*Map Cohomology Operations and Enumeration of Vector Bundles**

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1. Introduction. The present paper is a continuation of the work of Dold-Whitney [6] on the classification of vector bundles over a finite dimensional space. In the course of the present work, a new tool, map-cohomology operations, is described. This concept proves useful in the classification of maps by homotopies.

Basic to this work is the fact that the oriented vector bundles over a space X are in a 1 — 1 correspondence with $[X; BSO(n)]$, the homotopy classes of maps from X to $BSO(n)$ [4], while the stable classes of oriented bundles over X correspond to $[X; BSO]$.

Some work has been done toward classifying bundles over X using Postnikov systems. $[X; BSO]$ is an Abelian group given to us by K -theory; if $\dim X \leq 7$, we have an exact sequence:

$$0 \rightarrow H^4(X; \mathbb{Z}) \xrightarrow{d} [X; BSO] \xrightarrow{w} w_2(X) \rightarrow 0,$$

where $w_2(X) = \{x \in H^2(X; \mathbb{Z}_2) \mid \beta_2 x^2 = 0\}$, where $w[f] = f^*w_2$ and where $dz = [g]$, where $g^*w_4 = \pi z$. In [6], Dold and Whitney classify all oriented vector bundles over a 4-dimensional complex. In [7], James and Thomas classify n -dimensional realizations of a stable vector bundle over an n -dimensional complex if the bundle is oriented or if n is odd. In the present paper we arrive at formulas for counting realizations by a slightly different method.

Generally, if X is a finite dimensional complex and if ξ is a stable oriented bundle over X , and if n is sufficiently large, the number of realizations of ξ as an n -bundle (*i.e.*, n -plane bundle), given that such a realization exists, may be put into a 1 — 1 correspondence, in a non-canonical way, with K/L , where K is a cohomology group of X and L is a subgroup. We shall derive specific formulae for L in terms of ξ in certain cases (*cf.* 5.8, 5.10).

Throughout, we let β_2 denote the Bokstein of the coefficient sequence $\mathbb{Z} \rightarrow$

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$Z \rightarrow Z_2$ and we let $\pi: H^*(\ ; Z) \rightarrow H^*(\ ; Z_2)$ be the map induced by $Z \rightarrow Z_2$. We assume that all spaces have base points and that all homotopies are base-point preserving. We let $p_i: X_1 \times X_2 \rightarrow X_i$, for $i = 1$ or 2 , denote the projection and $\Delta: X \rightarrow X \times X$ denote the diagonal map.

2. Map-cohomology operations.

2.1 Let B be any space with base point, let p and q be integers and let M and N be Abelian groups. Let $\text{Map}(X, A; B)$ be the set of maps from X to B which send A to the basepoint. A map-cohomology operation of type $\{B, p, q, M, N\}$ is a collection of functions θ from $\text{Map}(X, A; B) \times H^p(X, A; M)$ to the set of subsets of $H^q(X, A; N)$. There is one such function for each cellular pair (X, A) , and they are subject to the following conditions:

- (i) If $f \sim f' \text{ rel } A$, then $\theta(f, x) = \theta(f', x)$.
- (ii) If $g: (X', A') \rightarrow (X, A)$ is any map, then $g^*\theta(f, x) \subset \theta(fg, g^*x)$.
We say θ is *homogeneous* if
- (iii) for any map f , $0 \in \theta(f, 0)$.

We define $D\theta$, the *domain* of θ , to be the set of pairs (f, x) where $\theta(f, x)$ is non-empty. If $\theta(f, x)$ is a one-element set for all (f, x) , we say that θ is *primary*. If ψ is a $\{B, p, r, M, G\}$ map-cohomology operation, we say that ψ is *secondary* to θ provided $(f, x) \in D\psi$ if and only if $0 \in \theta(f, x)$.

2.2 If θ is a $\{B, p, q, M, N\}$ map-cohomology operation, and if $e: C \rightarrow B$ is any map, then we define a $\{C, p, q, M, N\}$ map-cohomology operation $e^*\theta$ by the formula $e^*\theta(f, x) = \theta(ef, x)$.

2.3 If θ is a map-cohomology operation, we say that β is a *choice function subsidiary* to θ if β is defined on $D\theta$, and $\phi \neq \beta(f, x) \subset \theta(f, x)$ for all $(f, x) \in D\theta$.

3. Classification of liftings. Throughout this section we let $\{E, \pi, B, K, K\}$ be a principal K -bundle in the sense of Steenrod [4], where $K = K(G, n)$ is an Eilenberg-MacLane space and G is an Abelian group. We let $y: E \rightarrow BK = K(G, n+1)$ be the classifying map for the bundle, and let $k \in H^{n+1}(B, G)$ be the k -invariant, *i.e.* the image of the fundamental class of BK under y^* .

If (X, A) is any cellular pair, and if f and g are maps from X to E which send A to the basepoint of E , and if F is a homotopy $\text{rel } A$ of πf with πg , we let $\delta^n(f, g; F) \in H^n(X, A; G)$ denote the obstruction to lifting F to a homotopy $\text{rel } A$ of f with g . If $\{K, 0, *, K, K\}$ is the K -bundle over a one-point space $*$, and if $h: (X, A) \rightarrow (K, k_0)$ is any map, (where k_0 is the basepoint of K) then $\delta^n(h, c; 0) = h^*\alpha$, where $c: X \rightarrow K$ is the constant map, $0: X \times I \rightarrow *$ is the trivial homotopy, and α is the fundamental class of K .

We shall now let $f: X \rightarrow B$ be any map, where X is some cell complex with basepoint. Now f may be lifted to E if and only if $f^*k = 0$; we shall be concerned with the classification of these liftings up to homotopy.

3.1 Let $g: X \rightarrow E$ be a lifting of f . We define a subgroup L_f of $H^n(X; G)$ by

$L_f = \{\delta^n(g, g; F) \mid F \text{ is a homotopy of } f \text{ with itself}\}$. If $h: X \rightarrow E$ is any other lifting of f , then:

$$\delta^n(h, h; F) = \delta^n(h, g; fp_1) + \delta^n(g, g; F) + \delta^n(g, h; fp_1) = \delta^n(g, g; F),$$

thus L_f does not depend on the choice of g .

3.2 Theorem. *Let $g: X \rightarrow E$ be a lifting of f . Let W be the set of equivalence classes of liftings of f , where two liftings are equivalent iff they are homotopic maps. Let $\Phi: W \rightarrow H^n(X; G)/L_f$ be defined as follows: $\Phi[h] = \delta^n(h, g; fp_1) + L_f$. Then Φ is one-to-one and onto.*

Proof. If H is any homotopy of h_1 with h_2 , then $\delta^n(h_1, h_2; \pi H) = 0$. Thus $\delta^n(h_2, g; fp_1) - \delta^n(h_1, g; fp_1)$

$$= \delta^n(h_1, g; fp_1) + \delta^n(h_1, h_2; \pi H) = \delta^n(h_2, h_2; \pi H) \in L_f,$$

whence Φ is well-defined. If $\Phi[h_1] = \Phi[h_2]$, then $\delta^n(h_2, g; fp_1) - \delta^n(h_1, g; fp_1) = \delta^n(h_2, h_2; F)$ for some homotopy F of f with itself. We have

$$\delta^n(h_1, h_2; F) = \delta^n(h_1, g; fp_1) + \delta^n(g, h_2; fp_1) + \delta^n(h_2, h_2; F) = 0.$$

Thus F may be lifted to a homotopy H of h_1 with h_2 . This shows that Φ is one-to-one.

Finally, if a $H^n(X; G)$, we shall find a map $h: X \rightarrow E$ such that $\delta^n(h, g; fp_1) = a$. The bundle induced over E by the map $\pi: E \rightarrow B$ is simply a product bundle $p_1: E \times K \rightarrow E$. Let $\mu: E \times K \rightarrow E$ be the bundle map induced by π ; the following diagram is commutative:

$$\begin{array}{ccc} E \times K & \xrightarrow{\mu} & E \\ \downarrow p_1 & & \downarrow \pi \\ E & \xrightarrow{\pi} & B \end{array}$$

Let $q: X \rightarrow K$ be a map such that $q^*a = a$ (a = fundamental class). Let $h = \mu(g \times q)\Delta: X \rightarrow E$; then $\delta^n(h, g; fp_1) = \delta^n((g \times q)\Delta, (g \times c)\Delta; gp_1) = q^*a = a$, where $c: X \rightarrow K$ is the constant map. Thus Φ is onto.

4. Classification of liftings by map-cohomology operations. In this section, we describe the group L_f in terms of map-cohomology operations, where $f: X \rightarrow B$ is a map from a cell complex to a space with finitely many non-zero homotopy groups. Throughout, assume that m is a positive integer, and that we are given:

- (i) a positive integer $n(k)$ for each $0 < k \leq m$,
- (ii) an Abelian group G_k for each k ($0 < k \leq m$),
- (iii) a cell complex with basepoint E_k for each k , where E_0 is a one-point space,
- (iv) a map $y_k: E_{k-1} \rightarrow BK_k$ for each $0 < k \leq m$, where BK_k is an Eilenberg-MacLane space of type $(G_k, n(k) + 1)$ which is a topological group [3],

(v) a map $\pi_k : E_k \rightarrow E_{k-1}$ for each k ($0 < k \leq m$) such that $\{E_k, \pi_k, E_{k-1}, K_k, K_k\}$ is the principle K_k -bundle induced by the universal K_k -bundle over BK_k and the map y_k , where $K_k = \Omega BK_k$, the loops on BK_k .

For all k ($0 < k \leq m$), let $\lambda_k : K_k \rightarrow E_k$ be the inclusion map. We let $\mu_k : E_k \times K_k \rightarrow E$ be the bundle map induced by π_k as described in 3.2. We now let $f : X \rightarrow E_m$ be any map which sends A to the basepoint of E_m , where (X, A) is a cellular pair, and we let $f_k = \pi_{k+1} \cdots \pi_m f$ be the composite of f with the appropriate fiberings, for all $0 \leq k \leq m$. Let $(K, M) = (X \times I, X \times \partial I \cup A \times I)$. We shall compute $L = L_{f_{m-1}} \subset H^{n(m)}(X, A; G_m)$.

4.1 We say that a homotopy H of f_k with itself is an (f_k, z) homotopy for some $z \in H^{n(k)-1}(X, A; G_k)$ if $H = \mu_k(f_k p_1 \times F) \Delta$ (where $\Delta : S \rightarrow S \times S$ is always the diagonal map, for any space S) for some map $F : (K, M) \rightarrow K_k$ where $F^* \alpha = sz$ (s = suspension). Clearly any two (f_k, z) homotopies are homotopic rel M rel $f_{k-1} p_1$. (In general, if s_0 and s_1 are maps from any space X to any space Y , and if A is a subset of X , and if $g : Y \rightarrow Z$ is any map, where Z is any space, then we say that s_0 is homotopic to s_1 rel A rel g if there is a homotopy $\{s_t\}$ connecting s_0 with s_1 such that $s_t|_A = s_0|_A$ and $gs_t = gs_0$ for all $0 \leq t \leq 1$.) We say that any homotopy R of f_p with itself is (f, p) admissible; if, in addition, the composite $\pi_{k+1} \cdots \pi_p R$ is an (f_k, z) homotopy of f_k with itself, for some $z \in H^{n(k)-1}(X, A; G_k)$, we say that R is (f, p, k, z) admissible. For any (f, p) admissible homotopy R , let $D(f_{p+1}, R)$ denote $\delta^{n(p+1)}(f_{p+1}, f_{p+1}; R) \in H^{n(p+1)}(X, A; G_{p+1})$.

4.2 For each k ($0 < k < p \leq m$) and each $z \in H^{n(k)-1}(X; G_k)$ we let $\theta_{p,k}(f, z) = \{D(f_p, H) \mid H \text{ is an } (f, p-1, k, z) \text{ admissible homotopy}\}$.

Proposition. $\theta_{p,k}$ is a map-cohomology operation of type $\{E_m, n(k) - 1, n(p), G_k, G_p\}$.

Proof. Let H be a homotopy rel A of h^0 with h^1 , where h^0 and h^1 are any maps from X to E_m which send A to the basepoint. Let $H_r = \pi_{r+1} \cdots \pi_m H$, a homotopy of h_r^0 with h_r^1 , for each r ($0 < r \leq m$). Let z be any element of $H^{n(k)-1}(X, A; G_k)$, and let a be any element of $\theta_{p,k}(h^0, z)$. Now $a = D(h_p^0, R_0)$ for some $(h^0, p-1, k, z)$ admissible homotopy, R_0 . Let $S_0 = \mu_k(h_k^0 p_1 \times F) \Delta$ where $F : (K, M) \rightarrow K_k$ is any map such that $F^* \alpha = sz$. For each $t \in I$, let $S_t = \mu_k(h_k^t p \times F) \Delta$, a homotopy of h_k^t with itself, where $h_k^t(x)$ is defined to be $H(x, t)$ for all $x \in X$. Now the composition $\pi_{k+1} \cdots \pi_p : E_p \rightarrow E_k$ has the Covering Homotopy Extension Property for K relative to $M[1, III, 2]$, hence we may find a homotopy $\{R_t\}$ of R_0 such that $\pi_{k+1} \cdots \pi_{p-1} R_t = S_t$ for all $t \in I$, and each R_t is a homotopy of h_{p-1}^t with itself. Now R_1 is an $(h^1, p-1, k, z)$ admissible homotopy, hence $a = d(h_p^0, R_0) = D(h_p^1, R_1) \in \theta_{p,k}(h^1, z)$. Thus $\theta_{p,k}(h, z)$ does not depend on the homotopy class of h .

Let now $g' : (X', A') \rightarrow (X, A)$ be any map, where (X', A') is another cellular pair. Let $z \in H^{n(k)-1}(X; G_k)$ and $a \in \theta_{p,k}(f, z)$. Then $a = D(f_p, H)$ for some $(f, p-1, k, z)$ admissible homotopy H . Let R be the composite $\pi_{k+1} \cdots \pi_{p-1} H$; then $R = \mu_k(f_k p_1 \times F) \Delta$ where $F : (K, M) \rightarrow K_k$ is a map such that $F^* \alpha = sz$. Now

$\pi_{k+1} \cdots \pi_{p-1}(H(g \times 1)) = \mu_k(f_k p_1 \times F) \Delta(g \times 1) = \mu_k(f_k g p_1 \times F(g \times 1)) \Delta$, a homotopy of $f_k g$ with itself, and $(F(g \times 1))^* \alpha = s g^* z$, whence $H(g \times 1)$ is an $(f g, p-1, k, g^* z)$ admissible homotopy.

Thus $g^* a = D(f g, H(g \times 1)) \in \theta_{p,k}(f g, g^* z)$. We conclude that $g^* \theta_{p,k}(f, k) \subset \theta_{p,k}(f g, g^* z)$.

4.3 Proposition. For all $0 < k < m$, $\theta_{k+1,k}$ is a primary map-cohomology operation.

Proof. For any $z \in H^{n(k)-1}(X; G_k)$, let $a_1, a_2 \in \theta_{k+1,k}(f, z)$. For $i = 1$ or 2 , $a_i = \delta^{n(k+1)}(f_{k+1}, f_{k+1}; H_i)$, where $H_i = \mu_k(f_k p_1 \times F_i) \Delta, F_i^* \alpha = s z, F_1 \sim F_2 \text{ rel } L$, whence $H_1 \sim H_2 \text{ rel } L$. Thus $a_1 = a_2$.

4.4 Proposition. For all $0 < k < p < m$, $\theta_{p+1,k}$ is secondary to $\theta_{p,k}$.

Proof. Let $z \in H^{n(k)-1}(X; G_k)$. We need only show that $\theta_{p+1,k}(f, z)$ is non-empty iff $0 \in \theta_{p,k}(f, z)$. Note that $\theta_{p+1,k}(f, z)$ is non-empty iff there exists some (f, p, k, z) admissible homotopy.

Suppose H is (f, p, k, z) admissible. Then $\pi_p H$ is $(f, p-1, k, z)$ admissible. Now $0 = \delta^{n(p)}(f_p, f_p; \pi_p H) \in \theta_{p,k}(f, z)$.

On the other hand suppose $0 \in \theta_{p,k}(f, z)$. Then $0 = \delta^{n(p)}(f_p, f_p; S)$ for some $(f, p-1, k, z)$ admissible homotopy S . S may be lifted to a homotopy H which is (f, p, k, z) admissible.

4.5 Proposition. If H is any homotopy of f_k with itself where $\pi_k H = f_{k-1} p_1$, then H is an (f_k, z) homotopy for some $z \in H^{n(k)-1}(X; G_k)$.

Proof. We recall that $E_k = \{(e, \sigma) \in E_k \times \mathcal{L} \mid y_k e = \Pi \sigma = \sigma(1) \in BK_k\}$, where $\mathcal{L}$ is the path space of BK_k , that is, the space of all maps $\sigma: I \rightarrow BK_k$ such that $\sigma(0)$ is the base point $*$ of BK_k . We consider $\mathcal{L}$ to be a topological group under pointwise multiplication in BK_k . Then $K_k = \Omega BK_k \subset \mathcal{L}$ is a subgroup.

We may define $\mu = \mu_k: E_k \times K_k \rightarrow E_k$ as follows: if $((e, \sigma), \tau) \in E_k \times K_k$, let $\mu((e, \sigma), \tau) = (x, \sigma \cdot \tau)$.

$$\begin{array}{ccccc} E_k \star K_k & \xrightarrow{\mu} & E_k & \xrightarrow{p_2} & \mathcal{L} \\ & & \downarrow \pi & & \downarrow \pi \\ & & E_{k-1} & \xrightarrow{y_k} & BK_k \end{array}$$

Let $F: (K, M) \rightarrow K_k$ be the map given by $F(x, t) = (p_2 f x)^{-1} \cdot (p_2 H(x, t))$ for all $(x, t) \in K$, and let $z = F^* \alpha$. Then $\mu(f p_1 \times F) \Delta(x, t) = \mu(f x, F(x, t)) = \mu((\pi f x, p_2 f x), (p_2 f x)^{-1} (p_2 H(x, t))) = (\pi f x, p_2 H(x, t)) = (\pi H(x, t), p_2 H(x, t)) = H(x, t)$ for all $(x, t) \in K$. Thus H is an (f_k, z) homotopy.

4.6 Theorem. For any $1 < k < p \leq m$ and for any $z \in H^{n(k-1)-1}(X; G_{k-1})$, we have that $\theta_{p,k-1}(f, z)$ is a coset of the union of $\theta_{p,k}(f, x)$ over all $x \in H^{n(k)-1}(X; G_k)$.

Proof. Suppose that $a_1, a_2 \in \theta_{p,k-1}(f, z)$. Then $a_i = D(f_p, A_i)$, where A_i

is an $(f, p-1, k-1, z)$ admissible homotopy, for $i = 0$ or 1 . Then $A_1 - A_2$ is $(f, p-1, k-1, 0)$ admissible; hence $\pi_k \cdots \pi_p(A_1 - A_2) \sim fp_1$. Thus $A_1 - A_2$ is homotopic to an $(f, p-1, k, x)$ admissible homotopy (cf. 4.5) $H; a_1 - a_2 = D(f_p, H) \in \theta_{p,k}(f, x)$.

Suppose $a = D(f_p, A)$ and $b = D(f_p, B)$ for some $(f, p-1, k-1, z)$ admissible homotopy A and some $(f, p-1, k, x)$ admissible homotopy B , for some $x \in H^{n(k)-1}(X; G_k)$. Then B is $(f, p-1, k-1, 0)$ admissible; $A + B$ is $(f, p-1, k-1, z)$ admissible and then $a + b = D(f_p, A + B) \in \theta_{p,k-1}(f, z)$.

4.7 Theorem. Let $f: X \rightarrow E_m$ be a map. Then if $m = 1$, $L_f = 0$. If $m > 1$, L_f is the union of $\theta_{m,1}(f, z)$ over all $z \in H^{n(1)-1}(X; G_1)$.

Proof. If $m = 1$, any $(f, 0)$ admissible homotopy is trivial. If $m > 1$, and if $a \in L_f$, then $a = \delta^{n(m)}(f, f; R)$ for some homotopy R of f_{m-1} with itself. Now $\pi_2 \cdots \pi_{m-1}$ is an (f_1, z) homotopy for some $z \in H^{n(1)-1}(X; G)$ (cf. 4.5), thus R is $(f, m-1, 1, z)$ admissible and $a \in \theta_{m,1}(f, z)$.

If, on the other hand, $a \in \theta_{m,1}(f, z)$ for some z , then $a = D(f, R)$ for some $(f, m-1, 1, z)$ admissible homotopy R . R is $(f, m-1)$ admissible, thus $a \in L_f$.

5. Specific computations of L_f and application to enumeration of vector bundles. Through 5.1 and 5.2, we shall let X be a cell complex with base-point x_0 . Let QX be the space obtained from $X \times I$ by collapsing $x_0 \times I$ to a point and by identifying $(x, 0)$ with $(x, 1)$ for all $x \in X$. Let $J: X \times I \rightarrow QX$ be the identification map, and let $k: X \rightarrow QX$ be the inclusion map, where $kx = J(x, 0)$ for all $x \in X$. If $f: X \rightarrow Y$ is any map, where Y is any space, and if F is a homotopy rel x_0 of f with itself, let $UF: QX \rightarrow Y$ denote the unique map such that $(UF)J = F$.

5.1 Let $(K, M) = (X \times I, x_0 \times I \cup X \times \partial I)$. Let $j: QX \rightarrow (QX, kX)$ be the identity and let $J: (K, M) \rightarrow (QX, kX)$ be the collapsing map. With coefficients in a commutative ring, we have a diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & H^p(QX, kX) & \xrightarrow{j^*} & H^p(QX) & \xrightleftharpoons[(Up_1)^*]{k^*} & H^p(X) & \longrightarrow 0 \\ & \downarrow J^* & \simeq & & & & \\ & H^p(K, M) & \xleftarrow[\simeq]{s} & H^{p-1}(X) & & & \end{array}$$

where the top row is an exact sequence split by $(Up_1)^*$.

For any $a \in H^*(X)$, let $Qa = j^*(J^*)^{-1}sa \in H^*(QX)$ and let $qa = (Up_1)^*a \in H^*(QX)$. We then have that $(qa) \cup (qb) = q(a \cup b)$, $(Qa) \cup (Qb) = 0$, and $(qa) \cup (Qb) = Q(a \cup b)$, for all $a, b \in H^*(X)$.

5.2 Lemma. Let $\eta = \{E, \pi, B, K, K\}$ be a principal K -bundle, where $K = K(G, n)$, and let $y: B \rightarrow BK$ be the classifying map for η . Let $f: X \rightarrow E$ be any map and let $F: f \sim f$ be a homotopy. Then $(UF)^*y^*\alpha = -Q\delta^n(f, f; F)$, where α is the fundamental class of BK .

Proof. Let $x_0 \in X$ be the base point. Let $g: kX \rightarrow E$ be the unique map where $gkx = fx$ for all $x \in X$, and let $0: kx_0 \rightarrow E$ be the trivial map ($kx_0 \in kX \subset QX$). Now $s\delta^n(f, f; F) = \gamma^{n+1}(gJ; F)$, the obstruction to lifting F to a map on $X \times I$ which agrees with gJ on $X \times \partial I \cup x_0 \times I$ (s = suspension). Thus $(UF)^*y^*\alpha = -\gamma^{n+1}(0; UF) = -j^*\gamma^{n+1}(g; UF) = -j^*(J^*)^{-1}\gamma^{n+1}(gJ; F) = -j^*(J^*)^{-1}s\delta^n(f, f; F) = -Q\delta^n(f, f; F)$.

5.3 For $t = 3, 4, 5$ or 6 we construct a Postnikov Tower for $BSO(t)$ over BSO in the manner of Mahowald [9].

$$\begin{array}{ccccc}
 K_3 & \xrightarrow{\lambda_3} & E_3 & \xleftarrow{P_3} & BSO(t) \\
 & & \downarrow \pi_3 & & \downarrow i \\
 K_2 & \xrightarrow{\lambda_2} & E_2 & \xrightarrow{P_2} & BSO \\
 & & \downarrow \pi_2 & & \nearrow \\
 K_1 & \xrightarrow{\lambda_1} & E_1 & \xrightarrow{Y_2} & BK_3 \\
 & & & & \downarrow \\
 & & & & BK_2
 \end{array}$$

The map i is the classifying map for the universal t -bundle over $BSO(t)$. $K_1 = K(Z_2, 2)$, $K_2 = K(Z, 4)$, and $K_3 = K(G_3, n(3))$. G_3 , $n(3)$ and $P_2^*y_3^*\alpha$ (α = fundamental class of BK_3) are given as:

t	G_3	$n(3)$	$P_2^*y_3^*\alpha$
3	Z_2	3	w_4
4	Z	4	$\beta_2 w_4$
5	Z_2	5	w_6
6	Z	6	$\beta_2 w_6$

G_3 is simply $\pi_{n(3)}(V_t)$, the first non-zero homotopy group of V_t , the space of t -planes in R^∞ . P_2 is a 7-W.H.E. (weak homotopy equivalence) since $\pi_k(BSO) = 0$ for all $4 < k < 8$. P_3 is an $n(3)$ -W.H.E., and, if $t = 3$, a 4-W.H.E. since $\pi_4(V_3) = 0$.

We shall let σ_2 denote $(\lambda_1^*)^{-1}u_2 \in H^2(E_1; Z_2)$; $(\pi_2 P_2)^*\sigma_2 = w_2$, since $\pi_2 P_2$ is a 2-W.H.E. We have that $y_2^*v_5 = 2\beta_4 p\sigma_2 = \beta_2 \sigma_2^2$ [6], where $p: H^2(-; Z_2) \rightarrow H^4(-; Z_4)$ is the Pontrjagin square, and β_4 is the Bockstein of the sequence $Z \rightarrow Z \rightarrow Z_4$. For each $2 \leq i \leq 7$, we shall let o_i denote $(P_2^*)^{-1}w_i$.

5.4 Let $\nu_1 = \mu_1(1 \times \lambda_1^{-1}): E_1 \times E_1 \rightarrow E_1$. We may find a multiplication $\nu_2: E_2 \times E_2 \rightarrow E_2$ consistent with ν_1 , i.e. such that $\nu_2|_{E_2 \vee E_2} = 1 \vee 1$ and $\pi_2 \nu_2 = \nu_1(\pi_2 \times \pi_2)$.

Proof. The k -invariant, $\beta_2 \sigma_2^2$ is primitive, that is, $\nu_1^* \beta_2 \sigma_2^2 = \beta_2 \sigma_2^2 \times 1 + 1 \times \beta_2 \sigma_2^2$. Thus we may find a multiplication ν_2 as above [8, Thm. 1.3].

5.5 Let $(U, V) = (P_\infty \times I, P_\infty \times \partial I \cup p_0 \times I)$, where $p_0 \in P_\infty = K(Z_2, 1)$ is the base point. There is an embedding $i: P_\infty \rightarrow SO$ where $i^*\{k+1\} = u_1^k$ for all k ($\{k+1\}$ is defined in [5, p. 48]). ΩSO is of the homotopy type of BSO , hence i corresponds to a map $R: (U, V) \rightarrow BSO$. The transgression of $\{k+1\}$ is w_{k+1} , hence $R^*w_{k+1} = su_1^k$ for all $k \geq 1$. Let $S: (U, V) \rightarrow K_1$ be the map where $S^*u_2 = su_1$ (s = suspension homomorphism).

Since $(\pi_2 P_2 R)^* \sigma_2 = su_1$, $\pi_2 P_2 R$ is homotopic to $\lambda_1 S$; by the Covering Homotopic Extension Property [1, III, 2] we may choose a map $H: (U, V) \rightarrow E_2$ homotopic to $P_2 R$ such that $\pi_2 H = \lambda_1 S$. For each $2 \leq k+1 \leq 7$, $H^*o_{k+1} = su_1^k$.

5.6 Proposition. *Let $g: X \rightarrow E_3$ be any map and let $f = \pi g$. Let $z \in H^3(X; Z)$ and let $\theta_{3,2}(g, z) = \{\delta^{n(3)}(g, g; R) \mid R \text{ is } (g, 2, 2, z) \text{ admissible}\}$ (cf. 4.2). Then $\theta_{3,2}(f, z)$ consists of the element*

$$\begin{aligned} \pi z & \quad \text{if } t = 3, \\ 0 & \quad \text{if } t = 4, \\ Sq^2 \pi z + \pi z f^* o_2 & \quad \text{if } t = 5, \\ \beta_2(Sq^2 \pi z + \pi z f^* o_2) & \quad \text{if } t = 6. \end{aligned}$$

Proof. First we note that $\theta_{3,2}$ is a primary map-cohomology operation (cf. 4.3). Let $F: (U, V) \rightarrow K_2$ be a map where $F^*v_4 = sz$, and let $R = \mu_2(fp_1 \times F) \Delta: f \sim f$, an $(f, 2, 2, z)$ admissible homotopy. $F^*\pi v_4 = s\pi z$, whence $(UF)^*\pi v_4 = Q\pi z$. $UR = \mu_2(fUp_1 \times UF) \Delta: QX \rightarrow E_2$ and $\mu_2 o_4 = o_4 \times 1 + 1 \times \pi v_4$. Thus $(UR)^*o_4 = gf^*o_4 + Q\pi z$. Now $F^*o_2 = 0$, whence $(UF)^*o_2 = 0$. Since $\mu_2^*o_2 = o_2 \times 1$, we have $(UR)^*o_2 = gf^*o_2$.

Using the Wu formula for $Sq^i w_i$, we conclude:

If $t = 3$, then $f^*o_4 = 0$, whence $(UR)^*o_4 = Q\pi z$.

If $t = 4$, then $f^*\beta_2 o_4 = 0$, whence $(UR)^*\beta_2 o_4 = 0$.

If $t = 5$, then $f^*o_6 = 0$, whence $(UR)^*o_6 = Q(\pi z f^* o_2 + Sq^2 \pi z)$.

If $t = 6$, then $f^*\beta_2 o_6 = 0$, whence $(UR)^*\beta_2 o_6 = Q\beta_2(\pi z f^* o_2 + Sq^2 \pi z)$.

Apply 5.2 and we are done.

5.7 Proposition. *For any map $g: X \rightarrow E_3$ and any $x \in H^1(X; Z_2)$, let*

$$\begin{aligned} \beta_{3,1}(g, x) &= x^3 + xf^*o_2 & \text{if } t = 3, \\ &= \beta_2(x^3 + xf^*o_2) & \text{if } t = 4, \\ &= x^5 + x^3 f^*o_2 + x^2 f^*o_3 + xf^*o_4 & \text{if } t = 5, \\ &= \beta_2(x^5 + x^3 f^*o_2 + x^2 f^*o_3 + xf^*o_4) & \text{if } t = 6, \end{aligned}$$

where $f = \pi g$. Then $\beta_{3,1}$ is a choice function subsidiary to $\theta_{3,1}$ (cf. 4.2, 2.3).

Proof. Let $H: (U, V) \rightarrow E_2$ and $v_2: E_2 \times E_2 \rightarrow E_2$ be as given in 5.5 and

5.4 respectively. Let $h: X \rightarrow P_\infty$ be a map where $h^*u_1 = x$, and let $R = \nu_2(fp_1 \times H(h \times 1))\Delta: f \sim f$. A routine computation shows that $\pi_2 R = \mu_1(\pi_2 fp_1 \times S(h \times 1))\Delta$, where $S: (P_\infty \times I, P_\infty \times \partial I \cup p_0 \times I) \rightarrow K_1$ is the suspension. R is $(g, 2, 1, x)$ admissible, since $(S(h \times 1))^*u_2 = sx$.

Now for all $2 \leq i \leq 7$, $\nu_2^*o_i = \sum_{j=0}^i \sigma_j \times \sigma_{i-j}$, where $o_0 = 1$ and $o_1 = 0$. Routine computation shows that $(UR)^*o_4 = gf^*o_4 + Q(x^3 + xf^*o_2)$ and $(UR)^*o_6 = gf^*o_6 + Q(x^5 + x^3f^*o_2 + x^2f^*o_3 + xf^*o_4)$. We note that $gf^*y_3^*\alpha = 0$, where α is the fundamental class of BK_3 ; the theorem follows now from 5.2.

5.8 Theorem. *If X is a finite complex of dimension $k(t)$, where $k(3) = 4$ and $k(t) = t$ for $t = 4, 5$ and 6 , and if ξ is a stable oriented vector bundle over X , then ξ may be represented as a t -bundle (unless $t = 3$ and $w_4 = 0$), and these representations may be put into a $1 - 1$ correspondence with $H^t(X; G)/L$, where $G = \mathbb{Z}_2$ if $t = 3$ or 5 , $G = \mathbb{Z}$ if $t = 4$ or 6 and where L consists of all*

$$\begin{aligned} x^3 + xw_2 + \pi y & \quad \text{if } t = 3, \\ \beta_2(x^3 + xw_2) & \quad \text{if } t = 4, \\ x^5 + x^3w_2 + x^2w_3 + xw_4 + Sq^2\pi y + \pi yw_2 & \quad \text{if } t = 5, \\ \beta_2(x^5 + x^3w_2 + x^2w_3 + xw_4 + Sq^2\pi y + yw_2) & \quad \text{if } t = 6, \end{aligned}$$

where $x \in H^1(X; \mathbb{Z}_2)$, $y \in H^3(X; \mathbb{Z})$ and w_i is the i^{th} Stiefel Whitney class of ξ .

Proof. $P_3: BSO(t) \rightarrow E_3$ is a $k(t)$ -W.H.E.; thus $[X: BSO(t)] \simeq [X; E_3]$.

The theorem follows from 4.6, 4.7, 5.6 and 5.7.

5.9 Let E_1 and E_2 be as in 5.3, and let E_3 be the next stage in a Postnikov system for BSO . $G_3 = \mathbb{Z}$, $n(3) = 8$, and the k -invariant, y^*v_9 , is $\beta_2 o_4^2 \in H^0(E_2; \mathbb{Z})$. Let $P_3: BSO \rightarrow E_3$ be an 8-W.H.E. We let $\{E_4, \pi_4, E_3, K_4, K_4\}$ be a $K_4 = K(\mathbb{Z}_2, 7)$ bundle induced by the map $y_4: E_3 \rightarrow BK_4$, where $P_3^*y_4^*u_8 = w_8$. We let $P_4: BSO(7) \rightarrow E_4$ be a 7-W.H.E.; $\pi_8(V_7) = 0$, so P_4 is an 8-W.H.E. For each $2 \leq i \leq 8$, let $s_i = (P_3^*)^{-1}w_i$.

Let now $g: X \rightarrow E_4$ be any map, let $f = \pi g$, and let $(K, M) = (X \times I, X \times \partial I \cup x_0 \times I)$, where $x_0 \in X$ is the basepoint. If $x \in H^1(X; \mathbb{Z}_2)$, we may find a $(g, 3, 1, x)$ admissible homotopy $R_1 = \nu_3(fp_1 \times F_1)\Delta$, where $\nu_3: E_3 \times E_3 \rightarrow E_3$ is a multiplication consistent with ν_2 (cf. 5.4) and $F_1: (K, M) \rightarrow E_3$ satisfies the condition $F_1^*s_{i+1} = sx^i$ for all $2 \leq i+1 \leq 8$ (cf. 5.5). We conclude that $\delta^7(g, g; R_1) = x^7 + x^5f^*s_2 + x^4f^*s_3 + x^3f^*s_4 + x^2f^*s_5 + xf^*s_6$.

If $y \in H^3(X; \mathbb{Z})$, let $T: (K, M) \rightarrow K_2$ be a map where $T^*v_4 = sy$. Then $\lambda_2 T$ has a lifting $F_2: (K, M) \rightarrow E_3$, since $T^*\lambda_2^*\beta_2 o_4 = T^*(\beta_2 \pi v_4) = 0$. Let $R_2 = \nu_3(fp_1 \times F_2)\Delta: f \sim f$; computation gives us that R_2 is $(g, 3, 2, y)$ admissible and $\delta^7(g, g; R_2) = Sq^2\pi yf^*s_2 + \pi yf^*s_4$.

If $z \in H^7(X; \mathbb{Z})$, let $F: (K, M) \rightarrow K_3$ be a map with $F^*v_8 = sz$. Then $R_3 = \nu_3(fp_1 \times F)\Delta: f \sim f$ is $(g, 3, 3, z)$ admissible and $\delta^7(g, g; R_3) = \pi z$.

5.10 We may apply 4.6 and 5.9 to obtain:

Theorem. *If X is a finite complex of dimension 8, and if ξ is a stable oriented*

bundle over X , then ξ may be represented as a 7-bundle if and only if $w_8 = 0$, in which case the representations may be put into a 1 — 1 correspondence with $H^7(X; Z_2)/L$, where L is the subgroup of all elements of the form:

$$x^7 + x^5w_2 + x^4w_3 + x^3w_4 + x^2w_5 + xw_6 + Sq^2\pi yw_2 + \pi yw_4 + \pi z,$$

where $x \in H^1(X; Z_2)$, $y \in H^3(X; Z)$, and $z \in H^7(X; Z)$, and w_i is the i^{th} Stiefel Whitney class of ξ .

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