On Locally Supersonic Plane Flows with a Weak Shock Wave

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Summary. The aim of this paper is to provide a perfect fluid model for the observed phenomena of plane transonic flow past a given profile with an attached shock wave. The starting point of the present discussion is a simple lemma on the possible hodograph configurations of the images of the front and the rear of the shock. The whole work is carried through in the hodograph plane whereas Germain, see Küchemann and Sterne (1964), first discusses general homogeneous solutions by means of a non-linear ordinary differential equation and later introduces the hodograph method to complete the discussion for plane flow. In regard to the *homogeneous* solutions the present account includes (a) the derivation of a single analytic expression which contains effectively a complete two-parameter family of shock solutions, also (b) the first comprehensive account of the non-existence, within the family of homogeneous solutions, of normal or concave shock transitions. These alone are suitable in the physical flow problems.

The homogeneous solutions in themselves being unsuitable the author considers the generalisation to a wider class of solutions with a weak shock and proves two *uniqueness* theorems for these. The first suggests that there are solutions with arbitrarily good approximations to the normal shock, and the second that in certain cases the perturbation problem for a fixed profile is well set and so the corresponding flows will be stable.

Introduction. The existence of plane transonic flows with a local supersonic region terminated by a shock wave is now well established as an experimental fact, see for example Holder (1959) (1964). Moreover, the flow regime, in particular the location of the shock waves, has been found to depend in a stable manner on the free stream Mach number. This situation has been connected by Holder with both the motion of an ideal fluid, including a weak shock discontinuity, and the interaction of the shock wave with the wake and the boundary layer. Clearly the success of these experiments has depended

on a nice appreciation of the parts played in the motion of real fluids by both viscous and non-viscous properties of the fluid. However, in the case of subsonic flow and subject to well known limitations one can construct a perfect fluid model of the real flow whose behavior is similar in many respects. It is the object of this paper to discuss such a perfect fluid model for the case of plane transonic flows with a weak attached shock.

As far as exact theory is concerned, previous work on this subject seems to fall under two main headings, (i) perturbation theory for general transonic flow, (ii) local solutions for weak shocks. As for item (i) Morawetz (1956) (1957) (1958I) (1964) has shown that the perturbation problem for continuous, that is to say shock free flow is badly set and that this holds even if one admits certain weak discontinuities, Morawetz (1956), Appendix 3. Morawetz's conclusion has been sharpened in Manwell (1964) where the case of a fixed boundary stream line is considered in some detail with the conclusion that the cases for which the perturbation problem is well set are at best highly exceptional. In a footnote to this paper the author has conjectured that the perturbation problem for plane transonic flows with a weak shock may be well set. It is a main object of the present paper to establish certain general theorems pointing towards the truth of this conjecture.

However, it is first necessary to consider item (ii) the question of the local solutions of the equations of plane transonic flow with a weak shock. These were originally discussed by Frankl (1955). Frankl's solution is not exact since it contains limit lines and the general problem has since been treated in great detail by Germain (1956) (1958). The first complete account of Germain's researches is contained in Küchemann and Sterne (1964). At the outset both Frankl and Germain made use of the hodograph method but later Germain developed an idea due to Guderley and introduced another set of auxiliary variables which are, however, applicable only for homogeneous solutions. Although this method is very effective in establishing certain necessary conditions, in particular the limitation $1/2 < m < 5/6$ on the degree of homogeneity of the hodograph solutions for the stream function, it does not lead to a satisfactory solution of the equations of transonic flow even in the case of homogeneous solutions. In the present paper the author follows Germain's first paper in using only the hodograph method and, by means of a simple lemma on the configuration of the hodograph images of the front and the rear of the shock, is able to present a very complete discussion of the homogeneous solutions. In particular, the shock wave problem is reduced to a single analytic equation and it is shown that this equation contains a two parameter family of solutions. These are presented for two distinct types of shock transition with $2/3 \leq m < 5/6$. Germain, see Küchemann and Sterne (1964) presents only two isolated examples, one of each type with $m = 2/3, 4/5$, respectively and he describes his derivation as essentially numerical and involving successive approximation. (See also the solution of Lifsic and Ryzov (1964) for $n = 1.40225$, $m = .8287$ ($n = 1 + (3m)^{-1}$).)

Again, although Germain states in his first paper that 'normal' shocks cannot arise the proof tacitly assumes a very special hodograph configuration. This limitation is at least partially removed in his later work but he does not complete the argument for the non-existence of normal shocks. A considerable portion of the present paper is devoted to a systematic study of this aspect of the shock transitions. It is convenient to describe as *convex*, those transitions in which the velocity vectors at the shock turn in the same sense as does the velocity vector as we move along the profile in the direction of the flow. Similarly, those shocks in which the vector turns in the opposite sense will be described as *concave* transitions. These terms will not be confused with the description in Germain's work of *shock profiles* as concave in a certain sense. It will be shown below that there are no concave or normal shock transitions within a very general class of the homogeneous solutions.

The strong limitations on the homogeneous solutions leads us to consider a more general class of solutions of the equations of plane transonic flow with a weak shock. This is the class of potential flows having lines of finite discontinuity in the velocity components, the 'mechanical' shock conditions being satisfied exactly. In order to show that this proposed model is valid it is necessary to establish two properties, (a) there exist individual solutions of the equations which are of the type experimentally observed, (b) the perturbation problem for such solutions is correctly set. In regard to (a) it will be shown that the boundary value problem set in the hodograph plane and corresponding to the construction of an arbitrarily good approximation to the *normal* shock wave is uniquely defined by certain boundary conditions. It is convenient to describe this boundary value problem as the *direct problem*. These conditions include the shock conditions, with a slight approximation near the tip of the shock where the non-existence properties associated with homogeneity must apply, also certain auxiliary boundary conditions. The latter ensure that if the hodograph solution has the simplest structure of level lines, as commonly exhibited in the examples, then it will be free of limit lines and can be mapped into a physical flow with a weak shock. The shock need not be regarded as very small, as is the case for the homogeneous solutions. Let us note that the auxiliary condition, ensuring the mappability of the solution of the direct problem in the hodograph plane, holds equally for the homogeneous solutions of Section (2) which are completely mappable in any mixed region and for Frankl's solution which cannot be mapped in any mixed region extending to the shock. In Section (4) below a uniqueness theorem will be given for the general direct problem and it is known (*cf.* Morawetz (1958II) and Garabedian (1964)) that the 'a-b-c' method used in the proof strongly implies *existence* of these solutions. Also there is a close connection with the existence proofs of Manwell (1964). Section (4) deals with an approximation to the case of normal shocks but it seems likely that one may choose the hodograph images Λ_1 , Λ_2 of the front and the rear of the shock in a very general manner; the only obvious restriction is that they have sufficiently close contact at the

origin with the curves $\lambda = \lambda_i$ appropriate to the known homogeneous solutions of Section (2) below.

Again, in regard to item (b) it will be shown that there is a uniqueness theorem for the perturbation problem with weak shocks which are sufficiently near the normal case. Here, in contrast to the Morawetz theory for continuous flows the data may be set on the complete physical flow boundary; the perturbation of the shock itself is an unknown to be found in the course of the corresponding existence proof. As for the latter we recall the above remarks concerning the 'a-b-c' method and also note the analogy with known existence theorems for the Morawetz perturbation problem, see Manwell (1964). Finally we note the indication that the uniqueness theorems may fail unless we are sufficiently near the 'normal' case and that the supersonic to subsonic transitions are preferred to the supersonic to supersonic transitions since the latter appear to need an extra linear condition for the existence theorem to go through.

1. Equations for transonic flow with a weak shock wave. Let $u = u(x, y)$, $v = v(x, y)$, be solutions of the equations of plane transonic flow

$$(1.1) \quad u_y - v_x = 0, \quad (\rho u)_x + (\rho v)_y = 0,$$

and let us introduce the velocity potential function ϕ and the stream function ψ according to

$$(1.2) \quad d\phi = u dx + v dy, \quad d\psi = \rho v dx - \rho u dy.$$

Then equation (1.1) is replaced by the system

$$(1.3) \quad \phi_x = -\frac{1}{\rho} \psi_y = u, \quad \phi_y = \frac{1}{\rho} \psi_x = v.$$

Here we have used suffixes x, y to denote partial differentiation with respect to one of these two independent variables and a similar remark applies to the use of suffixes θ, q , below. Let us now write $u = q \cos \theta$, $v = q \sin \theta$ and seek x, y or ϕ, ψ as solutions in terms of the parameters (θ, q) . According to (1.2)

$$(1.4) \quad \begin{aligned} q dx &= \cos \theta d\phi + \frac{\sin \theta}{\rho} d\psi, \\ q dy &= \sin \theta d\phi - \frac{\cos \theta}{\rho} d\psi, \end{aligned}$$

and equating the quantities $x_{\theta q}$, $x_{q\theta}$ and $y_{\theta q}$, $y_{q\theta}$ one finds equations between ϕ, ψ in the form

$$(1.5) \quad \rho \phi_\theta = -q \psi_q, \quad \rho^2 q \phi_q = (\rho q)_\theta \psi_\theta.$$

When treating boundary value problems it is convenient to introduce quantities $s, k(s)$ defined by (1.6) (1.7)

$$(1.6) \quad q ds = -\rho dq, \quad s = 0, \quad q = c_*,$$

$$(1.7) \quad k(s) = \rho^{-3}(\rho q)_\theta = \rho^{-2}(1 - q^2/c^2).$$

Here c denotes the critical speed and $c^2 = dp/d\rho$. Then the canonical form of the system (1.5) is (1.8)

$$(1.8) \quad \phi_s = \psi_s, \quad \phi_* = -k(s)\psi_s.$$

It should be noticed that $s, k(s)$ are both positive for $q < c_*$ that is for subsonic flow, and change sign together across the sonic line $q = c_*$.

Weak shocks in transonic flow. If a potential flow solution of equation (1.3) exhibits a line of discontinuity in the velocity components such that ϕ and ψ are continuous the discontinuity will be described as a weak shock, and said to satisfy the 'mechanical' conditions. It is well known that the assumption of potential flow on both sides of the line of discontinuity is in general equivalent to ignoring a change of entropy of the third order in the shock strength and the corresponding introduction of vorticity into the field of flow. Here the description of a shock as 'weak' indicates that the strict physical equations of flow have been simplified. But as far as consistent equations are concerned the magnitude of the discontinuity need not be small in any sense. The uniqueness proofs of Section (5) are a good indication that such families of discontinuous potential flow solutions are amenable to a precise theory of boundary value problems, and that the linearizing assumptions of the homogeneous case are not essential.

In the hodograph method, the shock conditions being expressed by the continuity of ϕ and ψ between corresponding points of the hodograph images Λ_1, Λ_2 of, respectively the front and rear of the shock, it is then necessary to express the condition of continuity of the physical coordinates (x, y) as determined by equations (1.6). We find (1.9) and (1.10),

$$(1.9) \quad \left[\frac{\cos \theta}{q} \right] d\phi + \left[\frac{\sin \theta}{q} \right] d\psi = 0,$$

$$(1.10) \quad \left[\frac{\sin \theta}{q} \right] d\phi - \left[\frac{\cos \theta}{\rho q} \right] d\psi = 0.$$

Here the $[F]$ denotes $(F_2 - F_1)$ taken across the discontinuity. Let us first eliminate $d\phi/d\psi$ between (1.9) (1.10). We find

$$(1.11) \quad \frac{1}{\rho_1 q_1^2} + \frac{1}{\rho_2 q_2^2} - \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \frac{\cos(\theta_1 - \theta_2)}{q_1 q_2} = 0,$$

which will be described as the *shock polar* equation. Then, adding the ratios $(d\phi/d\psi)^2$ found from (1.9) (1.10) and simplifying by means of (1.11) we have

$$(1.12) \quad \left(\frac{d\phi}{d\psi} \right)^2 = -\frac{[R]}{[Q]}, \quad R = \rho^{-2} q^{-2}, \quad Q = q^{-2},$$

which expresses the ratios $(d\phi/d\psi)^2$ along the shock in terms of the magnitudes of the velocity on the two sides.

The shock wave problem will now be formulated more precisely in the hodo-

graph plane and as follows. It is required to construct a bounded solution of equations (1.5) in a mixed, subsonic/supersonic, domain comprising at least a partial neighborhood of the origin $\theta = s = 0$, taken at the tip of the shock. Let Λ_1 , Λ_2 be the images of the front and the rear of the shock. The mixed solution must satisfy the following. For chosen Λ_i equation (1.11) defines a pointwise correspondence between the two loci. Then, if the function ψ_1 , say, is prescribed as the unknown, equation (1.12) and the conditions $\psi_1 = \psi_2$, $\phi_1 = \phi_2$ determine the four functions $\psi_i(s_i)$, $\phi_i(s_i)$, $i = 1, 2$. Anticipating the result to be established immediately below, that the mixed solution contains at least one characteristic it is clear that, locally at least, we have a very plausible boundary value problem. *Given the loci Λ_i and equations (1.11) (1.12) with $\psi_1 = \psi_2$, $\phi_1 = \phi_2$ it is required to choose the unknown $\psi_1(s_1)$ in such a way that the solution (not necessarily its derivative) is continuous at the included characteristic drawn through the origin $\theta = s = 0$.* If we limit ourselves to the homogeneous solutions this boundary value problem, the *direct* problem as it will be called in this paper (to distinguish it from the perturbation problem associated with certain boundary conditions in the physical plane), is well set. If the Λ_i are suitably chosen and if we fix m , the degree of homogeneity in a certain range, there is apart from a trivial multiplier precisely one homogeneous solution. It is not surprising that for such a homogeneous problem there will be strong limitations on the parameter and the loci Λ_i . On the other hand, once we relax the condition of homogeneity there is an appropriate boundary value problem containing further auxiliary conditions which may be connected with an inhomogeneous integral equation. It seems likely that this problem can be solved with far less restriction on the Λ_i .

The solution of the direct problem in the hodograph plane does not necessarily solve the shock wave problem for which we have the further requirement that the hodograph solution shall be free of limit lines. In the case of certain families of homogeneous solution it will be shown directly that this condition of mappability is satisfied at all points. For the general problem we shall find a partial answer to the problem: the solutions of the direct problem are mappable if the level lines have the simple structure exhibited by most examples of the hodograph solution for plane transonic flow, briefly $d\phi > 0$ at the boundary $\psi = c$, and $d\psi > 0$ along the shock. It seems likely that these conditions will be satisfied provided the data in the inhomogeneous problem is chosen in a reasonable manner, or more specifically, if we solve the inhomogeneous problem with the sort of subsonic flow conditions encountered in the physical problems. Future discussion of the problem from this point of view appears to be matter for computation.

As a preliminary result we have the following:

Lemma (1). *For any hodograph solution of the weak shock wave problem which is connected with the subsonic flow the locus Λ_1 lies in the supersonic region which is bounded by the two characteristics Γ_1 , Γ_2 drawn through the origin; the*

locus Λ_2 lies either in the subsonic region or in a supersonic sector bounded by the sonic line and one of the Γ_i . Conversely, if we solve the direct problem in a connected mixed region and if $d\psi > 0$ along either Λ_i satisfying (1.23) then $g > 0$ at such a Λ_i .

It should be noted, cf. Frankl's solution, that $g > 0$ at Λ_1 (with $d\psi > 0$) does not ensure $g > 0$ in the complete mixed region. Here g is the Jacobian defined by

$$(1.13) \quad g = \frac{\partial(\psi, \phi)}{\partial(\theta, q)} = \rho q^2 \frac{\partial(x, y)}{\partial(\theta, q)} = \frac{\rho}{q} [\psi_s^2 + k(s)\psi_\theta^2].$$

With the choice of R, Q as in (1.12)

$$(1.14) \quad k(s) = \frac{dR}{dQ},$$

and so we may write

$$(1.15) \quad \left(\frac{d\phi}{d\psi}\right)^2 + k(s) = \frac{dR}{dQ} - \frac{[R]}{[Q]}.$$

If $R(Q)$ is concave upwards, as for example in the case of an adiabatic gas, the quantity on the right of (1.15) is negative at the front of the shock and positive at the rear. The same is true if we take $k(s) = s$ and then approximate to equation (1.12) as in (2.1) (2.2) below.

According to (1.8) we have identically in any hodograph solution

$$(1.16) \quad \left\{\frac{d\theta}{ds} - \frac{d\phi}{d\psi}\right\}\psi_s = \left\{k(s) + \frac{d\theta}{ds} \frac{d\phi}{d\psi}\right\}\psi_\theta,$$

for arbitrary arcs $\theta = \theta(s)$. The identities (1.17) (1.18) are easily deduced from (1.16)

$$(1.17) \quad \left\{k(s) + \frac{d\theta}{ds} \frac{d\phi}{d\psi}\right\}^2 \{\psi_s^2 + k(s)\psi_\theta^2\} = \psi_s^2 \left\{\left(\frac{d\phi}{d\psi}\right)^2 + k(s)\right\} \left\{\left(\frac{d\theta}{ds}\right)^2 + k(s)\right\},$$

$$(1.18) \quad \left\{\frac{d\theta}{ds} - \frac{d\phi}{d\psi}\right\}^2 \{\psi_s^2 + k(s)\psi_\theta^2\} = \psi_\theta^2 \left\{\left(\frac{d\phi}{d\psi}\right)^2 + k(s)\right\} \left\{\left(\frac{d\theta}{ds}\right)^2 + k(s)\right\}.$$

The Jacobian may not vanish in any mappable solution and so from (1.13)

$$(1.19) \quad \psi_s^2 + k(s)\psi_\theta^2 \neq 0.$$

The properties established above for the right hand member of (1.15) give us

$$(1.20) \quad \left(\frac{d\phi}{d\psi}\right)_i^2 + k_i(s_i) \leq 0, \quad i = 1, 2.$$

We now impose the condition that $d\psi \neq 0$ along either Λ_i . This leaves open the possibility that one of ψ_θ, ψ_s but not both might vanish. If for example $\psi_s \neq 0$ let us consider the identity (1.17). We can easily see that if both sides

of (1.17) vanish then necessarily in view of (1.19) (1.20)

$$(1.21) \quad k + \frac{d\theta}{ds} \cdot \frac{d\phi}{d\psi} = k + \left(\frac{d\theta}{ds}\right)^2 = 0$$

and for $k \neq 0$, we infer $d\phi/d\psi = d\theta/ds = |k(s)|^{1/2}$, which contradicts (1.15). It follows that along any Λ_i and for points at which $\psi_s \neq 0$ we have

$$(1.22) \quad \text{sgn } g = \text{sgn} \left[\left(\frac{d\phi}{d\psi}\right)^2 + k(s) \right] \text{sgn} \left\{ \left(\frac{d\theta}{ds}\right)^2 + k(s) \right\}.$$

Again, in case $\psi_s \neq 0$ the same result follows from (1.18). If we are given any solution of the weak shock equations in hodograph form and if the whole hodograph solution is connected with the subsonic region then it is necessary that the Jacobian should be positive everywhere. Hence from the inequalities (1.20) we deduce

$$(1.23) \quad \left\{ \left(\frac{d\theta}{ds}\right)^2 + k(s) \right\}_i \leq 0, \quad i = 1, 2.$$

The inequalities (1.23) impose a necessary limitation on the possible arcs Λ_i . Again, if one chooses the arcs to satisfy (1.23) we may reverse the argument to deduce from (1.20) (1.23) that if ψ increases steadily along either Λ_i in any solution of the direct problem then the Jacobian is indeed positive at this Λ_i . The Lemma is now proved. As we shall see below the condition $g > 0$ at Λ_2 is sufficient for the solution between a supersonic Λ_2 and the sonic line to be completely mappable but it is not sufficient in the case of the solution between Λ_1 and the sonic line. It is at this last point that Frankl's solution of the direct problem for normal shocks fails to give a physical solution.

2. The homogeneous solutions: general properties and the existence of supersonic to subsonic transitions. Following Germain (1956), see also Frankl (1955), let us replace the four relations $\psi_1 = \psi_2$, $\phi_1 = \phi_2$ and (1.9) (1.10) or (1.11) (1.12) by the first order approximations

$$(2.1) \quad \left(\frac{d\phi}{d\psi}\right)_1 = \left(\frac{d\phi}{d\psi}\right)_2 = \frac{\theta_1 - \theta_2}{s_2 - s_1} = \pm |\tfrac{1}{2}(s_1 + s_2)|^{1/2},$$

$$(2.2) \quad \psi_1 = \psi_2, \quad 2(\theta_1 - \theta_2)^2 + k'(0)(s_1 + s_2)(s_1 - s_2)^2 = 0.$$

To the same approximation $k(s) = sk'(0)$ and with a suitable choice of units $k'(0) = 1$, the equations for ϕ , ψ become

$$(2.3) \quad \phi_\theta = \psi_s, \quad \phi_s = -s\psi_\theta.$$

If $s < 0$ it is convenient to introduce characteristic coordinates according to

$$(2.4) \quad \begin{aligned} &\text{with} \quad \phi_\alpha = -|s|^{1/2} \psi_\alpha, \quad \phi_\beta = -|s|^{1/2} \psi_\beta, \\ &\quad \alpha = \tfrac{2}{3} |s|^{3/2} + \theta, \quad \beta = \tfrac{2}{3} |s|^{3/2} - \theta. \end{aligned}$$

According to Lemma (1) and in the case of solutions which are entirely connected with the subsonic region we have a Λ_1 in the quadrant $\alpha, \beta > 0$ and the line $\Gamma_1 : \beta = 0$ is necessarily included in the hodograph representation. In the sectors bounded by Λ_1 , Γ_1 and Γ_1 , $s = 0$, respectively, the homogeneous solutions may be represented, apart from a trivial numerical factor, in the forms (2.5) (2.6)

$$(2.5) \quad \phi_{(1)} = \phi + K_1 \phi^*, \quad \psi_{(1)} = \psi + K_1 \psi^*,$$

$$(2.6) \quad \phi_{(2)} = \phi - K_2 \phi^*, \quad \psi_{(2)} = \psi - K_2 \psi^*,$$

where explicitly

$$(2.7) \quad \psi = \alpha^m F\left(\frac{1}{6}, -m; \frac{5}{6} - m; -\frac{\beta}{\alpha}\right) \\ = \alpha^m \left(1 + \frac{\beta}{\alpha}\right)^{-1/6} F\left(\frac{1}{6}, \frac{5}{6}; \frac{5}{6} - m; \frac{\beta}{\alpha + \beta}\right),$$

$$(2.8) \quad \psi^* = \alpha^{-1/6} |\beta|^{m+1/6} F\left(\frac{1}{6}, m + \frac{1}{3}; \frac{7}{6} + m; -\frac{\beta}{\alpha}\right), \\ = \alpha^{-1/6} |\beta|^{m+1/6} \left(1 + \frac{\beta}{\alpha}\right)^{-1/6} F\left(\frac{1}{6}, \frac{5}{6}; \frac{7}{6} + m; \frac{\beta}{\alpha + \beta}\right),$$

also

$$(2.9) \quad \phi = -\left(\frac{3}{4}\right)^{1/3} \frac{3m}{3m+1} \alpha^{m+1/3} F\left(-\frac{1}{6}, -m - \frac{1}{3}; \frac{5}{6} - m; -\frac{\beta}{\alpha}\right) \\ = -\left(\frac{3}{4}\right)^{1/3} \frac{3m}{3m+1} \alpha^{m+1/3} \left(1 + \frac{\beta}{\alpha}\right)^{1/6} F\left(-\frac{1}{6}, \frac{7}{6}; \frac{5}{6} - m; \frac{\beta}{\alpha + \beta}\right),$$

and

$$(2.10) \quad \phi^* = \left(\frac{3}{4}\right)^{1/3} \alpha^{1/6} |\beta|^{m+1/6} F\left(-\frac{1}{6}, m; \frac{7}{6} + m; -\frac{\beta}{\alpha}\right) \\ = \left(\frac{3}{4}\right)^{1/3} \alpha^{1/6} |\beta|^{m+1/6} \left(1 + \frac{\beta}{\alpha}\right)^{1/6} F\left(-\frac{1}{6}, \frac{7}{6}; \frac{7}{6} + m, \frac{\beta}{\alpha + \beta}\right).$$

The proof of (2.7) ... (2.10) depends on a solution by series of the characteristic equations for ψ and ϕ , which are

$$(2.11) \quad 6(\alpha + \beta)\psi_{\alpha\beta} + \psi_\alpha + \psi_\beta = 0,$$

$$(2.12) \quad 6(\alpha + \beta)\phi_{\alpha\beta} - \phi_\alpha - \phi_\beta = 0.$$

For bounded solutions we have $m > 0$ and recalling Lemma (1), which shows that Γ_1 must be included in any solution which is partly connected with the subsonic region, a simple discussion of the Jacobian gives the further inequalities in (2.13)

$$(2.13) \quad 0 < m < \frac{5}{6}, \quad K_i \geq 0.$$

In the vicinity of the sonic line we may express ψ , ψ^* and ϕ , ϕ^* , respectively, by suitable combinations of (2.14) (2.15) and (2.16) (2.17) where λ , r are defined in (2.27) below.

$$(2.14) \quad \begin{aligned} Z_1 &= |\theta|^m F\left(-\frac{m}{2}, \frac{1-m}{2}; \frac{2}{3}; -\frac{4}{9} \frac{s^3}{\theta^3}\right), \\ &= r^m F\left(-\frac{m}{2}, \frac{1}{6} + \frac{m}{2}; \frac{2}{3}; \frac{\lambda^3}{1+\lambda^3}\right). \end{aligned}$$

$$(2.15) \quad \begin{aligned} Z_2 &= s |\theta|^{m-2/3} F\left(\frac{1}{3} - \frac{m}{2}, \frac{5}{6} - \frac{m}{2}; \frac{4}{3}; -\frac{4}{9} \frac{s^3}{\theta^3}\right), \\ &= s r^{m-2/3} F\left(\frac{1}{3} - \frac{m}{2}, \frac{1+m}{2}; \frac{4}{3}; \frac{\lambda^3}{1+\lambda^3}\right). \end{aligned}$$

$$(2.16) \quad \begin{aligned} \zeta_1 &= -\frac{m}{2} s^2 |\theta|^{m-1} F\left(\frac{1-m}{2}, 1 - \frac{m}{2}; \frac{5}{3}; -\lambda^3\right), \\ &= -\frac{m}{2} s^2 r^{m-1} F\left(\frac{1-m}{2}, \frac{2}{3} + \frac{m}{2}; \frac{5}{3}; \frac{\lambda^3}{1+\lambda^3}\right). \end{aligned}$$

$$(2.17) \quad \begin{aligned} \zeta_2 &= \frac{|\theta|^{m+1/3}}{m + \frac{1}{3}} F\left(-\frac{1}{6} - \frac{m}{2}, \frac{1}{3} - \frac{m}{2}; \frac{1}{3}; -\lambda^3\right), \\ &= \frac{r^{m+1/3}}{m + \frac{1}{3}} F\left(-\frac{1}{6} - \frac{m}{2}, \frac{m}{2}; \frac{1}{3}; \frac{\lambda^3}{1+\lambda^3}\right). \end{aligned}$$

These are valid for $\theta > 0$ and in the case $\theta < 0$ we merely change the sign of (2.16) (2.17). For example, according to the formula for the continuation of the hypergeometric series in β/α near the branch point $|\beta/\alpha| = 1$ we may replace (2.7) by

$$(2.18) \quad \begin{aligned} \psi &= A(m) \alpha^m F\left(\frac{1}{6}, -m; \frac{1}{3}; 1 + \frac{\beta}{\alpha}\right) \\ &\quad + B(m) \alpha^m \left(1 + \frac{\beta}{\alpha}\right)^{2/3} F\left(\frac{5}{6}, \frac{2}{3} - m; \frac{5}{3}; 1 + \frac{\beta}{\alpha}\right). \end{aligned}$$

From (2.18) we immediately deduce the alternative representation (2.19)

$$(2.19) \quad \psi = A(m) Z_1 - (4/3)^{2/3} B(m) Z_2, \quad \theta > 0.$$

Here both (2.18) (2.19) are completely defined by the same Cauchy data on $\alpha + \beta = 0 = s$. The coefficients are given by

$$(2.20) \quad A(m) = \frac{\Gamma(\frac{2}{3}) \Gamma(\frac{5}{6} - m)}{\Gamma(\frac{5}{6}) \Gamma(\frac{2}{3} - m)}, \quad B(m) = \frac{\Gamma(-\frac{2}{3}) \Gamma(\frac{5}{6} - m)}{\Gamma(\frac{1}{6}) \Gamma(-m)} > 0,$$

$$0 < m < \frac{5}{6}.$$

Replacing m by $-(1/3 + m)$ we find the corresponding representation for ψ^* with coefficients A^* , B^* where

$$(2.21) \quad A^*(m) = \frac{\Gamma(\frac{2}{3})\Gamma(\frac{7}{6} + m)}{\Gamma(\frac{5}{6})\Gamma(1 + m)} > 0, \quad B^*(m) = \frac{\Gamma(-\frac{2}{3})\Gamma(\frac{7}{6} + m)}{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3} + m)} < 0.$$

To determine the ϕ , ϕ^* we merely replace the Z_i by the corresponding ζ_i .

It is necessary to consider the analytic continuation of (2.19) across $\theta = 0$ for $s > 0$. This is readily achieved by writing the solutions for, say, the ψ in the vicinity of $\theta = 0$ in the form

$$(2.22) \quad \psi = P_1 z_1 + P_2 z_2,$$

where

$$(2.23) \quad \begin{aligned} z_1 &= s^{3m/2} F\left(-\frac{m}{2}, \frac{1}{3} - \frac{m}{2}; \frac{1}{2}; -\frac{9\theta^2}{4s^3}\right), \\ &= \left(\frac{3}{2}\right)^m r^m F\left(-\frac{m}{2}, \frac{1}{6} + \frac{m}{2}; \frac{1}{2}; \frac{\theta^2}{r^2}\right), \end{aligned}$$

and

$$(2.24) \quad \begin{aligned} z_2 &= \frac{3}{2} \theta s^{3/2(m-1)} F\left(\frac{1-m}{2}, \frac{5}{6} - \frac{m}{2}; \frac{3}{2}; -\frac{9\theta^2}{4s^3}\right), \\ &= \left(\frac{3}{2}\right)^m \theta r^{m-1} F\left(\frac{1-m}{2}, \frac{2}{3} + \frac{m}{2}; \frac{3}{2}; \frac{\theta^2}{r^2}\right), \end{aligned}$$

with $r^2 = \theta^2 + \frac{4}{3}s^3$.

We calculate P_1 , P_2 by means of the formula for the continuation of the hypergeometric function when its argument is in the vicinity of 1. The details are a little tedious, see (2.35) ... (2.40), but the result is rather simple. (The proof given here requires the formula for continuation of the hypergeometric function. An alternative proof makes use of the series solution $\psi(\theta, s) = \sum a_n \sin n\theta f(n^{2/3}s)$, the asymptotic formula for Bessel functions, and the properties of trigonometric series. Yet another proof which involves only elementary steps can be based on the definite integral solutions of Germain (1952) formula (39) as applied in Manwell (1958) (2.7) (2.12).) We find that $\psi \rightarrow \bar{\psi}$ which is represented by the form (2.19) with $\tilde{A}$, $\tilde{B}$; $\tilde{A}^*$, $\tilde{B}^*$ according to the scheme

$$(2.25) \quad \frac{\tilde{A}}{A} = -\frac{\tilde{B}^*}{B^*} = 2 \sin\left(\frac{1}{6} - m\right)\pi,$$

$$(2.26) \quad \frac{\tilde{A}^*}{A^*} = -\frac{\tilde{B}}{B} = 2 \cos m\pi,$$

and there is a similar representation for the potential functions.

In the case of solutions of the transonic flow equations with a shock discontinuity which may be represented by simple homogeneous functions the

loci Λ_i must be given by $\lambda = \text{constant}$ where

$$(2.27) \quad \lambda^3 = \frac{4}{9} \frac{s^3}{\theta^2}, \quad r^2 = \theta^2 + \frac{4}{9} s^3.$$

Here, since clearly $d\psi \neq 0$ along any locus $\lambda = \text{const.}$ other than the locus $\psi = 0$, we may apply Lemma (1) to give

$$(2.28) \quad \lambda_i^3 + 1 \leq 0, \quad i = 1, 2.$$

It is necessary to distinguish eight possible configurations of the Λ_i in the hodograph plane. These arise as follows. Let us denote the two possible positions of Λ_i in the quadrant $\alpha, \beta > 0$ for $\theta > 0$ and $\theta < 0$ respectively, by the numeral $I = 1, 2$. Again, the four possible positions of Λ_2 as we go round the origin in a clockwise direction starting in the sector $\theta < 0, s < 0$ and proceeding across the axes $s = 0, \theta < 0; \theta = 0, s > 0; s = 0, \theta > 0$, respectively, will be denoted by $J = 1, 2, 3, 4$.

The Cauchy problem with analytic data is well set on any suitable arc in the subsonic region and the only possible folds in the hodograph representation must lie in the supersonic region. According to the theory of hyperbolic equations, compare the solutions for the convergent divergent nozzle, the folds are necessarily characteristic. However, in the case of the homogeneous shock wave problem another peculiarity of the possible hodograph representations must be taken into account. The hodograph for the cases $I = 1, 2; J = 3, 4$ cannot cover the region simply. This depends on the fact proved in Lemma (3) below that, in a simple hodograph representation of the flow between Λ_1 and Λ_2 , the second branch of the level line $\psi = 0$ lies in $\theta < 0$. Fig. (1). It is then easily seen that in the cases $J = 3, 4$ for which Λ_2 lies in $\theta > 0$, if we are to avoid

FIGURE 1. Hodograph Plane for a Supersonic/Subsonic Transition of Type $I = 1, J = 2$

 FIGURE 2. Hodograph Plane: Proposed Solution of Type $I = 2, J = 3$

a fold in the subsonic region it is necessary to consider at least a two-sheeted representation: that part of the hodograph which corresponds to the flow immediately behind the shock wave lies over the original sheet which corresponds, in part, to the solution immediately to the front of the shock and in part, to certain of the flow lines which are in the neighborhood of the tip of the shock, but do not intersect it. The latter part of the solution is connected with the solution immediately behind the shock in such a way that we have to pass once (or more) round the origin in order to pass from one sheet to the other. Fig. (2). The reason for drawing attention to these non-simple hodograph representations is that, as we shall see below, they alone contain possible concave transitions. Also, it is desirable in connection with possible shock reflections to consider hodographs which are not entirely connected with the subsonic region. However, even with these generalizations the author has been able to rule out concave or normal shock transitions within the class of homogeneous solutions.

For the possible solutions of class $I = 1, 2; J = 1, 2$ we have the following:

Lemma (2). *In the case of the homogeneous solutions, if the hodograph is of one sheet and the configuration of the Λ_i defined by $I = 1, 2; J = 1, 2$, then the conditions $K_i \geq 0$ are necessary and sufficient that any solution of the shock wave problem is completely mappable into a physical flow without limit lines.*

Let us first consider the possibility that limit lines arise in the hodograph solution between the locus Λ_2 , supposing this to be supersonic of course, and the sonic line. The converse statement of Lemma (1) is applicable since ψ

increases along any locus $\lambda = \text{const.}$ other than the image of the level line $\psi = 0$, and so in any such solution of the shock wave problem we have $g > 0$ at Λ_2 . However, it was proved in Manwell (1952) that if $g > 0$ along such a non-characteristic locus as Λ_2 then $g > 0$ at all sonic or supersonic points which can be joined to Λ_2 by characteristics of both families drawn in the directions α, β increasing. Similarly, for points between Γ_1 and $s = 0$ we can check from the expansions that $g > 0$ in the neighborhood of Γ_1 and so infer $g > 0$ in the whole sector.

The other possibility is that limit lines might arise within the sector $\alpha, \beta > 0$ between Λ_1 and $\beta = 0$. Here we can check by differentiation of (2.5) with (2.7) (2.8) that $\psi_\beta > 0$. Since g is positive with the product ψ_α, ψ_β the question reduces to a consideration of the term ψ_α . Now ψ_α is positive near Γ_1 as we can see by expanding for small values of β . Also, it is positive at Λ_1 since there $g > 0$ according to Lemma (1). Hence, either ψ_α remains positive or it vanishes on certain arcs $\beta/\alpha = \text{const.}$ in $0 < \beta/\alpha < (\beta/\alpha)_1$. The situation at the locus nearest to Λ_1 is therefore that ψ_α changes sign so as to become positive along the direction of β and so s increasing. However, this is not possible because the proof of the Lemma of Manwell (1952) turns on the proposition that in these circumstances ψ_α changes sign in the opposite sense, to that stated. We conclude that the condition $K_1 \geq 0$ together with the shock conditions ensures $g > 0$ in the whole mixed region under consideration.

This discussion of the mappability of the homogeneous solutions makes use of their special properties and for general solutions of the direct problem, see Section (4), a different approach is necessary.

Germain's inequality $1/2 < m < 5/6$ for a supersonic to subsonic shock.

The second part of the inequality is contained in (2.13). If we postulate a simple hodograph representation between a supersonic Λ_1 and a subsonic Λ_2 the second part can be proved without making any use of the shock relations other than those embodied in Lemma (1). More precisely, if we assume the configuration $\Lambda_1 : \alpha, \beta > 0$, $\Lambda_2 : s > 0$, briefly denoted by $I = 1, 2; J = 2$, the following is a purely geometrical property of the homogeneous solutions.

Lemma (3). *If the homogeneous solution $\psi_{(2)} = \psi^m(\theta, s; K_2)$ for $K_2 \geq 0$, $0 < m < 5/6$ vanishes on at least two curves in the union of the region $s > 0$ and the sector bounded by $s = 0, \theta > 0$ and Γ_1 then $\psi_{(2)}$ vanishes precisely twice, once in the latter region and once in $s > 0, \theta < 0$; also $m > 1/2$.*

Let us first consider the cases $0 < m < 2/3$ which are sufficient as far as Germain's result is concerned. Any homogeneous solution can be represented by (2.22) with (2.23) (2.24) and in case $m < 2/3$, z_1 and z_2 are non-negative if $\theta > 0$. Hence the solution can vanish in $\theta > 0$ only if the P_i take opposite signs. However, we can easily show that

$$(2.29) \quad \frac{\partial z_1(\theta, r)}{\partial \theta} < 0, \quad \frac{\partial z_2}{\partial \theta}(\theta, r) > 0,$$

for any $0 < \theta < r$ and $0 < m < 1$. It follows that the root is simple and, obviously, in this case $\psi_{(2)}$ does not vanish in $\theta < 0$. A similar argument applies if the P_i are of the same sign in which case there is at most one simple root lying in $\theta < 0$. On the other hand, we find from (2.18)

$$(2.30) \quad \frac{\partial \psi_{(2)}}{\partial \beta} > 0, \quad \alpha > 0, \quad \alpha + \beta > 0,$$

showing that $\psi_{(2)}$ has at most one simple root in the sector bounded by $s = 0$ and Γ_1 . We now see that the complete locus $\psi = 0$ consists of one branch in this sector and another in $s > 0$. It follows that, under the conditions of Lemma (3), $\psi_{(2)}$ is zero or negative at $s = 0, \theta > 0$ and positive at $s = 0, \theta < 0$. Using (2.19) (2.25) (2.26) these last conditions become

$$(2.31) \quad A(m) - K_2 A^*(m) \leq 0,$$

$$(2.32) \quad (A - 2K_2 A^*) \cos m\pi - (3)^{1/2} A \sin m\pi > 0,$$

where for $0 < m \leq 2/3, A \geq 0$. However (2.31) (2.32) are inconsistent unless $\cos m\pi < 0$ which implies $m > 1/2$. Germain established $1/2 < m < 5/6$ for a general class of homogeneous solutions. His proof depends on a consideration of the trajectories of a non-linear differential equation.

To complete the Lemma we need to show that $\psi_{(2)}$ remains negative in $\theta > 0$. It is sufficient to prove $\psi_2(\theta = 0, s > 0) < 0$, which in turn follows from $P_{(1)} < 0$ for $1/2 < m < 2/3$. The last statement will be dealt with after we have proved the Lemma for the cases $2/3 \leq m < 5/6$.

In case $2/3 \leq m < 5/6$ we have (2.29), also z_2 is positive with $\theta > 0$ but z_1 changes sign in the range $0 < \theta^2/r^2 < 1$. However, by direct calculation we have

$$(2.33) \quad A(m) - K_2 A^*(m) \leq 0, \quad \psi(\theta > 0, s = 0) \leq 0,$$

and

$$B(m) - K_2 B^*(m) > 0,$$

showing that $\psi_{(2)}$ remains negative in $\theta \geq 0, s > 0$ according to (2.19) and (2.14) (2.15). The existence of a root in $\theta > 0, s \leq 0$ is obvious and its simplicity follows from (2.30). Again, for the behaviour of the solutions in $\theta < 0$ we have by (2.20) (2.21) and (2.25) (2.26)

$$(2.34) \quad \tilde{A}(m) - K_2 \tilde{A}^*(m) \geq 0, \quad \psi(\theta < 0, s = 0) \geq 0, \quad 2/3 \leq m < 5/6,$$

$$\tilde{B}(m) - K_2 \tilde{B}^*(m) > 0,$$

showing that there is certainly one root in $\theta < 0$. However, there will be one (simple) root if we can show that in the representation (2.22) with (2.23) (2.24) for $\theta < 0$ we have both $P_{(1)}, P_{(2)}$ negative. For in this case $\psi_{(2)}(|\theta|, r)$ is an increasing function of $|\theta|$. Now $P_{(1)}$ is negative on account of the behavior in $\theta > 0$ as $\theta \rightarrow 0$ and to prove $P_2 < 0$ we appeal to the explicit relations arising

in the continuation of $\psi_{(2)}$ across the line $\theta = 0$. These are as follows

$$(2.35) \quad 2c_1P_1 = (2/3)^m(A + \tilde{A}),$$

$$(2.36) \quad 2c_2P_2 = (2/3)^m(A - \tilde{A}),$$

$$(2.37) \quad 2d_1P_1 = -2^{2/3}(2/3)^m(B + \tilde{B}),$$

$$(2.38) \quad 2d_2P_2 = 2^{2/3}(2/3)^m(\tilde{B} - B),$$

the coefficients being determined as

$$(2.39) \quad c_1 = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3} - \frac{m}{2})\Gamma(\frac{1+m}{2})}, \quad d_1 = \frac{\Gamma(-\frac{1}{3})\Gamma(\frac{1}{2})}{\Gamma(-\frac{m}{2})\Gamma(\frac{1}{6} + \frac{m}{2})} > 0,$$

$$(2.40) \quad c_2 = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{6} - \frac{m}{2})\Gamma(1 + \frac{m}{2})} > 0, \quad d_2 = \frac{\Gamma(-\frac{1}{3})\Gamma(\frac{3}{2})}{\Gamma(\frac{1-m}{2})\Gamma(\frac{2}{3} + \frac{m}{2})} < 0.$$

From (2.26) (2.40) and equation (2.38) for $P_2(m)$ also (2.36) for $P_2^*(m)$ we can show $P_2 \leq 0$, $P_2^* > 0$ for $2/3 \leq m < 5/6$ and so

$$(2.41) \quad P_{(2)} = P_2(m) - K_2P_2^*(m) \leq 0.$$

The equality applies only if $m = 2/3$, $K_2 = 0$ in which case the solution is trivially $\psi = -s$: the Lemma applies with the two branches $\psi = 0$ along $s = 0$.

In a similar way, by using (2.35) (2.36) for the starred quantities and (2.37) (2.38) for the unstarred quantities with (2.26) we find that for $1/2 < m < 2/3$ we have $P_1 < 0$, $P_1^* > 0$, $P_2 > 0$, $P_2^* > 0$. These ensure $P_{(1)} < 0$ and so the second root must lie in $\theta < 0$.

Let us now consider the following:

Lemma (4). *A shock solution of the simplest type $I = 1, 2$; $J = 2$ is necessarily convex.*

This is obvious if $I = 1$ and in view of (2.1) is reduced to showing that $(d\phi/d\psi)_2 > 0$ in the other cases. For the homogeneous solutions with the Λ_i defined by $\lambda = \text{const.}$ it is sufficient to show that $\phi_{(2)}(\lambda_2) > 0$.

Let us write the hodograph equations in terms of λ, r as independent variables. We find

$$(2.42) \quad \frac{\partial\phi}{\partial r} = \frac{\lambda r}{\theta s} \frac{\partial\psi}{\partial\lambda}, \quad \frac{\partial\phi}{\partial\lambda} = -\frac{\theta s^2}{\lambda r} \frac{\partial\psi}{\partial r},$$

and the homogeneous solutions will be written briefly as

$$(2.43) \quad \phi = r^{m+1/3}g(\lambda), \quad \psi = r^mf(\lambda).$$

In case $\theta < 0$ we find further

$$(2.44) \quad \begin{aligned} f'(\lambda) &= -(3/2)^{2/3}(m + \frac{1}{3})(1 + \lambda^3)^{-5/6}g(\lambda), \\ g'(\lambda) &= (3/2)^{4/3}m\lambda(1 + \lambda^3)^{-7/6}f(\lambda). \end{aligned}$$

According to Lemma (3) $f(\lambda)$ of the solutions under discussion is positive between $\lambda = 0, \lambda = \lambda_2 > 0$. Also, it follows from (2.34) that $g(0)$ is positive and then the second of equations (2.44) ensures that $g(\lambda_2)$ is positive.

By a method which is similar to that used in Lemma (10) below, the author has shown elsewhere that the cases $I = 2; J = 2$ do not yield shock solutions. However, apart from this further result Lemma (4) can be easily strengthened to

Lemma (5). *Any shock solution of the simplest types $I = 1, 2; J = 1, 2$ must be convex.*

The cases having $I = 1$ need not be considered, $I = 2, J = 2$ is Lemma (4) and we now consider $I = 2, J = 1$. Equations (2.1) (2.2) can be arranged to give:

$$(2.45) \quad \operatorname{sgn}(\theta_1) \left| \frac{1+\mu}{\lambda_1} \right|^{3/2} - \operatorname{sgn}(\theta_2) \left| \frac{1-\mu}{\lambda_2} \right|^{3/2} = 3 \operatorname{sgn} \left(\frac{d\phi}{d\psi} \right) \mu',$$

$$\nu = 1, \quad s_2 \leq 0, \quad \nu = \frac{1}{2}, \quad s_2 > 0,$$

where it is convenient to set

$$(2.46) \quad \left| \frac{s_2}{s_1} \right| = \frac{1-\mu}{1+\mu}, \quad 0 < \mu \leq 1,$$

and we note that $s_2 > s_1$ in all shock transitions. In the present case $s_2 < 0$ and we have $|\lambda_1| > 1, |\lambda_2| < 1$ and since $\theta_i < 0$ it is easy to deduce from (2.45) the inequality

$$(2.47) \quad (1-\mu)^{3/2} - (1+\mu)^{3/2} < 3\mu \operatorname{sgn} \left(\frac{d\phi}{d\psi} \right).$$

However, this inequality is possible only if the right hand member is taken with the positive sign. The Lemma follows at once.

On convex and concave shock transitions. Let us consider a solution of the equations of plane transonic flow derived from the hodograph representation of equations (1.5) also (2.4). Let us further assume that we have a level line, say, $\psi = C$ which crosses the axis $s = 0$ for some $\theta, \cos \theta > 0$ and remains in one-one correspondence with the characteristics of equation (2.4). This correspondence is necessary if the solution is to be mappable into a physical flow that is to say one free of limit lines.

For the solution to be mappable it is sufficient if we know also that on such a correctly set level line the derivatives ψ_α, ψ_β do not vanish. In the vicinity of $\psi = C$ the stated conditions ensure $g > 0$, and it can be proved, (cf. Manwell (1952)), that the whole of the mixed solution between $\psi = C$ and the sonic line is mappable. Indeed the situation is extremely simple, for it turns out that given sufficient smoothness of the hodograph solution and the proper behavior at the chosen level line $\psi = C$ all the level lines between $\psi = C$, and $s = 0$ will have the same properties as does $\psi = C$.

We can now see that in the physical flow solution every such level line determines a *convex* streamline with respect to the mixed flow. With the notation of equations (1.1) (1.2) (1.3) let us work in the coordinate plane of θ , s . Along any locus $\psi = C$ the conditions

$$d\alpha \geq 0, \quad d\beta \leq 0, \quad d\psi = 0$$

ensure that ψ_α , ψ_β cannot take opposite signs, and we will take for convenience ψ_α , ψ_β positive. It is now a very easy consequence of equations (2.4) that when we move along the level line $\psi = C$ in the direction of θ decreasing then will $d\phi > 0$ and assuming $\cos \theta > 0$ we have $dx > 0$. Again, along any locus $\phi = \text{const.}$ moving towards the sonic line we have $d\alpha < 0$, $d\beta < 0$ and $d\psi < 0$, $dy > 0$. These last statements concerning the physical flow determine a pattern in which x increases and θ decreases along the boundary, while at the same time the ordinate y increases at every point as we move into the flow.

In the case of flows with weak shock discontinuities and, say, the supersonic/subsonic transition the hodograph representation of the local supersonic region is bounded only in part by the image of the contour and by the sonic line since now we have to include the image of the front of the shock. If we have a supersonic/supersonic transition the image of the local supersonic region is not connected, and the two separate sectors are bounded respectively by arcs of the boundary $\psi = \text{const.}$, the images of the front and rear of the shock and two sonic arcs. However, in spite of the considerable differences in both the structure of the hodograph solutions and the details of the analysis it turns out that within a very general class of *homogeneous* solutions the convexity noted above for continuous solutions persists to the case of shocks.

Germain (1956) claims to have disproved the possibility of a normal shock transition. It seems to the present writer that the only configurations considered in that paper are *a priori* impossible by Lemma (1) because the author has at no point mentioned analytic continuation across the line which is necessary even in the simplest cases. In a second paper Germain (1958) the homogeneous solutions are treated by an indirect method based on a first order non-linear differential equation. This method appears to be fairly general, for the example computed in the case of the index $m = 2/3$ is of type $I = 2$; $J = 1$, see Tricomi and Ferrari (1962). However, the second paper contains no further treatment of the impossibility of normal shocks. The present account disproves these for a general class of solution of the types $I = 1, 2$; $J = 1, 2$ also $I = 1, 2$; $J = 3, 4$ see Section 3 below.

Lemmas (6) (7) below contain some further results which will be needed in connection with the latter four cases, only. However, this analysis, compare Lemma (3), turns on the properties of the homogeneous solutions as such, and makes very little use of the shock equations (2.1) (2.2). These, along with the explicit expressions of (2.5) (2.6) are not introduced until (2.55) below.

Lemma (6). *If $g > 0$ in any bounded homogeneous solution defined both in $s > 0$ and in the complete quadrant $\alpha, \beta \geq 0$ then $0 < m < 2/3$.*

If we use the representation (2.5) with the second forms of (2.7) (2.8) and make use of the formulae for the continuation of the hypergeometric functions near $\alpha = 0$ the condition that $g > 0$ in the vicinity of $\alpha = 0$ may be written as

$$(2.48) \quad \frac{m}{m + \frac{1}{6}} \frac{\Gamma(\frac{5}{6} - m)^2}{\Gamma(\frac{2}{3} - m)\Gamma(1 - m)} - \tilde{K}_1 \frac{\Gamma(\frac{5}{6} - m)\Gamma(\frac{1}{6} + m)}{\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})} \geq 0,$$

Here, since $\tilde{K}_1 \geq 0$ either $0 < m < 2/3$ or $m = 2/3$, $\tilde{K}_1 = 0$ and in the latter case $\tilde{\psi}_{(2)}$ reduces to a multiple of s . It should be observed that it has not been assumed that the continuation $\tilde{\psi}_{(2)}$ must be positive at either of the characteristics; the condition $g > 0$ does of course ensure the non-vanishing of the solution along either characteristic.

Lemma (7). *If $\tilde{\psi}_{(2)} > 0$ at the Γ_1 and the hodograph solution is mappable in the complete quadrant then $1/3 < m \leq 2/3$; correspondingly if the continuation in $s > 0$ changes sign between the characteristics $0 < m \leq 1/3$.*

By some tedious but straightforward calculations we can show that the solution (2.6) has a unique continuation as far as Γ_2 and for $|\lambda_2| < 1$ is represented by

$$(2.49) \quad \tilde{\psi}_{(2)} = (C - K_2 C^*)\tilde{\psi} + (D - K_2 D^*)\tilde{\psi}^*,$$

where $(\bar{})$ denotes that α, β are to be interchanged. Here we have

$$(2.50) \quad C(m) = \frac{\cos(2m + \frac{1}{3})\pi}{\sin(m + \frac{1}{6})\pi}, \quad C^*(m) = -\frac{2\Gamma(\frac{7}{6} + m)^2 \sin(m - \frac{1}{3})\pi}{(m + \frac{1}{6})\Gamma(1 + m)\Gamma(\frac{1}{3} + m)},$$

$$(2.51) \quad D(m) = \frac{2\Gamma(\frac{5}{6} - m)^2 \sin(m - \frac{1}{3})\pi}{(m + \frac{1}{6})\Gamma(\frac{2}{3} - m)\Gamma(-m)}, \quad D^*(m) = -C(m).$$

It is easily checked that $C^2 + DC^* = 1$, $CD + DD^* = 0$ showing that a repetition of the continuation by way of the subsonic region according to (2.49) brings us back to the original representation of the solution in the vicinity of $\beta = 0$.

If $g > 0$ near Γ_2 for $\beta > 0$, $\alpha > 0$ we consider first the case $\tilde{\psi}_{(2)} > 0$ at Γ_2 . Then we have

$$(2.52) \quad C - K_2 C^* > 0, \quad K_2 \geq 0,$$

$$(2.53) \quad D + K_2 C \leq 0.$$

The case $K_2 = 0$ is easily disposed of since we know by Lemma (6) that $m \leq 2/3$ and the conditions (2.52) (2.53) lead to $7/12 < m \leq 2/3$. If however, $K_2 > 0$ we deduce from (2.52) (2.53)

$$(2.54) \quad D + K_2^2 C^* < 0$$

and this forces $m > 1/3$.

Similarly, if $\tilde{\psi}_{(2)}(\Gamma_2) < 0$ we have (2.52) (2.53) with the signs reversed, and for $K_2 > 0$ the inequality (2.54) is reversed too, which leads to $m < 1/3$.

Finally, if $K_2 = 0$ in the case $\tilde{\psi}(\Gamma_2) < 0$ the reversed inequalities lead to $1/12 < m \leq 1/3$. This completes the proof of Lemma (7). If we assume Germain's inequality $m > 1/2$ proved above for a special configuration, the case $\tilde{\psi}(\Gamma_i) < 0$ would be ruled out immediately. Also, it is easily seen that within the field of homogeneous solutions the corresponding hodograph representation would be folded throughout the plane and not only in a certain finite neighborhood of the origin. However, as it is a main point in the present paper that the homogeneous solutions need not represent the flow except in the vicinity of the tip of the shock it is important to have a direct proof that $\tilde{\psi}(\Gamma_i) < 0$ cannot arise. For the case of possible *concave* shock transitions, this is given in Lemma (9).

In order to complete the proof it turns out that we need to make a more detailed use of the shock conditions, equations (2.1) (2.2) with (2.5) (2.6) than hitherto. It is therefore convenient to first present a general reduction of these equations and to prove the existence of a class of homogeneous solutions. The reader will note that the remainder of Section (2) is independent of Lemmas (4) (5) (6) (7). These refer to general properties which are not required for the main result on the existence of shock transitions, Theorem 1 below. It is easy to show from equations (2.1) (2.2) (2.5) (2.6) that at the respective Λ_i

$$(2.55) \quad \phi_{(1)} = \phi + K_1 \phi^* = R_{12}(\psi + K_1 \psi^*) = R_{12} \psi_{(1)} ,$$

$$(2.56) \quad \phi_{(2)} = \tilde{\phi} - K_2 \tilde{\phi}^* = R_{12}(\tilde{\psi} - K_2 \tilde{\psi}^*) = R_{12} \psi_{(2)} ,$$

where

$$(2.57) \quad R_{12} = \frac{3m}{3m+1} \left| \frac{s_1 + s_2}{2} \right|^{1/2} .$$

Here we have chosen the case of convex shocks; in the case of concave shocks the sign of R_{12} must be reversed. From (2.55) (2.56) we find first

$$(2.58) \quad K_1 = \frac{R_{12} \psi_1 - \phi_1}{\phi_1^* - R_{12} \psi_1^*} \geq 0 ,$$

$$(2.59) \quad K_2 = \frac{\tilde{\phi}_2 - R_{12} \tilde{\psi}_2}{\tilde{\phi}_2^* - R_{12} \tilde{\psi}_2^*} \geq 0 ,$$

and using these we get

$$(2.60) \quad \psi_{(1),1} = \frac{\psi_1 \phi_1^* - \phi_1 \psi_1^*}{\phi_1^* - R_{12} \psi_1^*} > 0 ,$$

$$(2.61) \quad \psi_{(2),2} = \frac{\tilde{\psi} \tilde{\phi}^* - \tilde{\phi} \tilde{\psi}^*}{\tilde{\phi}^* - R_{12} \tilde{\psi}^*} > 0 .$$

It is not difficult to show from the relations (2.43) (2.44) that we can simplify the numerators in (2.60) (2.61) according to

$$(2.62) \quad (\psi \phi^* - \phi \psi^*)_1 = \left(\frac{3}{4} \right)^{1/3} \left(\frac{6m+1}{3m+1} \right) (\alpha_1 \beta_1)^{m+1/6} > 0 ,$$

$$(2.63) \quad (\tilde{\psi}\tilde{\phi}^* - \tilde{\phi}\tilde{\psi}^*)_2 = \left(\frac{3}{4}\right)^{1/3} \frac{6m+1}{3m+1} (\theta_2^2 + \frac{4}{9}s_2^3)^{m+1/6} > 0.$$

It follows from (2.58) (2.60) (2.62) that the condition $K_1 \geq 0$ with $\psi(\Lambda_i) > 0$ is satisfied if and only if

$$(2.64) \quad R_{12}\psi_1 - \phi_1 \geq 0.$$

Similarly, from (2.59) (2.61) (2.63) $K_2 \geq 0$ if

$$(2.65) \quad \tilde{\phi}_2 - R_{12}\tilde{\psi}_2 \geq 0.$$

According to Lemma (2) the conditions (2.64) (2.65) are sufficient to ensure that any solution of the four hodograph equations comprised in (2.1) (2.2), that is to say, any solution of the direct problem in the homogeneous case, can be mapped into a physical flow with a weak shock. More precisely, the solution satisfies Germain's homogeneous relations: for points sufficiently close to the origin, it satisfies the strict conditions for a weak shock as nearly as we please.

In solving equations (2.12) let us note that (2.45) (2.46) provide a convenient parametric representation of either of the two last (elementary) relations in (2.1) (2.2) respectively. The two constants K_i having been eliminated the direct problem is reduced to the solution of the simultaneous equations (2.45) and (2.66) below. The former is to be taken for the case $\theta_1 > 0$, $\theta_2 < 0$, $d\phi/d\psi > 0$, and equating (2.60) (2.61) we find, say,

$$(2.66) \quad \psi_{(1)} = \theta_1^m F_1^{(m)}(\lambda_1, \mu) = \psi_{(2)} = \theta_2^m F_2^{(m)}(\lambda_2, \mu).$$

It is rather easily seen that the λ_i and μ being connected by (2.45) (2.46) we have also

$$(2.67) \quad \left| \frac{\theta_1}{\theta_2} \right| = \left| \frac{\lambda_2}{\lambda_1} \right|^{3/2} \left(\frac{1+\mu}{1-\mu} \right)^{3/2}.$$

Since (2.45) and (2.67) are elementary relations the solution of the direct problem has been reduced, in effect, to the discussion of the single relation (2.66) involving any two of the three parameters λ_1 , λ_2 , μ and for a given index m . Let us now consider the main result of Section 2.

Theorem 1. *There exists a two-parameter family of solutions of the local shock wave problem whose type is $I = 1$; $J = 1, 2$ and in which $2/3 \leq m < 5/6$.*

These solutions are all convex and for $I = 1$; $J = 2$ we find the desired supersonic to subsonic transition. The proof will be given in detail for this case only. Working out (2.66) according to (2.60) (2.62) with (2.8) (2.10) we find

$$(2.68) \quad \left\{ \frac{\phi^* - R_{12}\psi^*}{|\alpha\beta|^{m+1/6}} \right\}_1 = \left(\frac{3}{4}\right)^{1/3} \alpha_1^{-m} \left(1 + \frac{\beta_1}{\alpha_1}\right)^{1/6} \left[F_{(\phi^*)} - \frac{3m}{3m+1} \left(\frac{\mu}{1+\mu}\right)^{1/2} F_{(\psi^*)} \right].$$

Here, $F_{(\psi^*)}$, $F_{(\phi^*)}$ denote the hypergeometric functions of argument $\beta/(\alpha + \beta)$ in (2.8) (2.10) respectively. Again, making use of (2.19) (2.21) with (2.14) (2.17) we have the expansion (2.69):

$$(2.69) \quad \left\{ \frac{\tilde{\phi}^* - R_{12}\tilde{\psi}^*}{(\theta^2 + \frac{4}{9}s^3)^{m+1/6}} \right\}_2 \\ = |\theta_2|^{-m} \left\{ \frac{3m}{3m+1} \left(\frac{3}{2} \right)^{1/3} \frac{|\tilde{A}^*|}{\Theta} \left(\frac{\mu}{1+\mu} \right)^{1/2} - \frac{3(\frac{4}{9})^{2/3} |\tilde{B}^*|}{3m+1} \right\} + O(\lambda_2).$$

It is convenient to set

$$(2.70) \quad \Theta = \left| \frac{1-\mu}{(1+\mu)\lambda_2} \right|^{1/2},$$

and then (2.45) becomes

$$(2.71) \quad \frac{1}{|\lambda_1|^{3/2}} = \frac{3\mu^{1/2}}{(1+\mu)^{3/2}} - \Theta^3.$$

The existence proof depends on two remarks: that we may determine a range of Θ , λ_1 for which both λ_2 and $(1-\mu)$ as determined by (2.70) (2.71) tend to zero together, also that (2.66) is an analytic relation between the variables λ_2 , $(1-\mu)$ in the vicinity of $\lambda_2 = 0 = 1-\mu$. We shall prove for a suitable index 'm' that the difference $\psi_{(2)} - \psi_{(1)}$ of (2.66) changes sign with values of $\Theta_2 = \Theta(\lambda_2, \mu)$ such that the value of $|\lambda_1|$ as determined by (2.71) satisfies $|\lambda_1| > 1$. In this way we show that there exists a set of values $\lambda_1 < -1$, $\lambda_2 > 0$, μ satisfying (2.66) subject to the auxiliary relations (2.45) (2.67). Then according to the implicit function theorem there is, for each suitable index m, a single infinity of such values. The existence of the individual solution for some sufficiently small positive λ_2 and a suitable choice of μ in the vicinity of $\mu = 1$ may be inferred by continuity with the relation (2.66) with (2.68) (2.69) taken for $\lambda_2 = 0 = 1-\mu$. In this case equation (2.66) reduces to (2.72):

$$(2.72) \quad |\tilde{A}^*| - \frac{2^{5/3}(1+1/\mu)^{1/2}}{3m} |\tilde{B}^*| \Theta \\ = 2^{-1/6} \left(\frac{\alpha_1 + \beta_1}{2\alpha_1} \right)^{m+1/6} \Theta^{3m+1} \left[\frac{3m+1}{3m} (2)^{1/2} F_{(\phi^*)} - F_{(\psi^*)} \right],$$

which, in view of (2.71) with $\mu = 1$ determines for each m a single relation for the unknown Θ (or $|\lambda_1|$). Corresponding to the range $1 < |\lambda_1| < \infty$ we have, say,

$$(2.73) \quad \Theta_1^3 = \frac{1}{8+6(2)^{1/2}} < \Theta^3 < \frac{3}{2(2)^{1/2}} = \Theta_\infty^3.$$

The expression $[\dots]$ on the right hand side of (2.72) is a decreasing function of $\beta/(\alpha + \beta)$ and, since its value for $\alpha \rightarrow 0$ is easily shown to be positive, it is certainly positive for all $0 < \beta/\alpha < 1$. (Similarly we find that $\psi(\Lambda_1)$ of (2.68) is positive for all $0 < \mu < 1$.) On the other hand the left hand member of (2.72)

will be negative for $\Theta = \Theta_\infty$ if we have (2.74)

$$(2.74) \quad \bar{\sigma}(m) = 2^{2/3} \frac{\bar{B}^*(m)}{m\bar{A}^*(m)} > \left(\frac{9}{8}\right)^{1/3} \doteq 1.04.$$

Furthermore we can show by using the series expansion for the logarithmic derivative that $\bar{\sigma}'(m) < 0$, $\bar{\sigma}(m) \geq \bar{\sigma}(5/6) = 1.16 > 1.04$. Hence, (2.74) is satisfied for all values of the parameter in $2/3 \leq m < 5/6$. Finally we must now show that for the other extreme value $\beta = 0$, $\Theta = \Theta_1$, the left hand member of (2.72) exceeds the right. Explicitly we require

$$(2.75) \quad |\bar{A}^*| \left[1 - \frac{2}{3} \left(1 + \frac{1}{\mu} \right)^{1/2} \bar{\sigma}(m)\Theta_1 \right] > \frac{\Theta_1^{3m+1}}{2^{m+1/3}} \left[(2)^{1/2} \left(1 + \frac{1}{3m} \right) - 1 \right].$$

Now, $|\bar{A}^*(m)|$ increases with m in $(2/3, 5/6)$ and $\bar{\sigma}(m)$ decreases. Hence, the left hand member (2.75) is an increasing function of m . Again, since $\Theta_1^3 < 2$ it is clear that the right hand member for fixed $\Theta = \Theta_1$ decreases with m . The desired inequality (2.75) will hold for all values of m if it is true for the smallest value $m = 2/3$. For this case a simple computation determines the values .0532, .0340 for the left and right hand members, respectively. The proof of the existence of solutions of the hodograph equations for the local shock wave problem is now complete.

To show that these solutions are mappable into physical flow we need to check the inequalities (2.64) (2.65). Now, in the particular case $m = 2/3$ we have $\psi = -s$, $\phi = -\theta$ showing that ψ_1 , $\bar{\phi}_2$ are positive and ϕ_1 , $\bar{\psi}_2$ negative for all configurations $I = 1$; $J = 2$. Also it is easily seen that the expressions in (2.64) (2.65) remain positive for sufficiently small $\lambda_2 > 0$ and all $\lambda_1 < -1$ provided $m - 2/3$ is sufficiently small and positive. The existence of the local solutions having the supersonic/subsonic transition is now complete; very slight changes in the preceding argument leads to the existence of a two parameter family of type $I = 1$; $J = 1$.

It seems likely that (2.66) could be solved for a suitable value of m giving equality in (2.64) (2.65); one would expect these last two relations to determine completely the four quantities λ_1 , λ_2 , μ , m . If this is so there is one privileged member of the two parameter family in which ψ is regular in the neighborhood of Γ_1 .

3. The non-existence of normal or concave shock transitions within the family of homogeneous solutions. According to Lemma (5) the transitions associated with simple solutions of type $I = 1, 2$; $J = 1, 2$ are convex. In this section we shall first consider the simple hodograph solutions of type $I = 1, 2$; $J = 3, 4$ and later extend the argument to more general cases.

It is obvious that the cases with $I = 2$; $J = 3, 4$ cannot be convex whatever the structure of the hodograph. The arguments of Lemma (5) with a trivial change of sign ensure $d\phi/d\psi < 0$ and so concavity for any solution of type $I = 1$; $J = 4$. We have one other possibility, normal shocks for $I = 1, 2$; $J = 3$.

Now there are two loci $\phi_1 = \phi_2 = c = 0$ say, passing through the tip of the shock which in the case of a normal transition must be the Λ_i . On the other hand recalling the remarks after equation (2.28) the hodograph must contain a complete quadrant $\alpha, \beta > 0$ within which there is always a third locus $\tilde{\phi} = 0$ and this is not permissible. (As for possible *convex* transitions of type $I = 1$; $J = 3$ we can show that the two loci $\phi = 0$ lie in the sectors bounded by Λ_1 , Γ_1 and $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ respectively. It follows easily that $\tilde{\phi}(\tilde{\Gamma}_1) > 0$, $\tilde{\psi}(\tilde{\Gamma}_i) < 0$, the hodograph remains permanently folded and $m \leq 1/3$ by Lemma (7). Such solutions do not fall into the class considered by Germain.)

We shall now continue the general discussion of Lemmas (3) (4) (5) (6) (7). It will be remarked that these previous results turned on geometrical properties of the homogeneous solutions and the elementary properties of the shock polar. In what follows we need less elementary facts derived from (2.55) *et seq.*

Lemma (8). *For all concave transitions of the simplest type $I = 1, 2$; $J = 3, 4$ we have $\tilde{\psi}(\tilde{\Gamma}_i) > 0$ also $1/3 < m \leq 2/3$.*

Equations (2.55) $\dots$ (2.65) for $R_{12} < 0$ lead to

$$(3.1) \quad \phi_1 + |R_{12}| \psi_1 \leq 0, \quad \phi_1^* + |R_{12}| \psi_1^* > 0,$$

also

$$(3.2) \quad \phi_2 + |R_{12}| \psi_2 \geq 0, \quad \phi_2^* + |R_{12}| \psi_2^* > 0; \quad \tilde{\psi}(\tilde{\Gamma}_i) > 0,$$

and again

$$(3.3) \quad \phi_2 + |R_{12}| \psi_2 \leq 0, \quad \phi_2^* + |R_{12}| \psi_2^* < 0; \quad \tilde{\psi}(\tilde{\Gamma}_i) < 0.$$

We can now show that the second of (3.3) is impossible and so $\tilde{\psi}(\tilde{\Gamma}_i) > 0$ is necessary. According to (2.8) (2.10) if Λ_2 is supersonic then both ϕ_2^* and ψ_2^* are non-negative. Again, if $\theta, s \geq 0$, $m \leq 2/3$ we have $\psi_2^* > 0$ according to (2.14) (2.15) (2.19) (2.21). It is therefore sufficient in this case to show that $\phi_2^* \geq 0$. Now the last certainly holds at $s = 0$, $\theta > 0$ and making use of (2.44) with a change of sign since $\theta > 0$ we find that ϕ_2^* decreases with λ for fixed r . Hence, ϕ_2^* is positive in $\theta \geq 0$ provided this applies as $\theta \rightarrow 0$. We then use (2.3) and (2.36) with (2.40) to show successively

$$(3.4) \quad \text{sgn } \phi^* = -\text{sgn } (\psi_\theta^*) = -\text{sgn } (P_2^*) = \text{sgn } (\cos m\pi - \tfrac{1}{2}),$$

and if $\phi^*(\theta = 0, s > 0)$ is to be negative, then $m > 1/3$. However, according to Lemma (7) we have $m \leq 1/3$ in the case under consideration. Hence, $\phi^*(\theta, s > 0)$ is certainly positive and (3.3) is impossible; Lemma (8) now follows from Lemma (7).

Lemma (9). *There are no normal or concave transitions of the simplest type $I = 1$; $J = 3$.*

Let us write the last two members of equations (2.1) as

$$(3.5) \quad 2\left(\frac{d\phi}{d\psi}\right)^2 = |s_1| - s_2 ,$$

$$(3.6) \quad (\theta_1 - \theta_2) \frac{d\psi}{d\phi} = |s_1| + s_2 .$$

Eliminating $|s_1|$ and writing

$$(3.7) \quad \frac{d\phi}{d\psi} = -\left(\frac{\lambda\theta}{s}\right)_2 S ,$$

we find

$$(3.8) \quad 1 - 2\lambda S - \frac{8}{9} S^3 = \frac{\theta_1}{\theta_2} > 0 .$$

Lemma (8) enables us to assert that $\tilde{\psi}(\tilde{\Gamma}_i) > 0$ but it remains to be shown that the loci $\psi = 0$ lie in $D_{(+)} : \theta^2 + (4/9)s^3 > 0$ adjacent to the first characteristic Γ_1 and not in the sector bounded by $\tilde{\Gamma}_1$ and Λ_2 , that is to say on the hodograph sheet found after a complete continuation round the origin. To rule out this last possibility let us write $\tilde{\psi} = r^m \tilde{f}(\lambda)$ and $\tilde{\phi} = r^{m+1/3} \tilde{g}(\lambda)$. Applying (2.44) with trivial changes of sign since now $\theta > 0$, we infer from the existence of two roots of $\tilde{f}(\lambda)$ at least one root of $\tilde{g}(\lambda)$ in the sector bounded by $\tilde{\Gamma}_1$, Λ_2 . However $\tilde{\psi}(\tilde{\Gamma}_1) > 0$ forces $\tilde{\phi}(\tilde{\Gamma}_1) < 0$ showing that there are two loci $\tilde{\phi} = 0$ between $\tilde{\Gamma}_1$ and Λ_2 which is impossible because we have already encountered two such loci in $D_{(+)}$ and the complete quadrant $\alpha, \beta > 0$; and normal shocks are in consequence excluded *a priori*. Finally, if we use (2.19) (2.20) (2.21) we can deduce from $\tilde{\psi}(\tilde{\Gamma}_1) > 0$ that $\tilde{\phi}(\theta, s = 0) < 0$ and so we may assert for the solution adjacent to Λ_2

$$(3.9) \quad \tilde{f}(\lambda) > 0, \quad \tilde{f}(\lambda)' < 0; \quad \tilde{g}(\lambda) < 0, \quad \tilde{g}(\lambda)' < 0 .$$

Writing out

$$(3.10) \quad S = -\left(\frac{3}{2}\right)^{1/3} \frac{3m+1}{3m} (1 + \lambda^3)^{1/6} \frac{\tilde{g}(\lambda)}{\tilde{f}(\lambda)} ,$$

we now deduce

$$(3.11) \quad S(\lambda) > 0, \quad S'(\lambda) > 0 .$$

It now becomes obvious that the inequality (3.8) holds for some $\lambda_2 > 0$ only if it is valid for $\lambda = 0$. Now we have for any $\tilde{K}_2 \geq 0$

$$(3.12) \quad \begin{aligned} S(0) \equiv S^{(m)}(0) &= \frac{2^{2/3}}{m} \frac{B + \tilde{K}_2(-B^*)}{A - \tilde{K}_2 A^*} , \\ &\geq 2^{2/3} \frac{B(m)}{A(m)} = 2^{2/3} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})\Gamma(\frac{2}{3})} \cdot \frac{\Gamma(\frac{2}{3} - m)}{\Gamma(1 - m)} , \quad \frac{1}{3} < m \leq \frac{2}{3} . \end{aligned}$$

The last function increases steadily in $1/3 < m \leq 2/3$ and

$$(3.13) \quad S^{(m)}(0) \geq S^{(1/3)}(0) = 2^{2/3} \frac{\Gamma(\frac{1}{3})^2 \Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6}) \Gamma(\frac{2}{3}) \Gamma(\frac{5}{3})} = 2^{2/3}(1.19) > 2^{2/3}.$$

Clearly the inequality (3.13) invalidates (3.8) in case $\lambda = 0$ and so (3.8) can never hold, which completes the non-existence proof for $I = 1; J = 3$.

Lemma (10). *There are no concave solutions of the simplest type $I = 1; J = 4$.*

The first inequality of (3.1) may be written as

$$(3.14) \quad (1 + \mu)^{1/2} F(-\tfrac{1}{6}, \tfrac{7}{6}; \tfrac{5}{6} - m; t_1) \geq F(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{5}{6} - m; t_1),$$

$$t_1 = \frac{\alpha_1}{\alpha_1 + \beta_1} \leq \tfrac{1}{2},$$

from which it is easily seen that

$$(3.15) \quad (1 + \mu) \geq 1 + \frac{4t_1}{5 - 6m},$$

$$(3.16) \quad t_1 \leq \frac{5 - 6m}{4} \mu \leq \frac{3\mu}{4}; \quad m > \tfrac{1}{3}.$$

Again the first inequality of (3.2) may be written

$$(3.17) \quad (1 - \mu)^{1/2} < (1 - t_2)^{1/3} \frac{F(\tfrac{1}{6}, -m; \tfrac{5}{6} - m; t_2)}{F(-\tfrac{1}{6}, -m - \tfrac{1}{3}; \tfrac{5}{6} - m; t_2)},$$

$$t_2 = |\beta/\alpha|_2,$$

from which we find

$$(3.18) \quad (1 - \mu)^{3/2} \leq (1 - t_2) \left/ \left(1 + \frac{6m + 1}{5 - 6m} t_2 \right) \right.,$$

and again, using (3.18)

$$(3.19) \quad \left| \frac{1 - \mu}{\lambda_2} \right|^{3/2} = (1 - \mu)^{3/2} \frac{1 + t_2}{1 - t_2} \leq \frac{1 + t_2}{1 + \left(\frac{6m + 1}{5 - 6m} \right) t_2} \leq 1.$$

Combining the preceding inequalities with the shock polar relation we shall find a contradiction. From (2.45), taking account of the signs,

$$(3.20) \quad \left| \frac{1 - \mu}{\lambda_2} \right|^{3/2} = \left| \frac{1 + \mu}{\lambda_1} \right|^{3/2} + 3\mu.$$

According to (3.19) the left hand member cannot exceed unity. However, we can easily show that the right hand member exceeds unity unless $t_1 = \mu = 0$ which is trivial. It is sufficient to express $|\lambda_1|$ in terms of t_1 and to make use

of (3.16). We find

$$\begin{aligned}
 (3.21) \quad (1 + \mu)^{3/2}(1 - 2t_1) + 3\mu &\geq (1 + \mu)^{3/2}(1 - \tfrac{3}{2}\mu) + 3\mu \\
 &= (1 + \mu)^{3/2}(1 - \mu) + \mu\{3 - \tfrac{1}{2}(1 + \mu)^{3/2}\} \\
 &\geq (1 - \mu) + \mu(3 - (2)^{1/2}) = 1 + \mu(2 - (2)^{1/2}) \geq 1,
 \end{aligned}$$

which completes the proof.

Lemma (11). *There are no normal or concave transitions of the simplest type $I = 2$; $J = 3$.*

As in Lemma (9) the impossibility of normal transitions is an immediate deduction from the occurrence of loci $\phi = 0$, $\phi' = 0$ in $D_{(+)}$, or the complete quadrant $\alpha, \beta > 0$. For concavity the first inequality of (3.1) leads to (3.22), compare (3.14),

$$(3.22) \quad (1 + 1/\mu)^{1/2} \geq \frac{F_{(\psi)}(t_1)}{F_{(\phi)}(t_1)} \geq \frac{F_{(\psi)}(\frac{1}{2})}{F_{(\phi)}(\frac{1}{2})}.$$

The last item of (3.22) depends on the monotonicity of the hypergeometric series of (2.7) (2.9) and $t_1 \geq 1/2$.

Let us now assume the following representation of the regular solution in the vicinity of $\theta = 0$ for $s < 0$,

$$(3.23) \quad \psi = \bar{P}_1(m)z_1 + \bar{P}_2(m)z_2,$$

where the z_i are the functions of (2.23) (2.24). It is easily seen that for $\theta = 0$ we have

$$(3.24) \quad \frac{F_{(\psi)}(\frac{1}{2})}{F_{(\phi)}(\frac{1}{2})} = \frac{m\bar{P}_1(m)}{\bar{P}_2(m)} = \frac{\Gamma(1 - \frac{m}{2})\Gamma(\frac{1}{3} - \frac{m}{2})}{\Gamma(\frac{1}{2} - \frac{m}{2})\Gamma(\frac{5}{6} - \frac{m}{2})},$$

where the expression involving the Γ -functions may be determined from the condition of regularity of the solution near Γ_1 . Again, it can be shown that the least value of (3.24) is $\tan \pi/3$ attained as $m \rightarrow 1/3$ from which it follows at once that $\mu \leq 1/2$.

Similarly, the first inequality of (3.2) may be written

$$(3.25) \quad \left(\frac{\mu}{1 - \mu}\right)^{1/2} > \frac{\left(\frac{4}{3}\right)^{2/3} \frac{B(m)}{mA(m)} \frac{F_4}{F_1} \frac{r^{1/3}}{s^{1/2}} + \left(\frac{1}{6} + \frac{m}{2}\right) \frac{F_3}{F_1} \frac{s^{3/2}}{r}}{1 - \left(\frac{4}{3}\right)^{2/3} \frac{B(m)}{A(m)} \frac{F_2}{F_1} \frac{s}{r^{2/3}}}.$$

A contradiction arises if we can show that the right hand member exceeds unity for then (3.25) gives $\mu > 1/2$. Now we have $F_1 < 1$, $F_2 > 1$, $F_3 > 1$, $F_4 > F_4(1)$, according to the expressions in (2.14) $\dots$ (2.17). Writing $\xi = s^{1/2}/r^{1/3}$

it will be sufficient if we have the inequality

$$(3.26) \quad \left(\frac{4}{3}\right)^{2/3} \frac{B(m)}{A(m)} \left[\frac{F_4(1)}{m\xi} + \xi^2 \right] > 1.$$

The latter certainly holds if it is true when the left hand member, regarded as a function of ξ takes its minimum. This leads to the condition that

$$(3.27) \quad 2^{2/3} 3^{1/3} \frac{B(m)}{A(m)} \left[\frac{F_4(1)}{m} \right]^{2/3} \\ = \frac{9}{5} \left\{ \frac{m^{1/3} (1+m)^{2/3} (1-m)}{(\frac{2}{3}-m)^{1/3}} \right\} \left[\frac{\Gamma(1+\frac{1}{3})^{5/3} \Gamma(1+\frac{5}{6}) \Gamma(1+\frac{1}{2})^{2/3}}{\Gamma(1+\frac{1}{6}) \Gamma(1+\frac{2}{3}) \Gamma(\frac{3+m}{2})^{2/3} \Gamma(\frac{4}{3}+\frac{m}{2})^{2/3}} \right]$$

should exceed unity. Now the elementary factor in the $\{ \}$ takes its least value, for $1/3 < m \leq 2/3$,

$$(3.28) \quad \frac{4}{5} \cdot 6^{1/3} = 1.454,$$

as $m \rightarrow 1/3$. An easy computation of the slowly varying factors shows that the quantity in (3.27) does exceed unity and the Lemma is proved.

Lemma 12. *There are no concave solutions of the simplest type $I = 2$; $J = 4$.*

In this case since Λ_2 lies in the supersonic region the inequality (3.22) and what follows in the preceding proof before (3.25) must be replaced by

$$(3.29) \quad (1 + \mu)^{1/2} > \frac{F_{(\psi)}(\frac{1}{2})}{F_{(\phi)}(\frac{1}{2})} > \tan \frac{\pi}{3},$$

which is impossible for $0 < \mu \leq 1$.

Multiple hodograph coverings and double shock transitions. In the previous discussion of the convexity or concavity of the shock transitions we have assumed the simplest possible hodograph representations which were compatible with the chosen configuration of the Λ_i . It will now be proved that the above non-existence statements for concave transitions hold more generally. Since the case that Λ_2 is subsonic seems to be the physically interesting one we will consider only $J = 2$ or $J = 3$ and we note at the outset that of the four cases $I = 1, 2$; $J = 2, 3$ the solutions of type $I = 1$; $J = 2$ are non concave *a priori*.

Lemma 13. *The multiple hodograph coverings do not lead to normal or concave transitions of type $I = 1, 2$; $J = 2$, and for $m > 1/3$.*

In the case of normal solutions the two loci $\phi = 0$ coincide with the Λ_i and then the third locus $\phi = 0$ which must occur in any complete quadrant $\alpha, \beta > 0$ is inadmissible. The restriction $m > 1/3$ does not apply to this part of the proof.

For possible concave solutions we recall Lemma (6) which gives $m \leq 2/3$

and then Lemma (7) shows that for $1/3 < m \leq 2/3$ the quantities $\psi(\Gamma_i)$ taken on the same sheet $D_{(+)} : \theta^2 + (4/9)s^3 \geq 0$ must have the same sign. Since ψ does not vanish in $D_{(-)} : \theta^2 + (4/9)s^3 \leq 0$ the two loci $\psi = 0$ lie on the same sheet $D_{(+)}$ including possibly the portion bounded by Λ_2 and the adjacent Γ_1 . In the latter case Lemmas (3) (4) may be applied and the transition is convex. If the loci $\psi = 0$ occur on some other sheet $D_{(+)}$, then we must have $\psi > 0$ between Γ_1 and Λ_2 and in particular for $J = 2$ we need $\psi > 0$ at $\theta = 0, s > 0$. However, ψ is negative on this line in view of $P_1 < 0, P_1^* > 0$ according to the remarks following (2.41) above.

Although no such simple generalization has been found for cases with $m \leq 1/3$ it should be pointed out that these do not fall into the class of solutions considered by Germain who establishes $m > 1/2$. Also such a possible homogeneous solution in itself appears unsuitable in the applications to the case of a body in an infinite stream since the part of the flow which does not meet the shock wave would have the character of the flow in a throat. The author conjectures that the restriction $m > 1/3$ in Lemma (13) is unnecessary.

Lemma (14). *The multiple hodograph coverings do not lead to normal or concave transitions of the type $I = 1, 2; J = 3$.*

As in Lemma (13) the normal cases are ruled out on account of the appearance of a third locus $\phi = 0$. For possible concave transitions we have Lemma (8), equations (3.3) (3.4) showing that if $\psi(\Gamma_i) < 0$ on the sheet including Λ_2 then $m > 1/3$. Also, since the solution contains at least one complete $D_{(-)}$, we have $1/3 < m \leq 2/3$ and the conditions $\psi(\Gamma_i) > 0$ or $\psi(\Gamma_i) < 0$ taken with $\psi > 0$ or $\psi < 0$ in any $D_{(-)}$, ensure that $\psi < 0$ at all Γ_i . Again, in view of the consequent changes of the sign of ϕ between the Γ_i taken on either a $D_{(+)}$ or a $D_{(-)}$, we have at least four loci $\phi = 0$ which is not permissible. If $\psi(\Gamma_i) > 0$, it may be shown that the detailed arguments for Lemma (9) (11) go over unchanged provided we consider the solutions adjacent to the Λ_i .

The above conclusions are not modified if we admit folds in the solution at the characteristics Γ_i . For example, if the solution as we approach Λ_1 is folded back at Γ_i then clearly $I = 1$, and for concavity we need consider only the case $J = 3$. However, the arguments for Lemma (14) remain in this case. Again, if the flow line approaching Λ_1 pass through a Γ_2 we have $\phi(\Gamma_2) > 0$ and this is not allowed since $d\phi \geq 0$ necessarily along any level line while the values taken by ϕ at both Λ_i are negative. In the case of the sheet immediately connected with Λ_2 the same remark applies for the flow lines which intersect Λ_2 . For the rest the first fold changes the sign of the Jacobian and since these lines must eventually return to the subsonic region we need a second change at Γ_i . Clearly this introduces an extra locus $\phi = 0$ which is not allowed.

Non-existence of a double shock configuration. Since the homogeneous solutions yield only convex transitions no two such solutions, suitably modified at finite distances from the origin but retaining the same general configuration

of the Λ_i can be combined to give a shock reflection at a smooth profile and one in which the original and the reflected shock spring from distinct points on the sonic line. The question now arises as to the possibility that the original and the reflected shock spring from the same point on the sonic line. Indeed, some of the photographs of Holder (1959) suggest that this is the experimental situation.

Let us now consider such a double shock transition to be associated with the two pairs of loci (Λ_1, Λ_2) and (Λ_3, Λ_4) each passing through the origin $\theta = 0 = s$. The only case of interest appears to be that in which Λ_2, Λ_3 are both supersonic since only in this case would it be possible to modify the homogeneous solutions very slightly so that one of the cross characteristics becomes a level line $\psi = c = \text{const}$. This would ensure that the physical images of the points of intersection of $\psi = c$ with Λ_2, Λ_3 coincide, and of course the images of the two shocks would be close to each other over their whole lengths. Analytically this new problem seems very promising since we might expect to be able to combine two *convex* transitions with the extra sheet enclosed by Λ_2, Λ_3 and so achieve 'normal' or 'concave' behaviour of the whole process. Unfortunately, a simple modification of the previous results rules out this possibility, at least in the simplest case.

Assuming a single unfolded sheet as we approach Λ_1 the latter lies in $|\lambda| > 1$. If $g > 0$ on the second sheet then Λ_2, Λ_3 lie in $|\lambda| < 1$ and $|\lambda| > 1$ respectively, if $g < 0$ these positions must be interchanged. It is easily seen that Λ_2, Λ_3 cannot enclose the characteristic Γ_2 because whether $g > 0$ or $g < 0$ this portion of the hodograph representation is incorrectly orientated with respect to that part of the hodograph which is connected with the subsonic region. On the other hand, taking $\psi > 0$ at Λ_1 and Γ_1 on the second sheet we have $\phi(\Gamma_1) < 0$. Now it is easily seen that with our convention, $d\phi = |s|^{1/2} (\psi_\beta d\beta - \psi_\alpha d\alpha)$ is positive along any level line as we approach Λ_1 , and so, for mappable solutions ϕ must increase between Λ_2 and Λ_3 . As we have seen that $\phi < 0$ at Γ_1 it is clear that $\phi(\Lambda_2) < 0$ also and the individual transition (Λ_1, Λ_2) is necessarily *concave*. If $g > 0$ on the second sheet Lemmas (10) (12) apply unchanged, save for the triviality that we have to cancel a positive factor in equation (3.2). If, on the other hand $g < 0$ then Λ_2 lies in $|\lambda| > 1$. Lemma (10) is unchanged as far as the proof refers to conditions at Λ_1 . On the other hand the inequality (3.19) is now obvious *a priori*. Again, (3.20) holds if $\theta_2 > 0$ while the possibility $\theta_1 > 0, \theta_2 < 0$ does not arise in concave transitions. Lemma (12) follows from the relations at Λ_1 the condition that Λ_2 is supersonic, and if also $m > 1/3$ see (3.24) *et seq*. However, from the behavior at Λ_4 and Lemma (3) we know that $m > 1/2$ and the proof goes through.

Comparison with previous solutions of the homogeneous problem. Apart from the derivations of the equations for the homogeneous problem as in (2.1) (2.2) above, and the introduction of the homogeneous functions along the lines of (2.7) (2.10) and (2.14) (2.17) the preceding discussion is independent

of the work of Germain. For the latter the reader is referred to Germain (1956) giving a preliminary discussion and Germain (1958) for a brief explanation of his derivation of the inequality $7/5 < n < 5/3$, $1/2 < m < 5/6$ in the present notation. A complete account of Germain's work was presented only recently, see Küchemann and Sterne (1964) pp. 241–262, and came into the present writer's hands only as this manuscript was being completed.

Germain first derives a general method for the homogeneous solutions which is based on a consideration of the trajectories of a certain ordinary first order differential equation. This approach seems to be the only known one for the case of axially symmetric flows but to derive full information in the case of plane flows Germain has recourse to the hodograph method. It seems to the author that in the case of plane flow there are considerable advantages in a direct discussion by the hodograph method. This is notably so in the construction of the solutions the complete derivation involving only the elementary Lemmas (1) (2) and then equations (2.55) (2.68). The final result is a single analytic relation containing a two-parameter family of solutions. In addition we have at some stage to carry out the routine process of the analytic continuation of the homogeneous solutions in the hodograph plane, see (2.25) (2.26) above for a very concise statement of the result of this continuation. Germain's paper presents this continuation too, but somewhat lengthily.

As for the final results the following comparison may be useful. Germain's solutions of the first type, see his Figures (34) (35) for $m = 2/3$ and $4/5$ respectively, are evidently of the type here designated by $I = 1$; $J = 1, 2$ respectively. His conjectured solutions of the second type, see his Figure (37) appear to be of the class $I = 1, 2$; $J = 1$. The author has examined this conjecture from the point of view of the present analysis and finds no difficulties with the necessary conditions, in particular the inequalities (3.1) (3.2) which were effective in disproving the existence of concave transitions in many cases. A curious point is that for $J = 1$ the inequality $m > 1/2$ does not show up in the present method although this inequality is so easily derived for the cases $J = 2$, see (2.31) (2.32) above. The author is not able to see how Germain's solutions of the third type have been derived and observes that Figure (38) appears to conflict with his statement (6.7.2) to the effect that the sonic line and the shock wave form a continuation of each other and are tangent to each other. It might be added that for possible (convex) solutions of type $I = 1, 2$; $J = 3$ and supposing the hodograph to wind round the origin, the configuration of the characteristics would be similar to that in Figure (38) but then the flow on the left of the figure would be partly subsonic.

It seems that Germain's methods have the advantage of being applicable in the case of both plane and axially symmetric flows and permit a general proof of the inequality $m > 1/2$ without too much consideration of the detailed structure of the homogeneous solutions. However, when the latter is the point at issue, in particular the vital question of the existence or non-existence of normal or concave shock transitions, the present approach goes farther.

FIGURE 3. Hodograph Plane: The Direct Problem for a General Nonhomogeneous Weak Shock Solution

Moreover, as we shall see below, the hodograph method enables us to deal with more general solutions of the weak shock equations and to discuss the stability of such solutions in the experimentally observed situation.

4. The direct problem in the hodograph plane. It has been shown above that within a very general class of the homogeneous solutions, normal or concave transitions are excluded. Let us now consider the case of more general weak shocks satisfying the mechanical conditions. For the sake of convenience in working out the details of the proof we shall continue to use the simplified forms (2.1) (2.2) but not in such a manner as to restrict the discussion to the homogeneous solutions, which is in reality to consider the case of infinitesimal shocks. It seems likely that the properties of the homogeneous solutions are relevant only in an arbitrarily small neighborhood of the origin.

A uniqueness theorem for the direct problem: The case of the nearly normal shock. It is easily made plausible that, apart from the peculiarities associated with the homogeneous solutions we ought to be able to specify the loci Λ_i with considerable freedom. In support of this idea we shall present a uniqueness proof for the case of nearly normal shocks. The method of proof follows in general lines Morawetz (1956) and such a proof by the 'a-b-c' process strongly implies existence of the solution, see Morawetz (1958II) also Garabedian (1964).

Let us consider solutions of the hodograph equation for

$$(4.1) \quad \mathcal{L}(\psi) \equiv k(s)\psi_{\theta\theta} + \psi_{ss} = 0,$$

defined in a strip $0 < \theta < \pi/2$ for $s \geq 0$ together with the continuation into $s < 0$ for a certain region. Let the solution satisfy the following boundary conditions. (Fig. (3).) On the boundaries of the mixed region for subsonic speeds

$$\begin{aligned}
 (4.2) \quad & \theta = 0, \quad 0 < s < a : \psi = 0, \\
 & \theta = 0, \quad s > a : \phi = 0, \\
 & \theta = \frac{\pi}{2}, \quad s > 0 : \phi = 0 (\psi = 0).
 \end{aligned}$$

On the partial boundary of the mixed region for supersonic speeds

$$\begin{aligned}
 (4.3) \quad & \theta = \frac{2}{3} |s|^{3/2}, \quad 0 < |s| < a : \psi = 0, \\
 & \theta = \theta_1 = \frac{2}{3} |a|^{3/2}, \quad a < |s| < a' : \phi = 0. \\
 (4.4) \quad & |s| = a', \quad \theta_1 < a < \theta_{11}, \quad \theta_{11} + \frac{2}{3} (a')^{3/2} = \pi/2 : \psi = 0.
 \end{aligned}$$

No data is prescribed on the characteristic $\alpha = \pi/2$ which closes the supersonic region of the hodograph plane. We also require

$$(4.5) \quad \psi(s_1, \theta_1) = +\psi(s_2, 0), \quad s_1 = -|s_2|.$$

If we suppose a to be some fixed small quantity it is readily seen that conditions (4.5) with $\phi_1 = \phi_2 = 0$ according to (4.2) (4.3), approximate as closely as we please to the precise conditions for normal shocks. According to equations (1.9) (1.10) these conditions are

$$\begin{aligned}
 (4.6) \quad & \theta = 0: \quad \phi(0, s_1) = \phi(0, s_2) = 0, \\
 & \psi(0, s_1) = \psi(0, s_2),
 \end{aligned}$$

where the relation between s_1, s_2 is to be determined from

$$(4.7) \quad q_1 \rho(q_1) = q_2 \rho(q_2).$$

In the uniqueness argument below it is convenient to take the approximation $s_1 = -(s_2)$, which is of the same order as equations (2.1) (2.2) above. However, it will be clear by continuity that the proof applies to some suitable finite interval of s and for any of the usual pressure density laws. It should be added that the use of (2.1) (2.2) does not restrict us to the homogeneous case.

Let us consider the identity derived from equation (4.1)

$$\begin{aligned}
 (4.8) \quad & 2 \iint (b\psi_\theta + c\psi_s) \mathcal{L}(\psi) d\theta ds = \oint (L ds + M d\theta) \\
 & + \iint [\psi_\theta^2 \{(kc)_s - kb_\theta\} - 2\psi_\theta \psi_s (b_s + kc_\theta) + \psi_s^2 (b_\theta - c_s)] d\theta ds,
 \end{aligned}$$

with

$$(4.9) \quad L = b(k\psi_\theta^2 - \psi_s^2) + 2kc\psi_\theta \psi_s,$$

and

$$(4.10) \quad M = c(k\psi_\theta^2 - \psi_s^2) - 2b\psi_\theta \psi_s.$$

The general line of the proof is to choose b, c so that the area integrands on the right of (4.8) are positive definite, and so that the line integrals, after application of the boundary conditions, become non-negative. The following choice of b, c is suggested by Morawetz (1956), compare her equations (48) ... (56)

$$(4.11) \quad k^{1/2}c - ib = \frac{i \cos \xi}{\sin^2 \xi + \sinh^2 a}, \quad s \geq 0,$$

where, say,

$$(4.12) \quad \xi = \theta + i \int_0^s |k|^{1/2} ds, = \theta + i\xi.$$

It is easily shown that in view of the Cauchy-Riemann relations the area integrand reduces to

$$(4.13) \quad c \frac{k'(s)}{k(s)} [\psi_s^2 + k(s)\psi_\theta^2].$$

Now the values taken by b, c along the boundaries of the subsonic region are as follows (4.14) (4.15) (4.16).

$$(4.14) \quad \theta = 0, \quad s > a: \quad c = 0, \quad b = \frac{\cosh \xi}{\sinh^2 \xi - \sinh^2 a} > 0,$$

$$\theta = 0, \quad 0 \leq s < a: \quad c = 0, \quad b < 0.$$

$$(4.15) \quad s = 0, \quad 0 \leq \theta < \frac{\pi}{2}: \quad c = 0, \quad b = -\frac{\cos \theta}{\sin^2 \theta + \sinh^2 a} < 0, \quad b'(\theta) \geq 0.$$

$$(4.16) \quad \theta = \frac{\pi}{2}, \quad s \geq 0: \quad b = 0, \quad k^{1/2}c = \frac{\sinh \xi}{\cosh^2 \xi + \sinh^2 a} > 0.$$

The boundary conditions on c ensure that it is positive at all interior points and so (4.13) is positive definite, provided $k'(s) > 0$. This last follows immediately from the fact that $\rho, c(\rho)$ are increasing functions of q^2 , see equations (1.6) (1.7). It should also be noted that the quantity in (4.13) is bounded as $s \rightarrow 0$. Applying the boundary conditions (4.2) and (4.4) (4.15) (4.16) we see that the line integrands for the boundaries $\theta = 0, \pi/2; s > 0$ are non-negative. The integrand for the arc $\theta = 0, a < s < b$ is

$$(4.17) \quad \int_a^b b_2(\psi_s^2)_2 ds_2 \geq 0.$$

In the supersonic region we follow Morawetz in choosing $c \equiv 0$, but it is necessary to replace her $b = b(\theta)$ by

$$(4.18) \quad b(\alpha) = b(\alpha, 0)$$

with

$$\alpha = \theta + \xi(s), \quad s \leq 0.$$

With this choice b, c are continuous in the mixed region. The area integrand for the supersonic region now becomes

$$(4.19) \quad b'(\theta + \xi)[|k|^{1/2} \psi_\theta - \psi_s]^2 = \frac{|k|}{4} b'(\alpha) \psi_\alpha^2,$$

where we observe that $b'(\alpha) \geq 0$ according to (4.15).

The line integral for $\Delta_1 : a < |s| < a'$ will be written as

$$(4.20) \quad - \int_a^{a'} b_1(\psi_s^2)_1 ds_1 \leq 0.$$

If we assume $|s_1| = s_2$ condition (4.5) reduces to $(\psi_{|s_1|})_1 = (\psi_s)_2$ and the sum of (4.17) and (4.20) is non-negative if $b_2 \geq b_1$ where we have explicitly

$$(4.21) \quad b_2 = \frac{\cosh \xi_2}{\sinh^2 \xi_2 - \sinh^2 a}, \quad b_1 = \frac{\cos \alpha_1}{\sin^2 \alpha_1 + \sinh^2 a},$$

with

$$\xi_2 = \xi(s_2), \quad \alpha_1 = \theta_1 + \xi(|s_1|),$$

see (4.12) (4.18). It is clear that if we suppose $|s_1|, s_2$ to be sufficiently small then these will be approximately equal and so the proof applies for certain small finite values of s_1, s_2 . Now $\alpha_1 > \xi_1 = \xi_2 = \xi$ (say) and so the required inequality follows from

$$(4.22) \quad b_1(\alpha_1) \leq b_1(\xi) \leq \frac{\cos \xi}{\sin^2 \xi} \leq \frac{\cosh \xi}{\sinh^2 \xi} \leq b_2(\xi), \quad 0 \leq \xi \leq \frac{\pi}{2},$$

in which the end results are trivial and the third inequality is an elementary property.

It can easily be shown that we have

$$(4.23) \quad \begin{aligned} L ds + M d\theta &= -b ds \left[\left(\frac{d\psi}{ds} \right)^2 + \left\{ |k| - \left(\frac{d\theta}{ds} \right)^2 \right\} \psi_\theta^2 \right] \\ &\quad - c d\theta \left[|k| \left(\frac{d\psi}{d\theta} \right)^2 + \left\{ 1 - |k| \left(\frac{ds}{d\theta} \right)^2 \right\} \psi_s^2 \right], \\ &= -(b ds + c d\theta) \left(\frac{d\psi}{ds} \right)^2, \quad k + \left(\frac{d\theta}{ds} \right)^2 = 0, \end{aligned}$$

the latter for characteristic arcs. Hence the integrand vanishes on the small characteristic arc through the origin and is non-negative on the free characteristic $\alpha = \pi/2$. To complete the discussion we observe that the line integrand vanishes on $|s| = a'$ according to (4.4).

The uniqueness proof for sufficiently smooth solutions now follows from $\psi_\theta = \psi_s \equiv 0 : s > 0$ and $\psi_\alpha \equiv 0 : s < 0$. The solution is necessarily constant in $s > 0$ and of the form $f(\beta)$ in $s < 0$. Letting $|s| \rightarrow 0$ we see that the solution vanishes in the whole mixed region.

There is no difficulty in regard to the uniqueness proof if we consider boundary conditions corresponding to the exact case of normal shocks. In this connection we recall that Frankl exhibited the following homogeneous solution the index being $m = 8/3$,

$$(4.24) \quad \psi = s^4 - 6s\theta^2,$$

$$(4.25) \quad \phi = 4s^3\theta - 2\theta^2,$$

which satisfies the direct problem for normal shocks and (2.1) (2.2). However, we find

$$(4.26) \quad \mathcal{J} = 16s^6 + 96s^3\theta^2 + 36\theta^4 = 81 \left[\lambda^3 + \frac{4}{3} - \frac{2}{(3)^{1/2}} \right] \left[\lambda^3 + \frac{4}{3} + \frac{2}{(3)^{1/2}} \right] \theta^4,$$

showing that $\mathcal{J}$ is negative in a certain neighborhood of $\Gamma_1 : \lambda = -1$, and this hodograph solution is not mappable into a physical solution of the weak shock equations. It is convenient to discuss this matter in detail and to explain why in many solutions found from the general boundary values problem as stated in equations (4.1) $\cdots$ (4.5), limit lines are unlikely to arise.

We first note that conditions (4.3) (4.4) ensure $\mathcal{J} \geq 0$ on the boundary of the supersonic region, excluding $\alpha = \pi/2$. If we knew $\mathcal{J} > 0$ it would follow from Manwell (1952) that in a finite neighborhood of the origin the solution would be mappable. The point at issue is simply the vanishing or non-vanishing of ψ_* . Put more picturesquely, we need to avoid the branching of level lines at supersonic points of the hodograph boundary. All that we are able to achieve in setting up the boundary value problem is to arrange that the boundary $\psi = 0$ (or $\phi = 0$) is correctly set with respect to the characteristics so that if there is no branching of the level lines then the solution is mappable. By analogy with the solutions of type $I = 1; J = 1, 2$, which have been constructed in the hodograph plane and shown to be mappable, it is to be anticipated that mappability depends essentially on choosing suitable *data* in the corresponding existence theorem. In regard to Frankl's solution two points should be noted, the hodograph does exhibit the desired simpler structure and avoids the branching of level lines but there is no locus corresponding to that in conditions (4.3) (4.4) along which we have $\mathcal{J} \geq 0$. On the contrary, we see from equation (4.26) that in the case of Frankl's solution we have $\mathcal{J} < 0$ on segments of all the level lines showing that this solution is not mappable in any complete neighborhood of the origin.

Extension of the uniqueness theorem. The previous case of the uniqueness theorem is of value in support of the contention that the admissible Λ_i can be chosen very freely. On the other hand, for a rigorous proof of the existence of the *physical* flows, based on this boundary value problem, it seems best to start from one example which is known to be mappable and then to change the data slightly. Clearly, since $\mathcal{J} > 0$ implies the same for adjacent solutions,

all the solutions of the new boundary value problem will be mappable. In this connection let us consider the extension of the uniqueness theorem in which the Λ_i correspond to a possible homogeneous solution. (Germain's example, his figure (35), has the same general character: the values of the λ_i are not specified.)

Let us choose the right hand member of (4.11) so that c vanishes on a suitable Λ_2 , and its smooth extension running into some curve $\theta = \text{const.}$ also $c = 0$ on a segment of the axis and $b = 0$ on $\theta = \pi/2$. We can construct a similar function having the behavior near the origin

$$(4.27) \quad k^{1/2}c - ib = i\zeta^{-2/p},$$

where

$$(4.28) \quad p = 1 + \frac{2}{\pi} \arctan (\lambda_2^{-3/2}).$$

It is easily seen that (4.13) remains positive and as in Morawetz (1956) $b'(\theta, 0) > 0$ follows from $c_s(\theta, 0) > 0$. Then the whole of the previous argument with verbal changes applies except for the line integrals over the loci Λ_i (in particular compare equation 41 of Morawetz (1956) for the locus $\varphi = 0$ in $\theta < 0$, $s > 0$). If we suppose these to be small, that is to say if we now take b as sufficiently small in comparison with $\pi/2$, we may use the values given by (4.27). Along Λ_2 we have

$$(4.29) \quad c = 0, \quad b = b_2 = (\theta_2^2 + \frac{4}{9}s_2^3)^{-1/p},$$

and at $s = 0$, $0 < \theta < \theta_1$ we find

$$(4.30) \quad c = 0, \quad b = -\theta^{-2/p},$$

giving

$$(4.31) \quad b_1 = \alpha_1^{-2/p} = [\theta_1 + \frac{2}{3} |s_1|^{3/2}]^{-2/p}.$$

Now using (1.12) (1.16) we find for the homogeneous case

$$(4.32) \quad |s_1|^{1/2} \psi_\theta \left[\left(\frac{\mu}{1+\mu} \right)^{1/2} \frac{1}{|\lambda_1|^{3/2}} + 1 \right] = \psi_s \left[\frac{1}{|\lambda_1|^{3/2}} + \left(\frac{\mu}{1+\mu} \right)^{1/2} \right],$$

and

$$(4.33) \quad s_2^{1/2} \psi_\theta \left[\left(\frac{\mu}{1-\mu} \right)^{1/2} \frac{1}{|\lambda_2|^{1/2}} - 1 \right] = \psi_s \left[\frac{1}{|\lambda_2|^{3/2}} + \left(\frac{\mu}{1-\mu} \right)^{1/2} \right],$$

for Λ_1 and Λ_2 respectively. We have also the shock polar relation

$$(4.34) \quad \left| \frac{1+\mu}{\lambda_1} \right|^{3/2} + \left| \frac{1-\mu}{\lambda_2} \right|^{3/2} = 3\mu^{1/2}.$$

In the present context the direct problem for normal shocks arises by allowing $|\lambda_1|^{-3/2}$ and $|\lambda_2|^{-3/2}$ to tend to zero $O(\mu^{1/2})$.

The condition that the line integrand taken for pairs of corresponding points

on Λ_1 , Λ_2 shall be positive may now be written as

$$(4.35) \quad \frac{b_2}{b_1} \frac{ds_1}{ds_2} \left(\frac{1 + \rho_1}{1 + \rho_2} \right)^2 \left[\frac{1 + 2\rho_2 - k_2 \left(\frac{ds_2}{d\theta} \right)^2 \rho_2^2}{1 + 2\rho_1 - k_1 \left(\frac{ds_1}{d\theta} \right)^2 \rho_1^2} \right] > 1;$$

where the quantities

$$(4.36) \quad \rho_i = \left(\frac{\psi_\theta}{\psi_s} \frac{ds}{d\theta} \right)_i,$$

are given by (4.32) (4.33). The inequality (4.35) is certainly satisfied for Λ_i sufficiently near the normal case. We have

$$(4.37) \quad \rho_i = O(\mu), \quad k \left(\frac{ds}{d\theta} \right)^2 = O(\mu^{-1}),$$

and then

$$(4.38) \quad \frac{b_2}{b_1} = \left(\frac{1 + \mu}{1 - \mu} \right)^{3/2} \frac{[1 + \lambda_1^{-3/2}]^{2/p}}{[1 + \lambda_2^{-3}]^{1/p}} = \frac{1 + 2}{p |\lambda_1|^{3/2}} + O(\mu),$$

showing that (4.35) is satisfied since the left hand member ultimately exceeds the right by a quantity $O(\mu^{1/2})$.

The uniqueness proof is now complete provided we impose conditions on the trial functions near the origin and the point at infinity which ensure convergence of the line integrals. In particular it seems that in general we must specify the leading term in the expansion of the solution in homogeneous functions near the origin. This need not be so for cases far from the normal but for these, in particular the solutions with small λ_2 , (cf. many examples given by theorem (I), the inequality (4.35) fails and the uniqueness theorem itself is probably untrue).

5. The perturbation problem for plane transonic flow with a weak shock. The boundary value problems of Section (4) were introduced in order to generalize the homogeneous solutions which failed to provide normal or concave transitions. In the case of flow past a given smooth convex contour the transitions must be respectively normal or concave according as the flow either follows the contour through the shock or breaks away from it with the formation of a free streamline. However the work in Section (4) serves only to provide *local* shock solutions; it will be observed that the hodograph boundary exhibits a 'gap' along which no data may be prescribed. We shall now consider a physically meaningful perturbation theory for an assumed solution which includes a weak shock.

In the perturbation theory we may consider either a family of smooth profiles for a given stream or a single smooth profile as the stream Mach number is continuously varied. For continuous shock free flow the problem was studied by Morawetz (1956) (1957) (1958I) and (1964). There are two possibilities:

- (a) that, subject only to smoothness conditions on the profile, continuous families of solutions can be constructed.
- (b) that this construction is in general impossible. The work of Morawetz, see Morawetz (1956) Appendix 3, which includes the case of certain weak discontinuities, shows fairly conclusively that (b) must hold. We then enquire as to:
- (c) which is the exceptional case, existence or non-existence?

For a class of contours which are smooth save for two isolated 'points of expansion' on the upper and lower surfaces respectively the discussion of Manwell (1964) answers (c) as follows: *the cases in which the perturbation problem for a complete closed profile is well set are at best, highly exceptional*. As the proof appeals to existence theorems for the sub-region of Morawetz's uniqueness proof, in particular to the occurrence of Tricomi singularities on the sonic line, the same result may be anticipated for general smooth profiles.

The main result of Section (5) is the discovery that for a class of contours of the same character as those discussed in Manwell (1964) the introduction of weak discontinuities as defined in Section (1) above leads to solutions in which the *perturbation problem appears to be correctly set*. The precise result is a *uniqueness* theorem for a general set of hodograph solutions with boundary conditions appropriate to a complete contour under perturbation of the stream conditions. Analytically the theorem proved is simply an extension of the preceding treatment of Morawetz's problem in which the free characteristics Γ_i completing the boundary of the mixed region are replaced by the arcs Λ_i . Corresponding to the non-negative character of the individual line integrals over the Γ_i , it will appear that the sum of these line integrals over the Λ_i has the non-negative property. Compare (4.27) (4.20) above. There is of course an important distinction between the Morawetz proofs where the Γ_i bound a supersonic sector removed arbitrarily from the whole flow field and the extension to the case of flow with weak shocks where the Λ_i bound a sector of the hodograph which plays no part in the physical flow solution. This distinction is an essential one showing why in the case of weak shocks, but not in the corresponding problem for continuous plane transonic flow, we are able to set data on the hodograph image of the complete flow boundary. This is the heart of the matter as to why continuous flows solutions are mathematically ill defined while those with weak shock discontinuities may be well set. A strict proof requires some existence theorems but by analogy with Morawetz (1958II) and Manwell (1964) these are strongly indicated by the uniqueness proof (*cf.* Garabedian (1964), Ch. 12 §1, 2).

From the standpoint of the theory of equations of mixed type and with the Russian terminology, Section (4) above extends the treatment of 'Frankl's problem', compare Bitsadze (1964) p. 135 *et seq* and in particular p. 142. Our main generalization is the relaxation of the requirement that the shock should be strictly normal.

In the same context of equations of mixed type the theory of the perturbation of plane transonic flow as developed below suggests yet another boundary value problem, one in which we have to consider values of the unknowns at all points of the mixed (hodograph) boundary. Along the arcs $\mathcal{C}_0 + \mathcal{C}_1 + \gamma'$ we set data explicitly while along the complementary arcs Λ_1, Λ_2 we require a linear relation between the unknowns. In contrast to Tricomi's and Frankl's problems there is no characteristic arc on which nothing is stated regarding the unknowns.

Formulation of the perturbation problem. We shall avoid the 'modified' hodograph plane of Manwell (1954) Morawetz (1956) (which involves a linearisation of the *equation* that can prove very troublesome, cf. Morawetz (1957)), and consider instead exact solutions of the hodograph equations with first order approximation to the boundary conditions. The formal simplicity of the modified hodograph plane can be retained by making use of the Legendre transforms Φ, Ψ where we have

$$(5.1) \quad \Phi(u, v) = \phi - x\phi_x - y\phi_y = \phi(x, y) - ux - vy,$$

$$(5.2) \quad \Psi(\rho u, \rho v) = \psi - x\psi_x - y\psi_y = \psi - \rho vx + \rho uy.$$

Let us now consider the correspondence between perturbations of $\phi(x, y)$, $\psi(x, y)$ and $\Phi(u, v)$, $\Psi(\rho u, \rho v)$ respectively. Here u, v , are the ordinary hodograph quantities and the image of a given point x, y in the hodograph plane is a point u, v which varies between two solutions with different boundary conditions. To study this correspondence in the case of small changes, let us first recall the well known identities

$$(5.3) \quad \Phi_u = -x = \Psi_{\rho v}, \quad \Phi_v = -y = -\Psi_{\rho u}.$$

Let $\delta\phi(x, y)$ denote any perturbation of the solution $\phi(x, y)$ of (1.3) and suppose $\delta\Phi(u, v)$ to be the corresponding change in the hodograph solution. In practice, of course, it is the solution in hodograph form, $\delta\Phi(u, v)$, say, which we can determine as a solution of the linear equations (5.3) (5.4). These latter are more conveniently written as

$$(5.4) \quad \rho q \Phi_a = \Psi_\theta, \quad q \Psi_a = -(\rho q)_a \Phi_\theta,$$

or again

$$(5.5) \quad \Psi_\theta = -\Phi_\sigma, \quad \Psi_\sigma = K(\sigma) \Phi_\theta,$$

with

$$(5.6) \quad \rho q d\sigma = -dq, \quad K(\sigma) = \rho(\rho q)_a.$$

Then the perturbation function $\delta\phi(x, y)$ is found by means of the identity (5.1) giving, on the assumption of a sufficiently smooth $\delta\Phi$

$$(5.7) \quad \delta\phi(x, y) = x \delta u + y \delta v + \delta\Phi(u, v) + \frac{\partial\Phi}{\partial u} \delta u + \frac{\partial\Phi}{\partial v} \delta v + O(\delta u^2, \delta u \delta v, \delta v^2).$$

However, the terms linear in δu , δv cancel by (5.3) and so

$$(5.8) \quad \delta\phi(x, y) = \delta\Phi(u, v) + O(\delta^2),$$

with an obvious meaning for the remainder. Equation (5.8) states that the perturbation of the potential function at a *given point* x, y may be found by calculating the Legendre transforms, Equation (5.1), of the ordinary hodograph potential function $\Phi(\theta, q)$ at the image θ, q of the point x, y as determined by the original hodograph mapping. A precisely similar result can be established for the stream function. If we use (5.2) and take account of (5.3) we find

$$(5.9) \quad \delta\psi(x, y) = \delta\Psi(\rho u, \rho v) + O(\delta^2).$$

In particular the preceding arguments show that the condition $\delta\psi(x, y)$ vanishes on a fixed profile $\mathcal{C}_\star$ say, is equivalent, to the first order in the small velocity changes, to the condition that

$$(5.10) \quad \delta\Psi(\rho u, \rho v) = 0, \quad u, v \in \mathcal{C},$$

where $\mathcal{C}$ is the hodograph image of $\mathcal{C}_\star$ in the *unperturbed* flow. Now both Ψ and $\Psi + \delta\Psi$ are solutions of the equations (5.4) (5.5) and these yield in the usual way exact solutions for the hodograph equations (1.5) above. We may now apply (5.8) (5.9) to the case of a perturbed weak shock front. The condition that $\phi(x, y)$, $\psi(x, y)$ are continuous at the shock front in both the given and the perturbed solution yields

$$(5.11) \quad [\delta\phi(x, y)]_1^2 + \left[\frac{\partial\phi}{\partial n}(x, y) \right]_1^2 \delta n = O(\delta^2),$$

$$(5.12) \quad [\delta\psi(x, y)]_1^2 + \left[\frac{\partial\psi}{\partial n}(x, y) \right]_1^2 \delta n = O(\delta^2),$$

where δn is measured normal to the original shock front. Eliminating δn we have, with $[]$ denoting $[]_1^2$,

$$(5.13) \quad \left[\frac{\partial\psi}{\partial n} \right] [\delta\phi] - \left[\frac{\partial\phi}{\partial n} \right] [\delta\psi] = 0$$

to the first order in δn , these quantities being calculated at the original shock front. At this front however, we have the conditions

$$(5.14) \quad [\rho u] \equiv [m] = 0 = [v],$$

where we take Ox along the normal, Oy along the tangent to the shock. Then according to (5.13) (5.14)

$$(5.15) \quad \left[\frac{\partial\psi}{\partial n} \right] / \left[\frac{\partial\phi}{\partial n} \right] = \frac{[\rho v]}{[u]} = \frac{v[\rho]}{m \left[\frac{1}{\rho} \right]} = \rho_1 \rho_2 \left(\frac{d\phi}{d\psi} \right)_\Lambda.$$

Writing r for the last expression calculated along the shock wave and recalling the relations (5.8) (5.9) we find the *perturbation condition for a shock wave* as

$$(5.16) \quad [\delta\Psi(\rho u, \rho v)] - r[\delta\Phi(u, v)] = 0.$$

The quantities in (5.16) are to be calculated at corresponding points θ_i , q_i of the hodograph images Λ_i as determined for the original solution according to the hodograph method.

Finally, since in view of (5.10) (5.16) it is clear that the perturbation theory should be set in terms of unknowns $\Phi(\theta, \sigma)$, $\Psi(\theta, \sigma)$ of equation (5.5), it is necessary to express the condition along a *free streamline* in terms of these functions. Let the flow be perturbed so that the streamline arc $\sigma = \sigma_1$ of $\mathcal{C}$ is changed into $\mathcal{C}' : \sigma = \sigma_1 + \delta\sigma_1$ and the condition $\psi = 0$ at $\mathcal{C}'$ may be expressed as

$$(5.17) \quad \left[\frac{\partial\psi}{\partial s}(\theta, s) \right]_{s=s_1} \delta s_1 + \delta\psi(\theta, s_1) = 0.$$

We must therefore express $\delta\psi(\theta, s)$ in terms of the corresponding Legendre transforms. Now for general solutions it follows from (5.1) (5.2) (5.3) that

$$(5.18) \quad \psi(\rho u, \rho v) = \Psi(\theta, \sigma) + \rho\Phi_\theta(\theta, \sigma).$$

The condition for the perturbation of a free streamline into an adjacent one may now be set as

$$(5.19) \quad \delta\Psi(\theta, \sigma_1) + \rho(\sigma_1) \frac{\partial}{\partial\theta} \{ \delta\Phi(\theta, \sigma_1) \} = - \frac{\partial\psi}{\partial s}(\theta, s_1) \delta s_1.$$

Here $\sigma_1 = \sigma(q_1)$, $s_1 = s(q_1)$ and δs_1 are constants and the right hand member is known in terms of the original solution, and to the extent of an arbitrary factor δs_1 .

An elementary proof of the Morawetz theorem for a suitably restricted boundary. If we agree to certain simplifying assumptions on the boundary $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2$ of a transonic hodograph solution the Morawetz perturbation problem, Morawetz (1956) can be treated by a simple modification of the discussion of Frankl's problem, Morawetz (1954). To achieve this it is necessary to impose a limitation on the slopes of $\mathcal{C}_i$ in the supersonic region, also to make $\mathcal{C}$ coincide with certain characteristic arcs γ'_i near the ends of the local supersonic region. It is not the purpose of this paper to consider the Morawetz problem itself. However, in view of the recent publication by this writer, of a greatly shortened, but by no means elementary proof of her (1956) theorem, see Morawetz (1964), it seems worth pointing out that the method to be given below appears capable of some improvement. In particular, it seems likely that the characteristic arcs γ'_i might be indefinitely reduced by a suitable modification of the multipliers $b(\theta, \sigma)$, $c(\theta, \sigma)$. Also, the introduction of some small characteristic arcs in the boundary is necessary when discussing the existence theorems by classical methods, cf. Manwell (1964). Again, the present

proof deals at once with the difficulty arising from the subsonic branch point of Morawetz (1956) and (1964), *cf.* Morawetz (1957).

Let us first introduce the function

$$(5.20) \quad W(\theta, \sigma) = \int_{\theta_0, \sigma_0}^{\theta, \sigma} \Phi \, d\theta - \Psi \, d\sigma,$$

where Φ, Ψ conveniently replace $\delta\Phi, \delta\Psi$ the perturbation functions to be found as solutions of (5.5) with conditions (5.10) (5.16) or (5.19). Then $W(\theta, \sigma)$ is a point function and we have

$$(5.21) \quad W_\theta = \Phi, \quad W_\sigma = -\Psi,$$

also

$$(5.22) \quad \mathcal{L}(W) \equiv KW_{\theta\theta} + W_{\sigma\sigma} = 0.$$

As in (4.8) (4.9) (4.10) above, let us consider the elementary identity

$$(5.23) \quad \begin{aligned} 2 \iint (bW_\theta + cW_\sigma) \mathcal{L}(W) \, d\theta \, d\sigma \\ = \iint [W_\theta^2 \{-Kb_\theta + (Kc)_\sigma\} \\ - 2W_\theta W_\sigma (b_\sigma + Kc_\theta) + W_\sigma^2 (b_\theta - c_\sigma)] \, d\theta \, d\sigma \\ + \oint L \, d\sigma + M \, d\theta, \end{aligned}$$

where

$$(5.24) \quad L = b(KW_\theta^2 - W_\sigma^2) + 2cKW_\theta W_\sigma,$$

$$(5.25) \quad M = c(KW_\theta^2 - W_\sigma^2) - 2bW_\theta W_\sigma.$$

We have to choose $b(\theta, \sigma), c(\theta, \sigma)$ as continuous, piecewise continuously differentiable functions making the area integral positive definite. The simplest choice seems to be

$$(5.26) \quad b = \theta, \quad c = c_i \sigma; \quad i = 0, 1, 2,$$

where the c_i are suitable positive constants and the suffixes refer to $\sigma > 0$; $\sigma < 0, \theta > 0$; $\sigma < 0, \theta < 0$ respectively. If $c_0 = 1$ the area integrands now reduce to

$$(5.27) \quad K'(\sigma)\sigma W_\theta^2, \quad \sigma \geq 0,$$

$$(5.28) \quad \{|K| (1 - c_i) - c_i |\sigma| K'(\sigma)\} W_\theta^2 + (1 - c_i) W_\sigma^2, \quad \sigma \leq 0.$$

For example, if $K(\sigma) = \sigma$ we may take

$$(5.29) \quad c_0 = 1; \quad c_i \leq \frac{1}{2}, \quad i = 1, 2.$$

Morawetz (1954) takes $c_0 = 1; c_i = 0, i = 1, 2$ with $W \equiv \psi(\theta, s)$, and arrives at a proof of uniqueness of Frankl's problem.

It is almost obvious that if the line integrals are non-negative then the solution of (5.21) reduces to zero. For example, if $c_i = 1/2$ then we get $\Phi \equiv 0$ in $\sigma \geq 0$ and $\Psi \equiv 0$ in $\sigma \leq 0$ from which, if Φ_σ is continuous the first of equations (5.5) determines zero Cauchy data for $\Phi(\theta, \sigma)$ at $\sigma = 0$. Hence Φ and so Ψ vanishes identically in the mixed domain. Now the line integral vanishes on the image of the physical axis of symmetry of the flow and converges near the subsonic branch point. It is non-negative on the arcs of $\mathbb{C}$ having $\Psi = 0$ provided

$$(5.30) \quad K(b \, d\sigma + c \, d\theta) \geq 0.$$

The condition (5.30) imposes little limitation on the open arcs in $\sigma > 0$ but in $\sigma < 0$ it is fairly restrictive. For example with the choice σ_i as in (5.29) *et seq* we should require

$$(5.31) \quad 2\theta\sigma \, d\sigma + \sigma^2 \, d\theta = d(\sigma^2\theta) \geq 0.$$

In particular, (5.31) can never hold as we approach the ends of the local supersonic region. Instead, however, we may complete the boundary by characteristic segments on which we take

$$(5.32) \quad \gamma'_i : \Psi = 0, \quad \Phi = a_i,$$

the first implying the second. We can always arrange that one of these constants is zero, say $a_1 = 0$. Then the general value of Φ on γ'_2 is a_2 times a known function, and so in the existence theory this part of the data appears as a known function. Therefore, there is no real lack of generality if we take the $a_i = 0$ making the corresponding line integrals vanish.

To deal with the characteristics through the origin we write

$$(5.33) \quad L \, d\sigma + M \, d\theta = b \, d\sigma \left[-\left(\frac{dW}{d\sigma}\right)^2 + W_o^2 \left\{ K + \left(\frac{d\theta}{d\sigma}\right)^2 \right\} \right] \\ + c \, d\theta \left[K \left(\frac{dW}{d\theta}\right)^2 - W_o^2 \left\{ 1 + K \left(\frac{d\sigma}{d\theta}\right)^2 \right\} \right].$$

Since $(d\theta/d\sigma)^2 = |K|$ on either characteristic the line integrand will be non-negative provided

$$(5.34) \quad b \, d\sigma + c \, d\theta \leq 0,$$

and clearly both terms are negative.

This completes the uniqueness proof for the Morawetz perturbation problem in certain cases. We have taken the point of intersection of the Γ_i at $\theta = 0$ where this corresponds to the axis of symmetry but by imitating the methods of Morawetz (1956) we could readily construct functions $b(\theta, \sigma)$, $c(\theta, \sigma)$ for the more general case. Alternatively, we may limit the discussion to the physically similar case of the flow round a corner, in which case the boundary $\theta = 0$, $\sigma > 0$ does not arise.

It is to be noted that the present argument while removing the difficulty

encountered by Morawetz (1956) in regard to the subsonic branch point introduces a similar one in connection with the characteristics γ'_i , however small these may be. However, the merit of the present approach is that at the expense of some restriction of the boundaries we are able to discuss not only continuous motion, but flow with weak shock discontinuities.

Some further points arise in regard to the existence theory. The simplification of the conditions in (5.32) to $a_i = 0$ is entirely harmless in this connection. On the other hand, closely connected with the choice of the a_i is the question of the Tricomi singularities at the intersection of the γ'_i with the sonic line. For physically meaningful solutions in the perturbation theory these singularities must vanish. In the closely analogous theorems for the Morawetz problem, see Manwell (1965), it was found that in order to satisfy this condition it is necessary to suppose one or more *free streamlines* to be associated with a given profile. In the case of shock discontinuities of the type $I = 1$, $J = 2$ there appears to be only one such Tricomi singularity and a single free streamline arc on upper and lower surfaces is to be anticipated.

In order to generalize the Morawetz perturbation problem to the case that the boundary arcs $\mathcal{C}_i$ are in part free streamlines let us first take (5.19) with zero data giving

$$(5.35) \quad \Psi(\theta, \sigma_1) + \rho(\sigma_1) \frac{d}{d\theta} \{\Phi(\theta, \sigma_1)\} = 0.$$

Applying (5.23) with $d\sigma = 0$ and (5.25) we find

$$(5.36) \quad \int M d\theta = - \int c \{ |K| \Phi^2 + \Psi^2 \} d\theta \\ - 2 \int \rho(\sigma_1) b(\theta) \Phi(\theta, \sigma_1) \frac{d\Phi}{d\theta} d\theta.$$

If $c \leq 0$ the first member is non-negative and the second after an integration by parts becomes

$$(5.37) \quad -\rho(\sigma_1) b(\theta) \Phi(\theta, \sigma_1)^2 + \rho(\sigma_1) \int b'(\theta) \Phi^2(\theta, \sigma_1) d\theta.$$

The latter is non-negative for any arc $\sigma = \sigma_1$, which terminates at the γ'_i since there we took $\Phi = 0$ and $b(\theta)$, $b'(\theta) > 0$ for $\theta > 0$ and a similar argument applies in $\theta < 0$.

The desired relaxation of the boundary values in the case of the free streamline is on account of the arbitrary *factor* applied to the known data on the right hand member of (5.19); this introduces one free constant into the perturbation problem.

Uniqueness of the perturbation problem for a class of plane flows with a weak shock discontinuity. It would be possible to consider the perturbation problem for the special hodograph region of Section (4) and with the notable distinction

$$(5.40) \quad \mathfrak{C} = \mathfrak{C}_0(\theta > 0, \sigma > 0): \quad b = 0, \quad f_2 = 0;$$

$$(5.41) \quad \sigma = 0, \quad 0 < \theta < \theta_{12}(\text{say}): \quad c = 0, \quad f_2 = -\pi/2.$$

Since these clearly ensure

$$(5.42) \quad c(\theta, \sigma) > 0, \quad \sigma > 0; \quad b'(\theta, \sigma = 0) > 0,$$

the area integrals for $\sigma > 0$ are positive definite. Also the line integrals over $\mathfrak{C}(\sigma > 0)$ are non-negative if we arrange

$$d\sigma \geq 0, \quad \theta \geq 0; \quad d\theta > 0$$

and take account of (5.39) (5.41).

In the supersonic region we may take

$$(5.43) \quad b = b(\alpha) \equiv b(\alpha, \sigma = 0), \quad c \equiv 0,$$

this choice making b and c continuous in the mixed region. The area integral is positive as in (4.19) and since $b'(\theta) > 0$ according to (5.42). The line integral over $\mathfrak{C}(\theta > 0, \sigma < 0)$ vanishes with $d\sigma \leq 0$ and we could generalize the permissible form of the hodograph boundary in $\sigma < 0$ by adding some $c(\theta, \sigma)$ which vanishes on $\sigma = 0$, compare (5.31).

However, to make the discussion as straight forward as possible this extension will be left over also the possibility of free streamline arcs, compare (5.35) *et seq.*

Let us now turn to the line integrals over the Λ_i . These will give a non-negative contribution if we have for pairs of corresponding points,

$$(5.44) \quad (L d\sigma + M d\theta)_1 + (L d\sigma + M d\theta)_2 \geq 0,$$

where, say,

$$(5.45) \quad L d\sigma + M d\theta = G(K\Phi^2 - \Psi^2) + 2H\Phi\Psi.$$

The following argument makes use of the remark that, on account of the single homogeneous relation (5.16), which may be written

$$(5.46) \quad \Psi_1 - r\Phi_1 = \Psi_2 - r\Phi_2,$$

we can reduce (5.44) to a sum of squares in *three* independent variables. The easiest way of carrying this out seems to be the following. Let us write for either Λ_i

$$(5.47) \quad G(K\Phi^2 - \Psi^2) + 2H\Phi\Psi = U(r\Phi - \Psi)^2 + V(\Psi + t\Phi)^2,$$

where U , V and t are real but as yet undefined. We can determine U from the condition that if the corresponding term in (5.47) is removed to the left hand side then the discriminant of both sides vanishes. We find in this way

$$(5.48) \quad U_i = -\left\{ \frac{KG^2 + H^2}{\Delta} \right\}_i, \quad i = 1, 2,$$

with the Δ_i as in (5.50) below. To ensure that the complete form (5.44) is non-negative we need the following conditions

$$(5.49) \quad V_i \geq 0, \quad U_1 + U_2 \geq 0.$$

The first of these is clearly the equivalent of

$$(5.50) \quad -(G + U)_i = \left\{ \frac{(Gr - H)^2}{\Delta} \right\}_i \geq 0,$$

and is satisfied if we have

$$(5.51) \quad \Delta_i = \{G(K - r^2) + 2rH\}_i > 0.$$

The second of (5.49) may be reduced to

$$(5.52) \quad (b_1^2 - |K_1| c_1^2)(|K_1| d\sigma_1^2 - d\theta_1^2)/\Delta_1 \geq (b_2^2 + K_2 c_2^2)(K_2 d\sigma_2^2 + d\theta_2^2)/\Delta_2.$$

To carry through the uniqueness proof we have to check on the inequalities (5.51) (5.52). Noting that c vanishes on the Λ_i we find

$$(5.53) \quad \Delta_1 = b_1 |d\sigma_1| (K_1 + r^2) + 2rb_1 d\theta_1,$$

in which each term is positive with $d\theta > 0$ and $d\sigma < 0$. Again we have

$$(5.54) \quad \Delta_2 = |b_2 d\sigma_2| (K_2 - r^2) - 2r |b_2 d\theta_2|.$$

Also, the inequality (5.49) may be written

$$(5.55) \quad |U_1| \geq |U_2|,$$

where

$$(5.56) \quad |U_i| = \frac{|b_i d\sigma_i| \left(1 + \frac{d\theta_i^2}{K_i d\sigma_i^2}\right)}{1 - \left\{r^2 + 2r \left|\frac{d\theta_i}{d\sigma_i}\right|\right\} K_i^{-1}}.$$

The inequalities (5.45) (5.55) are satisfied if we set $|\sigma_1| = \sigma_2$, $d\theta_i = 0$ which is formally the case of normal shocks. For then we have easily $|U_1| = |U_2|$ and $\Delta_i > 0$. It can be verified that the inequalities do hold for the precise analogue of the hodograph problem of Section (4); the small arcs near the origin which do not belong to the Λ_i contribute non-negative quantities.

Again, for possible homogeneous solutions which are sufficiently near the normal case in the sense of (4.34) *et seq* we have

$$(5.57) \quad b_1 = \{\theta_1 + \frac{2}{3} |\sigma_1|^{3/2}\}^{2/p} \{1 + O|\zeta_1|\},$$

$$(5.58) \quad b_2 = \{\theta_2^2 + \frac{4}{9} \sigma_2^3\}^{1/p} \{1 + O|\zeta_2|\},$$

with p as in (4.28). The inequalities are satisfied since for sufficiently small arc Λ_i

$$(5.59) \quad |U_1|/|U_2| = \left\{1 + \frac{2}{p |\lambda_1|^{3/2}}\right\} \{1 + O(\mu)\},$$

and the Δ_i are clearly positive.

To complete the uniqueness proof we must examine the convergence of the line integrals in the vicinity of the point at which the hodograph image of the physical boundary intersects the sonic line. As we approach this point the multipliers b , c increase without limit but, assuming the Tricomi approximation to equation (5.22) and making use of the estimates for the derivatives of $W(\theta, \sigma)$ according to Manwell (1964), it can be shown that the line integrand remains bounded both along a small normal arc and on a characteristic, these two arcs indenting the boundary near the singularity. The dominant part of the solution W is a homogeneous function of degree $4/3$ leading to a Lagrange potential of degree $1/3$ and a Tricomi singularity in the corresponding perturbation of the hodograph stream function.

An interesting comparison arises with the discussion in Manwell (1964) of certain existence theorems for the Morawetz perturbation problem. This being set in the sub-region of the complete flow in the case of no shock discontinuities it was found that there are in general two Tricomi singularities. The same situation is to be anticipated for shock transitions of type $I = 1$; $J = 1$, but for solutions of type $I = 1$; $J = 2$ the boundary cuts the sonic line only once and we anticipate one Tricomi singularity only. There is a close analogy between the existence theorem for the Morawetz problem and the perturbation problem for a symmetrical profile with a single weak attached shock on both upper and lower surfaces. As in the former problem we anticipate that the perturbation function will be bounded only if we may associate free streamlines with a given profile. However, if there is only one Tricomi singularity to be disposed of the supersonic to subsonic transition is the privileged one since, as distinct from the supersonic to supersonic case, a single free stream line associated with each surface would be sufficient. Again, it appears to be difficult to satisfy the inequality (5.51) in the case $J = 1$; for example if $m = 2/3$ it turns out that (5.51) conflicts with the condition on mappability (2.59) and it does not seem easy to modify the choice of b , c in the uniqueness proof to remedy this situation. We have here a clear indication from the theory as to the why the supersonic to subsonic regime which is observed in the experiments is the stable one.

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