2$, $f_3 = \delta(x)$ shows, we have no longer $K(f_3) = K(f_1) + K(f_2)$ in the case we consider, but replacing $K(f)$ by $K(f) + R^+$ where R^+ is the positive axis $x \geq 0$, we get equality. The extension of this result to several variables (Theorem 5.2) leads to some geometrical technicalities taken care of at the end of section 1.

3. Support functions of subharmonic functions on the plane. In order to prepare for plurisubharmonic functions we shall have to add some material to the preceding section. If η, ζ are complex numbers and $v(\eta\zeta + \zeta) \in V$, put $M(\eta, \zeta)v = M_*v(\eta\zeta + \zeta)$. The following lemma is a standard result.

Lemma 3.1. *If $v(z)$ is subharmonic and bounded from above in some band $y_1 < y < y_2$, then the function*

$$y \rightarrow f(y) = \sup_x v(x + iy)$$

is convex. If $v \in V$ and $v(z) \leq Ay$ for some real A when $y > 0$, then

$$(1) \quad y \uparrow \infty \Rightarrow y^{-1}f(y) \uparrow Mv.$$

If $v \in V$ is defined for $y \geq 0$, then

$$(2) \quad \eta \geq 0, \quad \operatorname{Im} \zeta \geq 0 \Rightarrow M(\eta, \zeta)v = \eta Mv.$$

Proof. Applying the maximum principle to the subharmonic function $v_*(z) = v(z) - \epsilon(x^2 - y^2)$, $\epsilon > 0$, in the band $y'_1 < y_2 < y'_2$, we see that v_* is

majorized there by a linear function of y taking the values $f(y) + \epsilon y^2$ at y'_1 and y'_2 . Hence, letting $\epsilon \rightarrow 0$, we see that f is convex. In particular, if $v \in V$ and $v(z) \leq Ay$, $t^{-1}f(t)$ increases to a limit c as $t \rightarrow \infty$. Lemma 2.2 gives $v(z) \leq yMv$ so that $c \leq Mv$. On the other hand, $v(z) \leq yc$ so that, by the same lemma, $c \geq Mv$. This proves (1). That (2) holds when v is harmonic follows from (2.6). Now if $v \in V$, then there exists a harmonic u with $Mu = 0$ such that $v(z) \leq u + yMv$. Applying (1) to $w = v - u$, we see that $M(\eta, \zeta)w = \eta Mw$. Hence, since $Mu = M(\eta, \zeta)u = 0$, we get (2).

We shall now define a useful class of weight functions on the real line and after that a class of functions subharmonic in the whole plane.

Definition. Let B be the class of continuous functions $R^1 \ni x \rightarrow \varphi(x) \geq 0$ such that $\varphi(x)$ increases with $|x|$ when $|x|$ is sufficiently large and such that $\int_{-\infty}^{+\infty} \varphi(x)(1+x^2)^{-1} dx < \infty$.

The class B admits positive linear combinations and changes of variable $x \rightarrow ax + b$, a, b real. Examples of functions in B are $\log(1+|x|)$, $|x|^a$, $0 < a < 1$, $|x|(\log(2+|x|))^{-1-\epsilon}$, $\epsilon > 0$. If $\varphi \in B$, then there exist $\varphi_1, \varphi_2 \in B$ such that

$$(3) \quad \varphi(x_1 + x_2) \leq \varphi_1(x_1) + \varphi_2(x_2).$$

In fact, it suffices to consider the case when $\varphi(x) = \varphi(-x) = \varphi(|x|)$ and when φ is not bounded. Then, if $|x_1| \leq |x_2|$ and $|x_2|$ is sufficiently large, we have $\varphi(|x_1 + x_2|) \leq \varphi(2|x_2|)$ and if $|x_2| \leq |x_1|$ and $|x_1|$ is sufficiently large we have $\varphi(|x_1 + x_2|) \leq \varphi(2|x_1|)$. Hence there is a constant c such that $\varphi(x_1 + x_2) \leq \varphi(2x_1) + \varphi(2x_2) + c$ and this proves (3). If $\varphi \in B$ and $u(z) = \int Q(z-t)\varphi_1(t) dt$, $y > 0$, then $u \geq 0$ is harmonic, $Mu = 0$ by (2.6) and

$$(4) \quad \varphi(x) \leq u(z) + o(y), \quad z = x + iy.$$

In fact, by (3),

$$(5) \quad \varphi(x) = \int Q(z-t)\varphi(x) dt \leq \int Q(z-t)\varphi_1(t) dt + \int Q(z-t)\varphi_2(x-t) dt,$$

where the last integral is independent of x , bounded when y is bounded and $o(y)$ as $y \rightarrow \infty$.

Definition. Let W be the set of all non trivial subharmonic functions w in the plane satisfying

$$(6) \quad w(z) \leq \varphi(x) + c|y|$$

for some $\varphi \in B$ and $c > 0$.

Example. When $f \neq 0$ is a distribution with compact support then $w(z) = \log |\hat{f}(z)|$ belongs to W . In fact, then (6) is true with $\varphi(x)$ a multiple of $\log(2+|x|)$.

By the properties of B , W admits positive linear combinations and changes of variable $z \rightarrow az + b$ and (4) and (6) show that W restricted to the upper

half-plane is contained in V . Hence

$$M(\eta, \zeta)w = M_z w(\eta z + \zeta)$$

is defined. In the example above, we have seen that $M(\pm 1, 0)w$ are the endpoints of the convex hull $K(f)$ of the support of f and that the maps $w \rightarrow M(\pm 1, 0)w$ are positive linear. We shall see now that $M(\eta, \zeta)w = M(\eta)w$ is independent of ζ and $\eta \rightarrow M(\eta)w$ is the support function of $K(f)$. The following result contains Titchmarsh's convolution theorem.

Theorem 3.1. *If $w \in W$, then $M(\eta)w = M(\eta, \zeta)w$ is independent of ζ and $\eta \rightarrow M(\eta)w$ is a support function. The map $W \ni w \rightarrow M(\eta)w$ is positive linear.*

Note. All the properties of the class W are not used in the proof, but they will come into play in the next section. Theorem 3.1 holds for the class W^* of subharmonic functions in the plane having positive harmonic majorants in all half-planes $\pm y > \text{const}$. An easy but somewhat lengthy argument shows that a subharmonic function w belongs to W^* provided it has positive harmonic majorants in two adjacent half-planes of this type.

Remark. The support function $g(\eta) = M(\eta)w$ of a function $w \in W^*$ can also be obtained by a simple limiting process. Putting $h(z) = g(y)$, $x + iy$, we have

$$(7) \quad w \in W^*, \quad R \rightarrow \infty \Rightarrow R^{-1}w(Rz) \, dx \, dy \rightarrow h(z) \, dx \, dy$$

in the weak convergence of measures. This observation is due to A. Beurling (unpublished) and I reproduce his proof here. Let $d\sigma(z) \geq 0$ be the measure $\Delta v(z) \, dx \, dy$, $Q(z) = y/\pi \, |z|^2$ the Poisson kernel and

$$G(z, z') = \pi^{-1} \log |z - \bar{z}'| \, |z - z'|^{-1}$$

Green's function for the upper half-plane. Then $\int_{y>0} y(1 + |z|^2)^{-1} \, d\sigma(z) < \infty$, $\int (1 + t^2) |w(t)| \, dt < \infty$ and we have the Riesz decomposition

$$w(z) = h(z) + \int Q(z - t)w(t) \, dt - \int_{y'>0} G(z, z') \, d\sigma(z'), \quad y > 0.$$

Let f be continuous with compact support. Then, multiplying by $f(z)$ and integrating, we get

$$(8) \quad \int_{y>0} (R^{-1}w(Rz) - h(z))f(z) \, dx \, dy = \int F_R(t)w(0, t) \, dt - \int_{y'>0} G_R(z') \, d\sigma(z'),$$

where

$$F_R(t) \equiv R^{-1} \int_{y>0} Q(Rz - t)f(z) \, dx \, dy = R^{-2}F_1(t/R),$$

$$G_R(z') \equiv R^{-1} \int_{y>0} G(Rz, z')f(z) \, dx \, dy = R^{-1}G_1(z'/R).$$

Simple estimates show that $F_1(t) = O((1 + t^2)^{-1})$ and that

$$G_1(z') = O(y'(1 + |z'|^2)^{-1}).$$

so that

$$F_R(t) = O((R^2 + t^2)^{-1}), \quad G_R(z') = O(y'(R^2 + |z'|^2)^{-1}).$$

Hence the right side of (8) tends to zero as $R \rightarrow \infty$, so that, applying the same reasoning to the lower half-plane, we get

$$(9) \quad R^{-1} \int w(Rz)f(z) \, dx \, dy \rightarrow \int h(z)f(z) \, dx \, dy.$$

Hence (7) is proved. The following application is also due to Beurling. Replacing f in (9) by Δf and noting that

$$\int h(z)\Delta f(z) \, dx \, dy = (b - a) \int f(x, 0) \, dx,$$

where $h(z) = by$, ay for $y \geq 0$, $y \leq 0$, we get

$$R^{-1} \int f(z) \, d\sigma(Rz) \rightarrow (b - a) \int f(x, 0) \, dx.$$

In other words,

$$(10) \quad R \rightarrow \infty \Rightarrow R^{-1} d\sigma(Rz) \rightarrow (b - a) \delta(y) \, dx \, dy$$

with weak convergence. If in particular, $w = \log |F(z)|$ where F is entire, then $d\sigma(z)$ is the sum of point masses $(2\pi)^{-1}$ at the zeros of F and (10) contains some classical results on the asymptotic distribution of the zeros of entire functions of completely regular growth (see [11] p. 150 and 242).

Proof. Lemma 3.1 shows that $M_\eta w(\eta z + \zeta) = M_\eta w(\eta z + \zeta')$ when $\eta \geq 0$, $\text{Im } \zeta \geq \text{Im } \zeta'$ and when $\eta \leq 0$, $\text{Im } \zeta \leq \text{Im } \zeta'$. Hence $M(\eta)w = M(\eta, \zeta)w$ does not depend on ζ and Lemma 3.1 also shows that $M(\eta)w = |\eta| M(\text{sgn } \eta)w$. Hence, to show that $\eta \rightarrow M(\eta)w$ is convex, it suffices to prove that $M(1)w + M(-1)w \geq 0$. The Poisson kernel Q has the property that

$$\text{Im } z, \quad \text{Im } z' > 0 \Rightarrow \int Q(z - s)Q(z' + s) \, ds = Q(z + z')$$

and we have

$$y + y' > 0 \Rightarrow w(z) \leq \int Q(z + iy' - s)w(-iy' + s) \, ds + (y + y')M(1)w.$$

In fact, $w(-iy' + s)ds$ is the boundary measure of w on the line $y = -y'$. By the same argument,

$$y' > 0 \Rightarrow w(-iy' + s) \leq \int Q(iy' + s - t)w(t) \, dt + y'M(-1)w.$$

Hence

$$w(z) \leq (y + y')M(1)w + y'M(-1)w + \int Q(z + 2iy' - t)w(t) dt.$$

Choosing z so that $w(z) > -\infty$, dividing by y' and letting $y' \rightarrow \infty$, we get the desired result.

4. Support functions of plurisubharmonic functions. Let $\mathbf{C}^n$ be complex n -space with coordinates $\zeta = (\zeta_1, \dots, \zeta_n)$. A function v defined in an open part Ω of $\mathbf{C}^n$ is said to be plurisubharmonic there if it is continuous from above and the functions $z \rightarrow v(\zeta + z\zeta')$ are subharmonic for all $\zeta \in \Omega$, $\zeta' \in \mathbf{C}^n$ and sufficiently small z . The main example is $v(\zeta) = \log |f(\zeta)|$, where f is holomorphic in Ω and not identically zero. We shall limit ourselves to the case $\Omega = \mathbf{C}^n$. If v is plurisubharmonic and if $d\rho(t)$ is the unit mass spread uniformly over $|t| = 1$, we have, in particular,

$$(1) \quad v(\zeta) \leq \int v(\zeta + t\zeta') d\rho(t).$$

A plurisubharmonic function is said to be trivial if it is identically $-\infty$ and nontrivial in the opposite case.

Every Borel function f which is locally bounded from above has a least upper continuous majorant, namely $f^*(\zeta) = \overline{\lim} f(\zeta')$, $\zeta' \rightarrow \zeta$. We shall need the following lemma: if f satisfies (1) then f^* is plurisubharmonic and $f^* = f$ almost everywhere. Since $f^* = f$ when f is trivial, we need only consider the case when f is non-trivial. It follows from (1) that $f(\zeta) \leq \int f^*(\zeta + t\zeta') d\rho(t)$ where, by Fatou's theorem, the right side is continuous from above and hence majorizes $f^*(\zeta)$. Hence f^* is plurisubharmonic. Iterations of (1) give

$$(1') \quad f(\zeta) \leq \int f(\zeta + t_1\zeta^{(1)} + \dots + t_n\zeta^{(n)}) d\rho(t_1) \dots d\rho(t_n).$$

Choosing $\zeta^{(1)}, \dots$ linearly independent, we see that f is locally integrable so that the right side of (1') is a continuous function of ζ . But then $f^*(\zeta)$ lies between the two sides of (1'). Also, if we replace $\zeta^{(1)}, \dots$ by $\epsilon\zeta^{(1)}, \dots$ and let $\epsilon \rightarrow 0$, the right side tends to $f(\zeta)$ almost everywhere. Hence $f = f^*$ almost everywhere and this finishes the proof. We shall also need the fact that a plurisubharmonic function v which is bounded from above is a constant. This follows if we observe that $z \rightarrow v(\zeta + z\zeta')$ is subharmonic in z and bounded from above and hence a constant. Finally, we note that if $g(\eta) = \sup_{\xi} v(\xi + i\eta)$, $\xi \in \mathbf{R}^n$, is locally bounded from above, then the function $\mathbf{R}^n \ni \eta \rightarrow g(\eta)$ is convex. In fact, Lemma 3.1 shows that the function $y \rightarrow h(\xi, y) = \sup_x v(\xi + \eta^{(1)}z + \eta^{(2)}(i-z))$ is convex and hence $y \rightarrow \sup h(\xi, y) = g(y\eta^{(1)} + (1-y)\eta^{(2)})$ is convex.

Our aim is to extend Theorem 3.1 to several variables. We first define the class corresponding to W .

Definition. Let W^n be the set of functions w , plurisubharmonic and non trivial in $\mathbf{C}^n$, with the property that $\varphi \in B$ and $c > 0$ exist such that

$$(2) \quad w(\zeta) \leq \varphi(|\operatorname{Re} \zeta|) + c |\operatorname{Im} \zeta|.$$

Here $|\zeta| = |\zeta_1| + \dots + |\zeta_n|$. As before, W^n admits positive linear combinations and also changes of variable $\zeta \rightarrow A\zeta + b$ where A is a real $n \times n$ matrix and $b \in \mathbf{C}^n$. In fact, by the properties of B , $\varphi(|A\zeta + b|) \leq \varphi_1(|A| |\zeta|)$ where $\varphi_1 \in B$ and $|A|$ is the operator norm. We also notice that (2) holds if and only if

$$(2') \quad w(\zeta) \leq \Sigma \psi(\operatorname{Re} \zeta_k) + c |\operatorname{Im} \zeta|$$

for some $\psi \in B$. If $w \in W^n$, $\eta \in R^n$, $z \in \mathbf{C}$, we have

$$(3) \quad w(\zeta + z\eta) \leq \Sigma \psi(\operatorname{Re} \zeta_k + x\eta_k) + c(|\operatorname{Im} \zeta| + |y| |\eta|)$$

so that the function $z \rightarrow w(\zeta + z\eta)$ belongs to W .

Definition. Put $M(\eta, \zeta)w = M_z w(\zeta + z\eta)$ and, for η fixed, let $M^*(\eta, \zeta)w$ be the least majorant of $M(\eta, \zeta)w$ which is continuous from above.

We shall prove the following extension of Theorem 3.1 to several variables.

Theorem 4.1. *If $w \in W^n$ then $M(\eta)w = M^*(\eta, \zeta)w$ is independent of ζ and $M(\eta, \zeta)w = M(\eta)w$ for almost all ζ . The function $\eta \rightarrow M(\eta)w$ is a support function and the maps $w \rightarrow M(\eta)w$ are positive linear.*

Proof. By (3) and the properties of B ,

$$(4) \quad w(\zeta + z\eta) \leq \Sigma \varphi_1(x\eta_k) + \Sigma \varphi_2(\operatorname{Re} \zeta_k) + c(|\operatorname{Im} \zeta| + |y| |\operatorname{Im} \eta|),$$

where $\varphi_1, \varphi_2 \in B$. Hence, to every compact $K \subset \mathbf{C}^n$ there is a harmonic $u(z) \in V$ with $Mu = 0$ such that

$$(5) \quad w(\zeta + z\eta) \leq u(z) + yM(\eta, \zeta)w, \quad \zeta \in K.$$

By Lemma 3.1,

$$y \uparrow \infty \Rightarrow y^{-1} \sup_x (w(\zeta + z\eta) - u(z)) \uparrow M(\eta, \zeta)w.$$

Now $\zeta, y \rightarrow \varphi_R(\zeta, y) = \max (w(\zeta + z\eta) - u(z)), |x| \leq R$, is continuous from above, since $\zeta, z \rightarrow w(\zeta + z\eta)$ is continuous from above. Further, $\varphi_R(\zeta, y)$ increases with R so that $\zeta, y \rightarrow y^{-1} \sup_x (w(\zeta + z\eta) - u(z))$ is a Borel function and, since this function increases with y , $\zeta \rightarrow M(\eta, \zeta)w$ is also a Borel function. By (3.4),

$$(6) \quad \varphi_1(x\eta_k) \leq u_k(z) + o(y),$$

where $u_k \in V$ is harmonic and $Mu_k = 0$. Hence, by (4) and Lemma 2.2, $M(\eta, \zeta)w \leq c |\operatorname{Im} \eta|$, so that the function $\zeta \rightarrow M(\eta, \zeta)w$ is bounded from above. Further, if K contains all $\zeta + t\zeta'$ with $t \in \mathbf{C}$, $|t| = 1$, we get from (5) that

$$w(\zeta + t\zeta' + z\eta) \leq u(z) + yM(\eta, \zeta + t\zeta')w$$

so that an integration gives

$$w(\zeta + z\eta) \leq u(z) + y \int M(\eta, \zeta + t\zeta') w \, d\rho(t).$$

Hence, by Lemma 2.2,

$$M(\eta, \zeta)w \leq \int M(\eta, \zeta + t\zeta') w \, d\rho(t)$$

so that the function $\zeta \rightarrow M^*(\eta, \zeta)w$ is plurisubharmonic and bounded from above. Hence $M(\eta)w = M^*(\eta, \zeta)w$ does not depend on ζ and $M(\eta)w = M(\eta, \zeta)w$ for almost all ζ . Since $w \rightarrow (\eta, \zeta)w$ is positive linear for all η and ζ , we conclude that also $w \rightarrow M(\eta)w$ is positive linear.

That the function $R^n \ni a \rightarrow M(a\eta, \zeta)w$ is positive homogeneous and convex follows from Theorem 3.1. Hence $a \rightarrow M(a\eta)w$ has the same properties and in order to show that the function $R^n \ni \eta \rightarrow M(\eta)w$ is convex, it suffices to show that it is convex in any two-dimensional half-space. Since this property is independent of the coordinate system, it suffices to show that $\eta \rightarrow M(\eta)w$ is convex when η belongs to the region $R_+^n : \eta_k > 0, \forall k$. Let $K \subset R_+^n$ be a compact ball with positive radius and choose $u(z) \in V$ harmonic such that $Mu = 0$ and $\psi(x) \leq u(z) + p(y)$ for some first degree polynomial p . If $\text{Im } \zeta \in K^0$, $\eta \in R_+^n$ and $y > 0$, the function $v(\zeta + z\eta) = w(\zeta + z\eta) - \sum u(\zeta_k + z\eta_k)$ is plurisubharmonic in ζ, z and, by virtue of (3), it is bounded by $c_1 + c_1 y \sum \eta_k$ for some $c_1 > 0$. Hence, by Lemma 2.2, we have $v(\zeta + z\eta) \leq c_1 + yM(\eta, \zeta)w \leq c_1 + yM(\eta)w$, so that

$$(7) \quad M(\eta)w \geq \overline{\lim_{y \rightarrow \infty}} y^{-1} v^*(\text{Im } \zeta + y\eta),$$

where

$$v^*(\text{Im } \zeta + y\eta) = \sup_{\text{Re } \zeta} v(\zeta + z\eta) = \sup_{\text{Re } \zeta, x} c(\zeta + z\eta),$$

is a convex function of $\text{Im } \zeta + y\eta$, and hence also of η . Hence the right side of (7) is a convex function of η . Lemma 3.1 gives

$$M(\eta, \zeta)w = \lim_{y \rightarrow \infty} y^{-1} \sup_x v(\zeta + z\eta).$$

But $M(\eta, \zeta)w = M(\eta)w$ for almost all ζ . Hence we have equality in (7) so that $\eta \rightarrow M(\eta)w$ is convex when $\eta \in R_+^n$. This finishes the proof.

5. Application to convolutions. Let $\mathfrak{D}$ denote the space of all infinitely differentiable functions $R^n \rightarrow R$ with compact supports, $\mathfrak{D}'$ the space of distributions and $\mathcal{E}'$ the space of distributions with compact supports. When $f \in \mathfrak{D}'$, let $S = S(f)$ be the support of f , $K(f) = K(S)$ its convex hull and, when S is compact, $H(\eta)$ the support function of S and K . When $f_1, f_2 \in \mathcal{E}'$, the convolution

$$(1) \quad f_3(x) = (f_1 * f_2)(x) = \int f_1(x-y)f_2(y) dy$$

is well defined and belongs to $\mathcal{E}'$ and $S(f_3) \subset S(f_1) + S(f_2)$. In particular, $K(f_3) \subset K(f_1) + K(f_2)$. Also, if $f \in \mathcal{E}'$, its Fourier-Laplace transform $\hat{f}(\zeta) = \int e^{-i x \zeta} f(x) dx$ is entire analytic and (1) implies $\hat{f}_3 = \hat{f}_1 \hat{f}_2$ and conversely. Our next lemma is a precise formulation of the Paley-Wiener theorem.

Lemma 5.1. *If $0 \neq f \in \mathcal{E}'$, then $w(\zeta) = \log |\hat{f}(\zeta)|$ belongs to W^n and its support function equals the support function of $S(f)$.*

Note. If $f = 0$, then $S(f)$ is empty and the lemma is true provided we define the support function of the empty set to be $-\infty$ everywhere and the support function of $w \equiv -\infty$ to be $-\infty$ everywhere.

Proof. Let $\psi(x) \in \mathcal{D}$ be equal to one on $K(f)$. Then $|\hat{f}(\zeta)|$ is majorized by a constant times the maximum of the derivatives of $e^{-i x \zeta} \psi(x)$ of order $\leq N$, an integer only depending on f . Choosing the support of ψ close to $K = K(f)$ it then follows that to every $\epsilon > 0$ there is a constant C_ϵ such that

$$w(\zeta) \leq C_\epsilon + N \log(1 + |\zeta|) + H(\text{Im } \zeta) + \epsilon |\text{Im } \zeta|,$$

where H is the support function of K . Hence if $\eta \in R^n$ and $y > 0$, using the inequality $A + |z| \leq \sqrt{2} |z + iA|$, ($A, y > 0$), we get

$$(2) \quad w(\zeta + z\eta) \leq C'_\epsilon + H(\text{Im } \zeta) + \epsilon |\text{Im } \zeta| \\ + N \log |i(1 + |\zeta|) + z|\eta| + y(H(\eta) + \epsilon |\eta|),$$

where $C'_\epsilon = C_\epsilon + N \log \sqrt{2}$. Here the term involving z is harmonic for $y > 0$, belongs to V and has mass 0 at infinity. Hence, by Lemma 1.2, the definition of the support function $G(\eta) = M(\eta)w$ of w and Theorem 2.1, we have $G(\eta) \leq H(\eta) + \epsilon |\eta|$ so that $G(\eta) \leq H(\eta)$. Further, by Lemma 1.2, we can replace the last term in (2) by $yG(\eta)$. Hence, taking $\zeta = \xi \in R^n$ and $y = 1$ we get

$$(3) \quad w(\xi + i\eta) \leq C + N \log(1 + |\zeta|) + G(\eta),$$

where C is a constant. Next, let $g \in \mathcal{D}$. Then, for all $\eta \in R^n$,

$$(4) \quad (2\pi)^n \int f(x)g(x) dx = \int \hat{f}(\xi + i\eta)g(-\xi - i\eta) d\xi,$$

where the left side denotes the value of f at g . Also, we have the estimate

$$|\hat{g}(\xi + i\eta)| \leq c_p(1 + |\xi + i\eta|)^{-p\varphi(\eta)},$$

where p is any integer and φ is the support function of $K(g)$. Putting $p = N + n + 1$ and replacing η by $y\eta$ and letting $y \rightarrow \infty$, we see that (4) vanishes whenever there is an η such that $G(\eta) + \varphi(-\eta) < 0$. But this is the case whenever $K(g)$ does not intersect the convex set L whose support function is G . In fact,

then L and $K(g)$ are separated by a hyperplane, i.e. there is an η such that

$$(5) \quad G(\eta) = \max_{x \in L} x\eta < \inf_{x \in K(g)} x\eta = -\varphi(-\eta).$$

Hence $K(f) \subset L$ so that $G(\eta) \geq H(\eta)$ and, consequently, $G(\eta) = H(\eta)$. This finishes the proof.

Remark. The last part of the proof has not used the full force of Theorem 4.1. Replacing $\mathfrak{D}$ by a linear subspace of non-quasi-analytic functions and $\mathfrak{D}'$ by the dual of that space, we can allow more than logarithmic growth of the Fourier-Laplace transforms and still use Theorem 4.1. The lemma remains true for instance for the generalized distributions of Beurling (see [1]).

In the following theorem, $K(f)$ is defined to be empty when f vanishes identically.

Theorem 5.1 (Titchmarsh-Lions). *If f_1, f_2 are distributions with compact supports then*

$$(5) \quad K(f_1 * f_2) = K(f_1) + K(f_2).$$

Proof. If f_1 or f_2 vanishes identically, (15) is trivial, so that we shall assume that neither f_1 nor f_2 vanishes identically. Then, if $f_3 = f_1 * f_2$ and H_j is the support function of $K(f_j)$, $j = 1, 2, 3$, Lemma 5.1 and Theorem 4.1 give the equivalent statement that $H_3 = H_1 + H_2$.

Remark. By a regularization of f_1 and f_2 , (5) follows from the special case when $f_1, f_2 \in \mathfrak{D}$. See Hörmander [4].

Finally, we shall prove a result by Lions [6] which is a purely geometrical consequence of Theorem 4.1. Let $C \subset R^n$ be a closed convex proper cone and let A_C be the collection of closed sets S contained in sets of the form $L + C$ where L is compact but may depend on S . Since $C + C = C$, A_C is closed under addition. Let $\mathfrak{D}'_C$ be all $f \in \mathfrak{D}'$ whose supports $S(f)$ belong to A_C . Choose L compact such that $S(f_1), S(f_2) \subset L + C$. Then if x belongs to some compact set K , $f_1(x - y) + f_2(y)$ vanishes unless $y \in (L + C) \cap (K - L - C)$ which is compact. Hence the convolution $f_3 = f_1 * f_2$ is defined whenever $f_1, f_2 \in \mathfrak{D}'_C$. An easy argument shows that $S(f_3) \subset S(f_1) + S(f_2)$ and hence $f_3 \in \mathfrak{D}'_C$ so that $\mathfrak{D}'_C$ is closed under convolution. When $f \in \mathfrak{D}'_C$, put $K_C(f) = K(f) + C$. We have

Theorem 5.2. *If $f_1, f_2 \in \mathfrak{D}'_C$, then*

$$K_C(f_1 * f_2) = K_C(f_1) + K_C(f_2).$$

Note. Taking $C =$ the origin, we get Theorem 4.1.

Proof. As before, we can assume that neither f_1 nor f_2 vanishes identically. Choose a sequence $\varphi_\nu \in \mathfrak{D}$ such that $0 \leq \varphi_\nu \leq 1$ and $\varphi_\nu(x) = 1$ when $|x| \leq \nu$ and $\varphi_\nu(x) = 0$ when $|x| \geq \nu + 1$. Put $f_{1\nu} = f_1 \varphi_\nu$, $f_{2\nu} = f_2 \varphi_\nu$, $f_{3\nu} = f_{1\nu} * f_{2\nu}$. Then $S_{1\nu} = S(f_{1\nu}) \subset S_1 = S(f_1)$ and $S_{2\nu} = S(f_{2\nu}) \subset S_2 = S(f_2)$ are compact

and increase with ν to S_1 and S_2 respectively. Hence $K(S_{1\nu})$ and $K(S_{2\nu})$ increase with ν to $K(S_1)$ and $K(S_2)$ respectively. Theorem 4.1 gives

$$(6) \quad K(S_{3\nu}) = K(S_{1\nu}) + K(S_{2\nu}).$$

Moreover, $S_3 = S(f_3) \subset S_1 + S_2$ implies that

$$(7) \quad K(S_3) \subset K(S_1) + K(S_2).$$

Put $S = S_3$, $S_\nu = S_{3\nu}$. By (6), $K(S_\nu)$ increases and, since $f_{3\nu}$ is independent of ν on any given compact set L when ν is large enough, we have $S_\nu \cap L = S \cap L$ when ν is large enough. Further, by (6) and (7), there exists a compact set L such that $S \subset L + C$, $S_\nu \subset L + C$ for all ν . Hence the hypothesis (1.5) is true and we conclude that

$$K(S_\nu) \subset K(S) + C$$

for all ν . Hence

$$K(S_{1\nu}) + K(S_{2\nu}) \subset K(S) + C$$

so that

$$K(S) + C \supset K(S_1) + K(S_2).$$

Combining this with (7) and observing that $C + C = C$, we get the desired result.

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