Local C^n -operational Calculus

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1. Let X be a Banach space, and let W be a linear manifold in X . Denote by $\mathfrak{I}(W)$ the algebra of all linear transformations of X with domain W and range contained in W . For a compact interval Δ , let $C^n(\Delta)$ be the Banach algebra of all complex valued functions of class C^n on Δ ($n = 0, 1, 2, \dots$) with the norm

$$\|f\|_{n,\Delta} = \sum_{j=0}^n \sup_{\Delta} |f^{(j)}|/j!.$$

We consider an operator (= a bounded everywhere defined linear transformation of X) T with *real* spectrum $\sigma(T)$, and we fix throughout a compact interval $\Delta = [a, b]$ which contains $\sigma(T)$ in its interior. If n is a non-negative integer and W is an invariant linear manifold for T , a C^n -operational calculus for T on W is an algebra homomorphism $f \rightarrow T(f)$ of $C^n(\Delta)$ into $\mathfrak{I}(W)$ with the following properties:

- (i) $T(f) = I \mid W$ for $f(t) \equiv 1$;
- (ii) $T(f) = T \mid W$ for $f(t) \equiv t$; and
- (iii) for each $x \in W$, the mapping $f \rightarrow T(f)x$ of $C^n(\Delta)$ into X is continuous.

In (i) and (ii), I stands for the identity operator and $T \mid W$ is the restriction of T to W .

Let $\mathfrak{B}$ denote the Borel field of the real line $\mathbf{R}$. If $W \subset X$ is a linear manifold, a *generalized spectral measure* on W is a mapping $E(\cdot) : \mathfrak{B} \rightarrow \mathfrak{I}(W)$ with the following properties:

- 1. $E(\mathbf{R}) = I \mid W$, and
- 2. for each $x \in W$, $E(\cdot)x$ is a regular strongly countably additive vector measure on $\mathfrak{B}$ (cf. [3]).

Note that, for each $x \in W$, $E(\cdot)x$ is necessarily bounded (cf. [1; III. 4.5]).

Let P denote the algebra of all polynomial functions on Δ . For $n = 0, 1, \dots$, we write

$$\|x\|_n = \sup \{ \|p(T)x\|; p \in P, \|p\|_{n,\Delta} \leq 1 \}, \quad x \in W,$$

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and

$$W_n = \{x \in X; |x|_n < \infty\} = W_n(T).$$

The following properties of $|x|_n$ and W_n are obvious:

1. $|x|_n \geq |x|$ for all $x \in X$;
2. $(W_n, |\cdot|_n)$ is a normed linear space;
3. if S is an operator which commutes with T , then W_n is an invariant linear manifold for S , and as an operator on $(W_n, |\cdot|_n)$, S is continuous with bound $\leq |S|$.

We can state now our main result.

Theorem. 1. *There exists a C^n -operational calculus $T(\cdot)$ for T on W_n .*

2. W_n is "maximal" and $T(\cdot)$ is "unique" in the following sense: if W' is an invariant linear manifold for T , and $T'(\cdot)$ is a C^n -operational calculus for T on W' , then $W' \subset W_n$ and $T'(\cdot)x = T(\cdot)x$ for $x \in W'$.

3. If X is reflexive, there exists a unique generalized spectral measure E_n on W_n with support in $\Delta = [a, b]$, (i.e., for each $x \in W_n$, the vector measure $E_n(\cdot)x$ is supported by Δ) such that

$$(*) \quad T(f)x = \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (T - aI)^j x + \frac{(T - aI)^n}{n!} \int_a^b f^{(n)}(t) dE_n(t)x$$

for all $x \in W_n$ and $f \in C^n(\Delta)$. Furthermore, E_n commutes with every operator which commutes with T .

For $n = 0$, the first expression on the right side of (*) disappears.

2. Proof of the theorem. Part 2 is immediate: $T'(p) = p(T)|_{W'}$ for $p \in P$, and since $T'(\cdot)x$ is continuous for each $x \in W'$, there exists a positive function $M(\cdot)$ on W' such that $|p(T)x| \leq M(x) |p|_{n,\Delta}$ for all $p \in P$ and $x \in W'$, i.e., $W' \subseteq W$. For $x \in W'$, the mappings $T(\cdot)x$ and $T'(\cdot)x$ clearly coincide, since they are continuous on $C^n(\Delta)$ and agree on the dense subset P .

Parts 1 and 3 will be proved simultaneously. As expected, we define $T(p) = p(T)|_{W_n}$ for $p \in P$. For $x \in W_n$ fixed, the mapping $p \rightarrow T(p)x$ of $(P, |\cdot|_{n,\Delta})$ into X is linear and continuous, with norm equal to $|x|_n$, and since P is dense in $C^n(\Delta)$, it extends uniquely to a continuous linear mapping $T(\cdot)x$ of $C^n(\Delta)$ into X (with norm equal to $|x|_n$). For $f \in C^n(\Delta)$ fixed, $T(f)$ is a linear transformation of X with domain W_n .

For $x \in W_n$ and x^* in the dual space X^* of X , the mapping $f \rightarrow x^*T(f)x$ is a continuous linear functional on $C^n(\Delta)$ with norm $\leq |x^*| |x|_n$. It follows as in the proof of Lemma 2.8 in [2] that there exists a unique regular Borel measure $\mu(\cdot | x, x^*)$ on $\Delta = [a, b]$ such that

$$(1) \quad x^*T(f)x = \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} x^*(T - aI)^j x + \int_a^b f^{(n)}(t) d\mu(t | x, x^*)$$

for all $f \in C^n(\Delta)$, and

$$(2) \quad \text{var } \mu(\cdot | x, x^*) \leq K |x^*| |x|_n$$

where $K = (1 + b - a)^n/n!$.

Let $m \geq n$ be an integer, and take $f(t) = (t - a)^m$ in (1); thus, for $x \in W_n$ and $x^* \in X^*$,

$$(3) \quad x^*(T - aI)^m x = m(m-1) \cdots (m-n+1) \int_a^b (t-a)^{m-n} d\mu(t | x, x^*).$$

Let S be an operator commuting with T . If $x \in W_n$, then $Sx \in W_n$, and we may therefore apply (3) both to x and to Sx . Since $x^*(T - aI)^m Sx = (S^*x^*)(T - aI)^m x$, we obtain

$$\int_a^b (t-a)^k d\mu(t | Sx, x^*) = \int_a^b (t-a)^k d\mu(t | x, S^*x^*)$$

for $k = 0, 1, 2, \dots$; hence

$$(4) \quad \mu(\cdot | Sx, x^*) = \mu(\cdot | x, S^*x^*)$$

for all $x \in W_n$ and $x^* \in X^*$.

For $p \in P$ and $x \in W_n$, we have

$$\begin{aligned} & \sup \{ |q(T)p(T)x|; q \in P, |q|_{n,\Delta} \leq 1 \} \\ & \leq \sup \{ |(pq)(T)x|; |pq|_{n,\Delta} \leq |p|_{n,\Delta} \} \leq \sup \{ |h(T)x|; h \in P, |h|_{n,\Delta} \leq |p|_{n,\Delta} \}, \\ & \text{i.e.,} \end{aligned}$$

$$(5) \quad |p(T)x|_n \leq |p|_{n,\Delta} |x|_n, \quad p \in P, \quad x \in W_n.$$

Let g be a bounded Borel function on Δ . Then, for $p \in P$, $x \in W_n$ and $x^* \in X^*$,

$$\begin{aligned} \left| \int_a^b g(t) d\mu(t | x, p(T)^*x^*) \right| &= \left| \int_a^b g(t) d\mu(t | p(T)x, x^*) \right| \\ &\leq K |x^*| |p(T)x|_n \sup_{\Delta} |g| \\ &\leq K |x^*| |x|_n |p|_{n,\Delta} \sup_{\Delta} |g|, \end{aligned}$$

where we used (2), (4) and (5). Now, for $f \in C^n(\Delta)$,

$$\begin{aligned} |x^*p(T)T(f)x| &= |p(T)^*x^*T(f)x| \\ &= \left| \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} x^*p(T)(T - aI)^j x + \int_a^b f^{(n)}(t) d\mu(t | x, p(T)^*x^*) \right| \\ &\leq K_1 |x^*| |x|_n |p|_{n,\Delta} \sum_{j=0}^{n-1} \frac{|f^{(j)}(a)|}{j!} + K |x^*| |x|_n |p|_{n,\Delta} \sup_{\Delta} |f^{(n)}| \end{aligned}$$

where $K_1 = \max \{1, |T - aI|, \dots, |(T - aI)^{n-1}|\}$. Hence

$$(6) \quad |T(f)x|_n \leq M |x|_n |f|_{n,\Delta}, \quad x \in W_n, \quad f \in C^n(\Delta)$$

where $M = \max \{1, |T - aI|, \dots, |(T - aI)^{n-1}|, (1 + b - a)^n\}$. In particular,

$T(f) \in \mathfrak{I}(W_n)$ for all $f \in C^n(\Delta)$. The mapping $f \rightarrow T(f)$ of $C^n(\Delta)$ into $\mathfrak{I}(W_n)$ is clearly linear. It is also multiplicative on P . A routine argument, using (6) and the density of P in $C^n(\Delta)$, shows that the mapping is multiplicative on $C^n(\Delta)$. This completes the proof of Part 1.

Suppose now that X is reflexive, let δ be a fixed Borel subset of Δ and let $x \in W_n$. The mapping $x^* \rightarrow \mu(\delta |x, x^*)$ is a continuous linear functional on X^* , with norm $\leq K |x|_n$. Therefore $\mu(\delta |x, x^*) = x^*F(\delta)x$, where $F(\delta)x$ is some element of X . The mapping $x \rightarrow F(\delta)x$ is a linear transformation of X with domain W_n , and $|F(\delta)x| \leq K |x|_n$ for all δ . We will regard $F(\cdot)$ as defined on $\mathfrak{B}$ (in the usual way). By Theorem IV.10.1 in [1], $F(\cdot)x$ is a strongly countably additive vector measure on $\mathfrak{B}$, with support contained in Δ . We now rewrite Equation (1) in the form

$$(7) \quad T(f)x = \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (T - aI)^j x + \int_a^b f^{(n)}(t) dF(t)x$$

($f \in C^n(\Delta)$, $x \in W_n$). For x and S as in (4), we have

$$x^*F(\cdot)Sx = S^*x^*F(\cdot)x = x^*SF(\cdot)x$$

for all $x^* \in X^*$, i.e., $SF(\delta) = F(\delta)S$ for all $\delta \in \mathfrak{B}$. Fix $x \in W_n$ and let g be a Borel function on $\mathbf{R}$, which is bounded on Δ . The element $y = \int_{\mathbf{R}} g(t) dF(t)x$ is well-defined and the estimates following (5) show that for every $x^* \in X^*$ and $p \in P$,

$$|x^*p(T)y| \leq K |x^*| |x|_n |p|_{n,\Delta} \sup_{\Delta} |g|,$$

i.e., $|y|_n \leq K |x|_n \sup_{\Delta} |g|$. Thus $y = \int_{\mathbf{R}} g(t) dF(t)x \in W_n$. Taking in particular $g = c_\delta$, the characteristic function of $\delta \in \mathfrak{B}$, we see that $F(\delta)W_n \subset W_n$, i.e.,

$$F(\delta) \in \mathfrak{I}(W_n)$$

(in fact, $|F(\delta)x|_n \leq K |x|_n$ for all $\delta \in \mathfrak{B}$ and $x \in W_n$). By the choice of Δ , the point a is not in $\sigma(T)$. Define $E_n(\cdot) = n!(T - aI)^{-n}F(\cdot)$. Taking $f(t) = (t - a)^n/n!$ in (7), we see that $E_n(\mathbf{R}) = I/W_n$. This completes the proof of the theorem.

3. The Weyr characteristic.

3.1. Definition. Let $\{X_\lambda; \lambda \in \Lambda\}$ be a collection of closed invariant subspaces for T . Let $T_\lambda = T|_{X_\lambda}$. Write

$$w_0(\lambda) = w_0(\lambda; T) = \dim W_0(T_\lambda)$$

and $w_k(\lambda) = w_k(\lambda; T) = \dim \{W_k(T_\lambda) | W_{k-1}(T_\lambda)\} (k \geq 1)$, where $W_k | W_{k-1}$ is the quotient space of W_k modulo W_{k-1} and "dim" denotes the algebraic dimension. The set $\{w_k(\lambda); k = 0, 1, \dots; \lambda \in \Lambda\}$ will be called the *Weyr characteristic* of T with respect to $\{X_\lambda; \lambda \in \Lambda\}$.

3.2. Let $\Lambda = \sigma_0(T)$, the set of isolated points of $\sigma(T)$. Let $E(\lambda) = E(\lambda; T)$ be the spectral projection associated with $\lambda \in \sigma_0(T)$ (cf. [1, p. 573]). If T is *compact* (but not quasi-nilpotent), its Weyr characteristic (without any qualification)

is understood to be taken with respect to the collection $\{E(\lambda)X; \lambda \in \sigma_0(T)\}$. This is so in particular if T is an arbitrary operator in finite dimensional space.

We recall that in the latter situation, the *classical* Weyr characteristic is the set $\{w_k^*(\lambda); k = 0, 1, \dots; \lambda \in \sigma(T)\}$ where $w_k^*(\lambda)$ is equal to the number of Jordan cells of size larger than k corresponding to the eigenvalue λ .

3.3. Proposition. *For X finite dimensional, the Weyr characteristic of T coincides with the classical Weyr characteristic. In particular, the set*

$$\sigma(T) \cup \{w_k(\lambda); 0 \leq k \leq n; \lambda \in \sigma(T)\} \quad (n = \dim X)$$

is a complete set of similarity invariants for T .

Proof. Let $x_1, \dots, x_n$ be a basis of X with respect to which T has a Jordan canonical matrix representation. Thus, there exist integers $0 = n_0 < n_1 < \dots < n_k = n$ and an enumeration $\lambda_1, \dots, \lambda_k$ of $\sigma(T)$ (with possible repetitions) such that

$$(1) \quad \begin{aligned} Tx_{n_j} &= \lambda_j x_{n_j} \\ Tx_q &= \lambda_j x_q + x_{q+1} \quad n_{j-1} < q < n_j; \quad j = 1, \dots, k. \end{aligned}$$

Let $\lambda \in \sigma(T)$, and suppose, without loss of generality, that $\lambda_j = \lambda$ for $j = 1, \dots, l$ ($1 \leq l \leq k$). Then $\sigma(T_\lambda) = \{\lambda\}$ and $W_0(T_\lambda)$ is the linear span of the eigenvectors $x_{n_1}, \dots, x_{n_l}$ (cf. [4, p. 251]). Thus $w_0(\lambda) = l = w_0^*(\lambda)$. For $1 \leq j \leq l$ fixed and $n_{j-1} < q < n_j$, $(T - \lambda I)^{n_j - q + 1} x_q = 0$ but $(T - \lambda I)^m x_q \neq 0$ if $0 \leq m \leq n_j - q$. Therefore (cf. [1, p. 559, Theorem 8]) $x_q \in W_{n_j - q}(T_\lambda)$ and $x_q \notin W_{n_j - q - 1}(T_\lambda)$.

Let $[x]_h$ denote the coset containing x in $X \mid W_{h-1}(T_\lambda)$ ($h = 1, 2, \dots$). Then, for j and q fixed as above,

$$0 \neq [x_q]_{n_j - q} \in W_{n_j - q}(T_\lambda) / W_{n_j - q - 1}(T_\lambda).$$

It is easily seen that $[x_r]_{n_j - q} = 0$ for $q < r \leq n_j$, and

$$[x_r]_{n_j - q} \notin W_{n_j - q}(T_\lambda) / W_{n_j - q - 1}(T_\lambda)$$

for $n_{j-1} \leq r < q$. Thus, for $h = 1, 2, \dots$, the quotient space $W_h(T_\lambda) \mid W_{h-1}(T_\lambda)$ has the following properties:

- (i) if $n_j - h > n_{j-1}$, it contains *exactly one* coset $[x_q]_h \neq 0$ with $n_{j-1} \leq q \leq n_j$, namely, the coset with $n_j - q = h$, i.e., with $q = n_j - h$;
- (ii) if $n_j - h \leq n_{j-1}$, it contains *no* coset $[x_q]_h$ with $n_{j-1} \leq q \leq n_j$.

For $x_q \notin W_{h-1}$, the cosets $[x_q]_h$ are linearly independent. It follows from (i) and (ii) that $w_h(\lambda)$ is equal to the number of integers j ($1 \leq j \leq l$) for which (i) holds, i.e., for which $n_j - n_{j-1} > h$. But $n_j - n_{j-1}$ is the size of the j -th Jordan cell corresponding to λ . Hence $w_h(\lambda) = w_h^*(\lambda)$ ($h = 1, 2, \dots$). Q.E.D.

3.4 Definition. Let T_1 and T_2 be operators on X . Let $\{X_\lambda^1; \lambda \in \Lambda\}$ and $\{X_\lambda^2; \lambda \in \Lambda\}$ be collections of closed invariant subspaces for T_1 and T_2 respectively, such that the sums $X^i = \sum_\lambda X_\lambda^i$ ($i = 1, 2$) are direct. We say that T_1 and T_2

are *quasi-similar* with respect to $\{X_\lambda^i; \lambda \in \Lambda; i = 1, 2\}$ if

- (i) $T_i^*|(X^i)^\perp$ is quasi-nilpotent ($i = 1, 2$); and
- (ii) there exist one-to-one continuous linear maps Q_λ of X_λ^1 onto X_λ^2 ($\lambda \in \Lambda$) such that the diagrams below commute for all $\lambda \in \Lambda$:

$$\begin{array}{ccc} X_\lambda^1 & \xrightarrow{T_1} & X_\lambda^1 \\ Q_\lambda \downarrow & & \downarrow Q_\lambda \\ X_\lambda^2 & \xrightarrow{T_2} & X_\lambda^2 \end{array}$$

For compact (but not quasi-nilpotent) operators T_i with the same spectrum, quasi-similarity (without qualification) is understood to be with respect to the collection

$$\{E(\lambda; T_1)X, E(\lambda; T_2)X; \lambda \in \sigma_0(T_1) = \sigma_0(T_2)\}.$$

3.5 Proposition. *Let T_1 and T_2 be compact (but not quasi-nilpotent) operators with the same (real) spectrum. Then T_1 and T_2 are quasi-similar if and only if their Weyr characteristics coincide.*

Proof. This is an immediate consequence of Proposition 3.3 and of the spectral theory of compact operators (cf. [1] and Lemma 1.6 in [3]).

3.6. The above proposition is essentially a finite dimensional result. In fact, the usefulness of the Weyr characteristic as a criterion for similarity or quasi-similarity, seems very doubtful in “purely” infinite dimensional situations. For example, if T_1 and T_2 are hermitian quasi-similar operators on Hilbert space, then $T_i^*|(X^i)^\perp = T_i|(X^i)^\perp$ ($i = 1, 2$) are hermitian and quasi-nilpotent, hence equal to 0. Thus $(\sum_\lambda X_\lambda^i)^\perp \subset \text{kernel } T_i = K_i$. Supposing $X_\lambda^i \perp X_\mu^i$ for $\lambda \neq \mu$, we have $X = K_i \oplus \sum_\lambda \oplus X_\lambda^i$. Since $T_i|X_\lambda^i$ ($i = 1, 2$) are hermitian, the existence of Q_λ as in Definition 3.4 implies the existence of isometries U_λ of X_λ^1 onto X_λ^2 such that $U_\lambda T_1 = T_2 U_\lambda$ on X_λ^1 ($\lambda \in \Lambda$). Hence there exists a partial isometry U of X intertwining between T_1 and T_2 . However the Weyr characteristics of T_i degenerate to $\{\dim X_\lambda^i, 0, 0, \dots; \lambda \in \Lambda\}$ ($i = 1, 2$). Obviously, these dimensions do not provide in general an efficient test for the equivalence of T_i under a partial isometry.

We should mention in this connection that an entirely different generalization of the classical Weyr characteristic has been given in [5] for a suitable class of spectral operators, and, for this class, the characteristic does provide a complete set of “semi-similarity” invariants.

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