*A Strong Mixing Condition and A Central Limit Theorem on Compact Groups**

M. ROSENBLATT

1. Introduction. A particular type of "strong mixing" implies the validity of the central limit theorem for dependent processes under appropriate further restraints [2, 6, 7]. Necessary conditions and sufficient conditions for this type of strong mixing in the case of Gaussian stationary processes have been derived [3, 4]. This paper obtains necessary and sufficient conditions for strong mixing and a related concept of uniform ergodicity in the case of a stationary process on a compact commutative group generated by taking the product of independent, identically distributed elements of the group. Sufficient conditions for strong mixing are obtained when the group is not commutative. The conditions are then used to obtain a central limit theorem on the group.

2. Preliminaries. Let $\{X_k; k = 0, \pm 1, \dots\}$ be a stationary Markov process with instantaneous state space Ω and a Borel field $\mathfrak{B}$ of events $B \subset \Omega$. The one step transition probability function is

$$(1) \quad P(x, B) = P\{X_{k+1} \in B \mid X_k = x\}, \quad x \in \Omega, \quad B \in \mathfrak{B}.$$

The instantaneous invariant measure

$$(2) \quad m(B) = P(X_k \in B)$$

satisfies

$$(3) \quad \int_{\Omega} P(x, B) m(dx) = m(B) = P(B).$$

In order that all operations required be meaningful, it is required that the transition probability function be measurable ($\mathfrak{B}$) in x for each $B \in \mathfrak{B}$ and a probability measure in $B \in \mathfrak{B}$ for each $x \in \Omega$. The higher step transition probability functions are given recursively by

$$(4) \quad \begin{aligned} P_{j+1}(x, B) &= P\{X_{k+j+1} \in B \mid X_k = x\} \\ &= \int P_j(y, B) P(x, dy), \quad j = 1, 2, \dots \end{aligned}$$

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The transition probability function induces an operator T acting on, for example, the bounded measurable ($\mathfrak{B}$) function f

$$(5) \quad \int P(x, dy)f(y) = (Tf)(x).$$

We shall be interested in the action of the operator T on various linear spaces of functions.

The Markov process $\{X_k; k = 0, \pm 1, \dots\}$ can be taken to be defined on the infinite product space $\dots \times \Omega \times \Omega \times \dots$. Let $\mathfrak{B}_n$ be the backward Borel field generated by $X_k, k \leq n$, and $\mathfrak{F}_n$ the forward Borel field generated by $X_k, k \geq n$. It will be convenient to assume that $\mathfrak{B}$ is the Borel field generated by any single random variable $X_k, k = 0, \pm 1, \dots$. The stationary Markov process is said to be *strongly* or *uniformly mixing* if

$$(6) \quad \sup_{\substack{B \in \mathfrak{B}_m \\ F \in \mathfrak{F}_{m+n}}} |P(B \cap F) - P(B)P(F)| \downarrow 0$$

as $n \rightarrow \infty$ (see [2, 6, 7]). Let τ be the *shift transformation* taking X_k into $X_{k+1}, k = 0, \pm 1, \dots$. By extension this yields a transformation taking bounded measurable functions on the sample space of the process into bounded measurable functions on the sample space. We shall also use the same symbol τ to denote the induced set transformation which takes, for example, sets $F \in \mathfrak{F}_m$ into sets $\tau F \in \mathfrak{F}_{m+1}$. Cogburn [1] has introduced the concept of *uniform ergodicity* which is related to strong mixing. The stationary Markov process is said to be uniformly ergodic if

$$(7) \quad \sup_{\substack{B \in \mathfrak{B}_0 \\ F \in \mathfrak{F}_0}} \left| \frac{1}{n} \sum_{k=1}^n P(B \cap \tau^k F) - P(B)P(F) \right| \downarrow 0$$

as $n \rightarrow \infty$. Actually the concepts of strong mixing and uniform ergodicity are meaningful for general stationary (or even nonstationary) processes. We shall, however, be concerned with them for stationary Markov processes. Notice that the term strong mixing as employed here is considerably stronger than the condition usually referred to under this name in ergodic theory.

3. An equivalent operator formulation. A simple argument is used in this section to give equivalent formulations for strong mixing and uniform ergodicity for stationary Markov processes in terms of the operator T . A similar discussion is to be found in the work of Cogburn [1].

Let us first look at strong mixing. For any $B \in \mathfrak{B}_m$ and $F \in \mathfrak{F}_{m+n}, n > 0$,

$$(8) \quad |P(B \cap F) - P(B)P(F)| \\ = \left| \int P(B | x_m) m(dx_m) \int [P_n(x_m, dx_{m+n}) - m(dx_{m+n})] P(F | x_{m+n}) \right|,$$

so that (6) holds if and only if

$$(9) \quad \sup_{F \in \mathfrak{F}_n} \int m(dx) \left| \int [P_n(x, dy) - m(dy)] P(F | y) \right| \downarrow 0$$

as $n \rightarrow \infty$. But this is in turn equivalent to

$$(10) \quad \sup_{\|f\|_\infty \leq 1} \int m(dx) \left| \int [P_n(x, dy) - m(dy)] f(y) \right| \downarrow 0$$

as $n \rightarrow \infty$. It is obvious that (10) implies (9). The converse implication follows by approximating f by step functions. The proof of Lemma 1 is complete.

Lemma 1. *A stationary Markov process with transition operator T and invariant probability measure m is strongly mixing if and only if*

$$(11) \quad \sup_{f \perp 1} \frac{\|T^n f\|_1}{\|f\|_\infty} \rightarrow 0$$

as $n \rightarrow \infty$, where $\|\cdot\|_p$ is the L_p norm with respect to the measure m and $f \perp 1$ means that $\int f(x)m(dx) = 0$.

Lemma 2 gives the analogous result for Cogburn's uniform ergodicity and it can be obtained by a similar argument.

Lemma 2. *A stationary Markov process with transition operator T and invariant probability measure m is strongly mixing if and only if*

$$(12) \quad \sup_{f \perp 1} \frac{\left\| \frac{1}{n} \sum_{i=1}^n T^i f \right\|_1}{\|f\|_\infty} \rightarrow 0$$

as $n \rightarrow \infty$.

The trivial norm inequality

$$\|f\|_1 \leq \|f\|_p \leq \|f\|_\infty, \quad 1 \leq p \leq \infty,$$

leads to the simple but occasionally useful result that follows.

Lemma 3. *Consider a stationary Markov process with transition operator T and invariant probability measure m . If*

$$(13) \quad \sup_{f \perp 1} \frac{\|T^n f\|_p}{\|f\|_p} \rightarrow 0$$

as $n \rightarrow \infty$ for some p , $1 \leq p \leq \infty$, the process is strongly mixing. If

$$(14) \quad \sup_{f \perp 1} \frac{\left\| \frac{1}{n} \sum_{i=1}^n T^i f \right\|_p}{\|f\|_p} \rightarrow 0$$

as $n \rightarrow \infty$ for some p , $1 \leq p \leq \infty$, the process is uniformly ergodic.

One can readily see that any persistent stationary Markov chain with no periodic states is strongly mixing. A simple example of such a chain indicates

that the apparently stronger (and now verifiably stronger) condition (13) need not be satisfied. Let the chain have states $0, (1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2), \dots$ with the following transition diagram

where $\sum_1^\infty p_i = 1$, $p_i > 0$ for all j and $\sum j p_i < \infty$. If α is the stationary probability distribution, then

$$\alpha_0 p_i = \alpha_{(i,0)}, \quad j = 1, 2, \dots,$$

$$\alpha_0 = \left(1 + \sum_1^\infty j p_i\right)^{-1},$$

$$\alpha_{(j,k)} = \alpha_{(j,k+1)}, \quad k = 0, 1, \dots, j-2.$$

Let if be the vector with $if_s = 0$ unless $s = (j, j-1), (j, j-2)$ when

$$if_{(j,j-1)} = (2\alpha_0 p_j)^{-1/q} = -if_{(j,j-2)}.$$

Then

$$\|if\|_q = 1, \quad \|T^{j-2}if\|_q = 1$$

so that

$$\sup_{j \geq 1} \frac{\|T^j if\|_q}{\|if\|_q} = 1, \quad 1 \leq q \leq \infty.$$

Product of independent random variables on a compact group. Let G be a compact topological group. Take ν to be a regular probability measure on G whose support is not contained in any proper closed subgroup of G . Here $\mathfrak{B}$ is taken to be the Borel field generated by the topology on G . Consider the tran-

sition probability function given by

$$(15) \quad P(g, B) = \nu(g^{-1}B), \quad B \in \mathcal{B}, \quad g \in G.$$

The n step transition function $P_n(g, B)$ generated corresponds to the product of n independent identically distributed elements of G with common distribution ν :

$$(16) \quad P_n(g, B) = \nu^{(n)}(g^{-1}B).$$

Notice that $\nu^{(n)}$ is the n^{th} convolution of ν with itself. The transition function is here set up in terms of a right product but the argument in terms of left products is parallel. The unique invariant probability measure for (15) under the conditions given above is the uniform or Haar measure m on the group.

The remarks we make are most natural in the case of a commutative group. Let $\gamma(g) = (g, \gamma)$ denote the value of a character γ at the point $g \in G$. The set of all continuous characters of G forms the dual group Γ . For any set function η of bounded variation, a Fourier-Stieltjes transform $\hat{\eta}$ on Γ is given by

$$(17) \quad \hat{\eta}(\gamma) = \int_G (-g, \gamma) \eta(dg).$$

If η is absolutely continuous with respect to m , $\eta = f dg$, then the Fourier-Stieltjes transform of η is the Fourier transform of f and we write $\hat{\eta} = \hat{f}$. Γ is given the weak topology induced by the Fourier transforms $\hat{f}$ and since G is compact, this implies that Γ is *discrete* (see [8]). Let us now consider the action on a function $f \in L_2$, $f \perp 1$, by the operator T^n induced by (16)

$$(18) \quad \begin{aligned} (T^n f)(g) &= \int_G f(k) \nu^{(n)}(g^{-1} dk) = \int_G \sum_{\gamma \neq 0} \hat{f}(\gamma)(k, \gamma) \nu^{(n)}(g^{-1} dk) \\ &= \sum_{\gamma \neq 0} \hat{f}(\gamma) \int_G (gk, \gamma) \nu^{(n)}(dk) = \sum_{\gamma \neq 0} \hat{f}(\gamma) [\hat{\nu}(\gamma)]^n(g, \gamma) \end{aligned}$$

where by $\gamma = 0$ we mean the character $\gamma(g) \equiv 1$ on G . Then

$$(19) \quad \|T^n f\|_2^2 = \sum_{\gamma \neq 0} |\hat{f}(\gamma)|^2 |\hat{\nu}(\gamma)|^{2n}$$

and it is clear that (13) is satisfied with $p = 2$ if

$$(20) \quad \sup_{\gamma \neq 0} |\hat{\nu}(\gamma)| < 1.$$

On the other hand, if (20) is not satisfied, we can find a sequence of characters $\gamma_n \neq 0$ (and hence orthogonal to 1 and of absolute value one as functions on G) such that $|\hat{\nu}(\gamma_n)| \rightarrow 1$. Taking $f_k(g) = \gamma_k(g)$ we see that

$$(21) \quad 1 \geq \sup_{f \perp 1} \frac{\|T^n f\|_1}{\|f\|_\infty} \geq \sup_k |\hat{\nu}(\gamma_k)|^n = 1, \quad n = 1, 2, \dots$$

and the process is not strongly mixing. A similar argument indicates that the process is uniformly ergodic if and only if

$$(22) \quad \inf_{\gamma \neq 0} |\hat{\nu}(\gamma) - 1| > 0.$$

Theorem 1. *Let G be a compact Abelian group and ν a regular probability measure on G with support contained in no proper closed subgroup. The stationary Markov process generated by transition function*

$$(23) \quad P(g, B) = \nu(g^{-1}B)$$

and invariant measure the Haar measure is strongly mixing if and only if

$$(24) \quad \sup_{\gamma \neq 0} |\hat{\nu}(\gamma)| < 1.$$

The process is uniformly ergodic if and only if

$$(25) \quad \inf_{\gamma \neq 0} |\hat{\nu}(\gamma) - 1| > 0.$$

In the case of a noncommutative compact group it is not clear how to obtain a natural and meaningful necessary and sufficient condition for a stationary Markov chain generated by transition function (15). However, one can obtain a sufficient condition by making use of Lemma 3.

Let $\{M^{(r)}(g), r \in R\}$ be a maximal collection of non-equivalent, finite dimensional, unitary continuous irreducible representations of the compact group G . The matrix elements of $M^{(r)}(g)$ are denoted $m_{ij}^{(r)}(g)$, $i, j = 1, \dots, n(r)$ where $n(r)$ is the order of $M^{(r)}(g)$. The collection of all matrix elements $m_{ij}^{(r)}(g)$ obtained from such a maximal collection are a complete orthogonal system in L^2 on G with respect to the Haar measure on G by the Peter-Weyl theorem [5]. Actually

$$(26) \quad \int m_{ij}^{(r)}(g) \overline{m_{kl}^{(s)}(g)} dg = \delta_{r,s} \delta_{i,k} \delta_{j,l} n(r)^{-1}.$$

As before, let ν be a regular probability measure on G . Let $M^{(r)}(\nu)$ be the Fourier-Stieltjes coefficient of ν relative to $M^{(r)}(g)$, that is,

$$(27) \quad M^{(r)}(\nu) = \int M^{(r)}(g) \nu(dg).$$

Given any square $n \times n$ matrix $A = (a_{ij})$, A^* denotes the transpose conjugate of A and

$$(28) \quad \|A\| = \sup_x \frac{\|Ax\|_2}{\|x\|_2},$$

the norm of A as a linear operator on n -vectors x with $\|x\|_2^2 = \sum |x_i|^2$. Notice that $\|A\|$ is the square root of the maximal eigenvalue of AA^* or A^*A .

Theorem 2. *Let ν be a regular probability measure on the compact group G whose support is not contained in any proper closed subgroup of G . Let T be the transition operator generated by the transition function $\nu(g^{-1}B)$. Then*

$$(29) \quad \sup_{f \perp 1} \frac{\|T^n f\|_2}{\|f\|_2} \rightarrow 0$$

as $n \rightarrow \infty$ if and only if for some integer $k \geq 1$

$$(30) \quad \sup_{r \neq 0} \|\{M^{(r)}(\nu)\}^k\| \leq \rho < 1$$

where by $M^{(0)}(g)$ we understand the trivial representation consisting of the function identically one. Here the L^2 norms are understood to be taken with respect to the Haar measure of G .

Consider any function $f \perp 1$ in L^2 with respect to the Haar measure on G . We can write

$$f = \sum_{r \neq 0} \sum_{i,j} f_{i,j}^{(r)} m_{i,j}^{(r)}(g)$$

with

$$\sum_{r \neq 0} \sum_{i,j} |f_{i,j}^{(r)}|^2 / n(r) < \infty.$$

Now

$$(T^k f)(g) = \int f(gh) \nu^{(k)}(dg) = \sum_{r \neq 0} \text{tr} [f^{(r)} M^{(r)}(g) \{M^{(r)}(\nu)\}^k]$$

where $\text{tr}(A)$ denotes the trace of the matrix A and the matrix $f^{(r)} = (f_{i,j}^{(r)}; i, j)$. In the preceding relation $\nu^{(k)}$ stands for the k^{th} convolution of ν with itself. The relation follows from the multiplicative property of a representation of the group. But then

$$(31) \quad \|T^k f\|_2^2 = \sum n(r)^{-1} \text{tr} [f^{(r)} * f^{(r)} \{M^{(r)}(\nu) * \}^k \{M^{(r)}(\nu)\}^k]$$

while

$$(32) \quad \|f\|_2^2 = \sum_{r \neq 0} \text{tr} [f^{(r)} * f^{(r)}] n(r)^{-1}.$$

If condition (30) is satisfied, then it follows that

$$(33) \quad \sup_{f \perp 1} \frac{\|T^k f\|_2}{\|f\|_2} \leq \rho < 1$$

and hence that (29) is valid. Now suppose that (30) is not satisfied. Then there is a sequence of integers $k_n \rightarrow \infty$ such that for corresponding $r_n \neq 0$

$$(34) \quad \|\{M^{(r_n)}(\nu)\}^{k_n}\| \rightarrow 1$$

as $n \rightarrow \infty$. Let $v^{(n)}$ be the eigenvector of $\{M^{(r_n)}(\nu) * \}^{k_n} \{M^{(r_n)}(\nu)\}^{k_n}$ with maximal eigenvalue. Take $f^{(r_n)}$ to be the matrix with all rows the vector $v^{(n)}$. Then the sequence of functions $f_n \perp 1$

$$_n f = \text{tr} \{f^{(r_n)} M^{(r_n)}(g)\}$$

will be such that

$$\|T^{k_n} f\|_2 / \|f\|_2 \rightarrow 1$$

as $n \rightarrow \infty$ and hence

$$\sup_{f \perp 1} \frac{\|T^k f\|_2}{\|f\|_2} \equiv 1$$

for all k . The proof is complete.

Notice that in the case of commutative compact groups, if (29) is satisfied then already

$$\sup_{f \perp 1} \frac{\|Tf\|_2}{\|f\|_2} < 1.$$

However, this needn't be the case for noncommutative groups, specifically so for the rotation group.

A central limit theorem. In this section we consider the stationary Markov process $\{X_k, k = 0, \pm 1, \dots\}$ on a compact group G generated by the transition function $P(g, B) = \nu(g^{-1}B)$, with ν a regular probability measure on G , and the invariant measure the Haar measure on the group. Let

$$(35) \quad f_{k,n}(\cdot), \quad k = 1, \dots, k_n, \quad n = 1, 2, \dots \quad (k_n \rightarrow \infty \text{ as } n \rightarrow \infty)$$

be a triangular array of real-valued Borel functions on G . Under appropriate conditions on this family of functions asymptotic normality for the sums

$$(36) \quad S_n = \sum_{k=1}^{k_n} f_{k,n}(X_k),$$

when appropriately normalized, is obtained as $n \rightarrow \infty$. For convenience, it is assumed that

$$(37) \quad E f_{k,n}(X) \equiv 0.$$

Lemma 4. Let G be commutative. Assume that the functions $f_{k,n}$ are uniformly bounded in k, n with

$$(38) \quad E |S_n|^2 \cong k_n \sigma^2, \quad \sigma^2 > 0,$$

as $n \rightarrow \infty$. If the process $\{X_n\}$ is strongly mixing

$$(39) \quad \lim_{n \rightarrow \infty} P\{k_n^{-1/2} \sigma^{-1} S_n \leq x\} = \Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-(u^2/2)} du.$$

The proof of Theorem 1 implies that $\{X_k\}$ is strongly mixing if and only if

$$\sup_{f \perp 1} \frac{\|Tf\|_2}{\|f\|_2} = \alpha < 1.$$

But then for any bounded $f, g \perp 1$

$$\begin{aligned} |EfT^k h| &\leq \{E|f|^2\}^{1/2} \{E|T^k h|^2\}^{1/2} \\ &\leq \alpha^k \|f\|_2 \|h\|_2 \end{aligned}$$

if $k \geq 0$. Similarly for bounded functions $f_0, f_1, f_2, f_3 \perp 1$ and $k_1, k_2, k_3 \geq 0$

$$|Ef_0 T^{k_1} f_1 T^{k_2} f_2 T^{k_3} f_3| \leq \alpha^{k_1+k_2+k_3} \|f_0\|_2 \|f_1\|_2 \|f_2\|_2 \|f_3\|_2.$$

The estimates of the second and fourth order moments of partial sums of $f_{k,n}$'s (n fixed) are then good enough so that one can apply Liapounov's conditions in a proof of the central limit theorem for dependent random variables paralleling either the proof of theorem 1.6 in [2] or the proof in [7].

Theorem 3. *Let G be commutative. Assume that the functions $f_{k,n}$ are uniformly square integrable in k, n with*

$$(40) \quad E|S_n|^2 \cong k_n \sigma^2, \quad \sigma^2 > 0,$$

as $n \rightarrow \infty$. If the process $\{X_n\}$ is strongly mixing (or equivalently if condition (24) is satisfied)

$$(41) \quad \lim_{n \rightarrow \infty} P\{k_n^{-1/2} \sigma^{-1} S_n \leq x\} = \Phi(x).$$

Notice that the uniform square integrability of the $f_{k,n}$'s implies that $E|f_{k,n}(X)|^2 \leq C < \infty$ uniformly in k, n . Given $\epsilon > 0$ let $A_{k,n}(\epsilon) = \{g \mid -N_{k,n}(\epsilon) \leq f(g) \leq M_{k,n}(\epsilon)\}$ where $N_{k,n}(\epsilon), M_{k,n}(\epsilon) > 0$ are determined so that if

$${}_e f_{k,n}(g) = \begin{cases} f_{k,n}(g) & \text{if } g \in A_{k,n}(\epsilon) \\ 0 & \text{otherwise} \end{cases}$$

then

$$\int {}_e f_{k,n}(g) dg = 0, \quad \int dg |f_{k,n}(g) - {}_e f_{k,n}(g)|^2 < \epsilon.$$

Because of the uniform square integrability the numbers $N_{k,n}(\epsilon), M_{k,n}(\epsilon)$ can be taken to be uniformly bounded in k, n . Let

$$\begin{aligned} r_{k,n}(g) &= f_{k,n}(g) - {}_e f_{k,n}(g), \\ {}_e S_n &= \sum_{i=1}^{k_n} {}_e f_{k,n}(X_i), \quad {}_e a_n^2 = E|{}_e S_n|^2, \\ {}_e S'_n &= S_n - {}_e S_n. \end{aligned}$$

Now

$$E|{}_e S'_n|^2 = \sum_{i,l=1}^{k_n} E\{r_{i,n}(X_i) r_{l,n}(X_l)\}.$$

Since

$$\begin{aligned} |E r_{i,n}(X_i) r_{l,n}(X_l)| &\leq \{E|{}_e r_{i,n}(X)|^2\}^{1/2} \{E|T^{l-i} {}_e r_{l,n}(X)|^2\}^{1/2} \\ &\leq \alpha^{l-i} \{E|{}_e r_{i,n}(X)|^2\}^{1/2} \{E|{}_e r_{l,n}(X)|^2\}^{1/2} \leq \alpha^{l-i} \epsilon \end{aligned}$$

if $l > j$ it is clear that

$$\lim_{n \rightarrow \infty} \epsilon a_n^2 / k_n = \sigma^2.$$

By Lemma 4 we know that $\epsilon a_n^{-1} S_n$ is asymptotically normally distributed with mean zero and variance one. On letting $\epsilon \rightarrow 0$ one obtains the desired conclusion by a simple standard approximation argument.

In the case of a possibly noncommutative compact group the following result can be obtained.

Theorem 4. *Assume that the functions $f_{k,n}$ are uniformly square integrable in k, n with*

$$(42) \quad E |S_n|^2 \cong k_n \sigma^2, \quad \sigma^2 > 0,$$

as $n \rightarrow \infty$. If the Fourier-Stieltjes coefficients of ν relative to a complete irreducible set of unitary representations satisfy for some integer $s \geq 1$

$$(43) \quad \sup_{r \neq 0} ||\{M^{(r)}(\nu)\}^s|| \leq \rho < 1,$$

then

$$(44) \quad k_n^{-1/2} \sigma^{-1} S_n$$

is asymptotically normally distributed with mean zero and variance one.

The proof is essentially the same as that given for Theorem 3 (making use of Lemma 4). Condition (43) assures us that

$$\sup_{f \perp 1} \frac{||T^s f||_2}{||f||_2} \leq \rho < 1.$$

We make use of this in a proof paralleling that of Theorem 3.

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University of California, San Diego
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