Some Sharp Estimations of Coefficients of Univalent Functions

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I. Let $f(z) = z + a_2 z^2 + \dots$ belong to the class S , i.e., let $f(z)$ be holomorphic and univalent in the unit circle K . Put $G = f(K)$ and let D denote the image of G under the transformation $w = 1/\zeta$. D is an unbounded, simply connected domain which contains the point $w = \infty$. The complement of D in the w -plane is a bounded continuum E with capacity $C(E) = 1$. The point $z = \infty$ which lies outside of G goes onto $w = 0$ under the transformation $w = 1/\zeta$ and therefore the origin 0 of the coordinate system belongs to E .

The coefficients a_n , $n = 2, 3, \dots$, of $f(z) \in S$ can be expressed as polynomials of the moments $s_k = \int w^k d\mu$, $k = 1, 2, \dots$, where μ is the natural mass distribution on E (see [2]). Further if $\tilde{\varphi}(w)$ is the function of the class Σ^{-1} (the class of functions inverse to those holomorphic, univalent and normalized in the exterior of the unit circle) which maps D onto $|z| > 1$ then

$$\log \frac{\tilde{\varphi}(w)}{w} = -\left(\frac{s_1}{w} + \frac{s_2}{2w^2} + \dots + \frac{s_n}{nw^n} + \dots\right),$$

so

$$\frac{\tilde{\varphi}'(w)}{\tilde{\varphi}(w)} = \frac{1}{w} + \frac{s_1}{w^2} + \dots + \frac{s_n}{w^{n+1}} + \dots.$$

The function $\varphi \in S^{-1}$ which is inverse to f is such that

$$\varphi(w) = \tilde{\varphi}\left(\frac{1}{w}\right)^{-1}$$

and thus

$$\frac{w\varphi'(w)}{\varphi(w)} = \frac{w\left[\tilde{\varphi}\left(\frac{1}{w}\right)^{-1}\right]'}{\tilde{\varphi}\left(\frac{1}{w}\right)^{-1}}$$

$$\begin{aligned}
&= \frac{1}{w} \frac{\tilde{\varphi}'\left(\frac{1}{w}\right)}{\tilde{\varphi}\left(\frac{1}{w}\right)} \\
&= \frac{1}{w} [w + s_1 w^2 + s_2 w^3 + \cdots + s_n w^{n+1} + \cdots].
\end{aligned}$$

Let

$$[f(z)]^t = \sum_{r=-\infty}^{\infty} a_r^{(t)} z^r.$$

E. Jabotinsky [4] has shown that

$$\sum_{t=-\infty}^{\infty} a_0^{(t)} w^{-t-1} = \frac{\varphi'(w)}{\varphi(w)}.$$

Therefore, comparing coefficients, we see

$$a_0^{(-n)} = s_n.$$

And since it is known that [6]

$$|a_0^{(-n)}| \leq \binom{2n}{n},$$

we have

$$|s_n| \leq \binom{2n}{n}.$$

The inequality is sharp and equality holds for Koebe's function.

If we let $\bar{0}$ denote the conformal mass center of the μ -distribution we obtain the formula

$$s_k = s_k(\bar{0}) + \binom{k}{1} s_{k-1}(\bar{0}) s_1 + \binom{k}{2} s_{k-2}(\bar{0}) s_1^2 + \cdots + s_1^k,$$

where $s_j(\bar{0})$ is the j^{th} moment with respect to the point $\bar{0}$.

II. The coefficient a_5 is given by the formula (see [1])

$$(1) \quad a_5 = -\frac{s_4(\bar{0})}{4} + \frac{3}{4} s_2^2(\bar{0}) + 2a_2 a_4 - \frac{a_2^4}{2} - \frac{a_3^2}{2}.$$

We show that

$$(2) \quad \left| -\frac{s_4(\bar{0})}{4} + \frac{3}{4} s_2^2(\bar{0}) \right| \leq \frac{3}{2} \quad \text{for } f(z) \in S,$$

equality holding for Koebe's function. From (1) and (2) together with the inequality

$$(3) \quad \left| \frac{a_2^4}{2} - \frac{a_3^2}{2} \right| \leq \frac{7}{2} \quad \text{for } f(z) \in S,$$

it follows that

$$|a_5 - (2a_2a_4 - a_2^4)| \leq 5 \quad \text{for } f(z) \in S.$$

Proof of the inequality (2). To every function $f(z) \in S$ corresponds a function

$$F(z) = \frac{1}{f\left(\frac{1}{z}\right)},$$

$$F(z) = z - a_2 + \frac{a_1^{(-1)}}{z} + \frac{a_2^{(-1)}}{z^2} + \frac{a_3^{(-1)}}{z^3} + \dots,$$

which is univalent in $|z| > 1$. The coefficients of $F(z)$ are given by

$$\begin{aligned} a_1^{(-1)} &= a_2^2 - a_3 = \frac{s_2(\bar{0})}{2}; \\ a_2^{(-1)} &= -a_4 + 2a_2a_3 - a_2^3 = \frac{s_3(\bar{0})}{3}; \\ a_3^{(-1)} &= -a_5 + 2a_2a_4 + a_3^2 - 3a_2^2a_3 + a_2^4 \\ &= \frac{s_4(\bar{0})}{4} - \frac{3}{8}s_2^2(\bar{0}). \end{aligned} \quad (4)$$

It was proved [3] that

$$|a_3^{(-1)}| \leq \frac{1}{2} + e^{-6} < 0.51 \quad \text{for } f(z) \in S. \quad (5)$$

We now consider those functions $f(z) \in S$ for which

$$|a_3^{(-1)}| \geq \frac{2}{9} = 0.22 \dots \quad (6)$$

In terms of $a_1^{(-1)}$ and $a_3^{(-1)}$

$$\left| -\frac{s_4(\bar{0})}{4} + \frac{3}{4}s_2^2(\bar{0}) \right| = \left| \frac{3}{2}a_1^{(-1)^2} - a_3^{(-1)} \right| = \frac{1}{2} |3a_1^{(-1)^2} - 2a_3^{(-1)}|.$$

Therefore

$$\left| -\frac{s_4(\bar{0})}{4} + \frac{3}{4}s_2^2(\bar{0}) \right| \leq \frac{1}{2} \{3 |a_1^{(-1)}|^2 + 2 |a_3^{(-1)}|\}. \quad (7)$$

From the area theorem we have

$$|a_1^{(-1)}|^2 + 3 |a_3^{(-1)}|^2 \leq 1, \quad (8)$$

and so (from (6), (7) and (8)) it follows that

$$\left| -\frac{s_4(\bar{0})}{4} + \frac{3}{4}s_2^2(\bar{0}) \right| \leq \frac{1}{2} \{3 |a_1^{(-1)}|^2 + 9 |a_3^{(-1)}|^2\} \leq \frac{3}{2}. \quad (2)$$

Therefore the inequality (2) is true for functions $f(z) \in S$ which are "far" from the Koebe function.

Let $y(\alpha) = |-s_4(\bar{0})/4 + \alpha s_2^2(\bar{0})|$ be the function which corresponds to an arbitrary fixed function $f(z) \in S$. Without loss of generality we can assume that $s_2(\bar{0}) \geq 0$. For Koebe's function

$$(9) \quad y(\alpha) = \begin{cases} 4\alpha - \frac{3}{2} & \text{for } \alpha \geq \frac{3}{8} \\ -4\alpha + \frac{3}{2} & \text{for } \alpha \leq \frac{3}{8}. \end{cases}$$

In the general case $y(\alpha)$ is a hyperbola

$$(10) \quad y^2 - \left[\alpha s_2^2(\bar{0}) - \frac{\operatorname{Re}(s_4(\bar{0}))}{4} \right]^2 = \left[\frac{\operatorname{Im}(s_4(\bar{0}))}{4} \right]^2.$$

The points of the intersection of (9) and (10) are given by the equation

$$g(\alpha) = \alpha^2[16 - s_2^4(\bar{0})] - \alpha \left[12 - s_2^2(\bar{0}) \frac{\operatorname{Re}(s_4(\bar{0}))}{2} \right] + \frac{9}{4} - \frac{|s_4(\bar{0})|^2}{16} = 0.$$

Since $|s_4(\bar{0})| \leq 6$ (see [1]), we have $g(0) \geq 0$,

$$g\left(\frac{3}{8}\right) = - \left[\left| \frac{s_4(\bar{0})}{4} \right| - \frac{3}{8} s_2^2(\bar{0}) \right]^2 \leq 0.$$

Therefore one of the roots α_1 is $\leq 3/8$, the other root α_2 is $> 3/8$. For every function $f(z) \in S$ which is different from the Koebe function

$$0 < \sqrt{\Delta} < 12 - s_2^2(\bar{0}) \frac{\operatorname{Re}(s_4(\bar{0}))}{2},$$

where Δ is the discriminant of $g(\alpha)$, and

$$\alpha_2 < \frac{12 - s_2^2(\bar{0}) \frac{\operatorname{Re}(s_4(\bar{0}))}{2}}{16 - s_2^4(\bar{0})}.$$

If $\operatorname{Re}(s_4(\bar{0}))/2 \geq 3/4$, then $\alpha_2 < 3/4$. Hence

$$(2) \quad \left| -\frac{s_4(\bar{0})}{4} + \frac{3}{4} s_2^2(\bar{0}) \right| \leq \frac{3}{2}$$

for those functions $f(z) \in S$ for which $\operatorname{Re}(s_4(\bar{0})) \geq 3/2$.

Observe that if $\operatorname{Re}(s_4(\bar{0})) \geq 3/2$ then

$$|-a_3^{(-1)}| = \left| -\frac{s_4(\bar{0})}{4} + \frac{3}{8} s_2^2(\bar{0}) \right| \leq 0.22 \dots$$

And

$$2a_2a_4 - \frac{a_2^4}{2} - \frac{a_3^2}{2} = (2a_2a_4 - a_2^4) + \left(\frac{a_2^4}{2} - \frac{a_3^2}{2} \right);$$

therefore

$$(11) \quad \left| \frac{a_2^4}{2} - \frac{a_3^2}{2} \right| = \frac{1}{2} |a_2^2 + a_3| |a_2^2 - a_3| \leq \frac{7}{2} \quad \text{for all } f(z) \in S.$$

So finally from (1), (2) and (11) we have

$$(12) \quad |a_5 - (2a_2a_4 - a_2^4)| \leq 5 \quad \text{for all } f(z) \in S.$$

Remark 1. J. A. Jenkins proved [5] the following theorem: If $f(z) \in S$ and $a_j = 0$ for $j \leq m$, m -arbitrary fixed natural number, then for $m \geq 2$

$$(13) \quad |a_{2m-1} - \beta a_m^2| \leq \begin{cases} \frac{4\beta - m - 1}{(m-1)^2} & \text{for } \beta \geq \frac{m}{2}, \\ (m-1)^{-1} \left[1 + 2 \exp \left(-2 \frac{m + 2\beta - 2}{m - 2\beta} \right) \right], & \frac{m}{2} > \beta > 1 - \frac{m}{2}, \\ \frac{1 + m - 4\beta}{(m-1)^2} & \text{for } \beta \leq 1 - \frac{m}{2}. \end{cases}$$

In particular for $m = 3, \beta = 0$

$$|a_5| \leq \frac{1}{2} + e^{-2/3}.$$

Let $\bar{S}$ be a subclass of S such that $f(z) \in \bar{S}$ if and only if the corresponding continuum E contains the point $\bar{0}$. From (13) for $m = 3, \beta \leq -1/2$ we have

$$|a_5 - \beta a_3^2| = \left| -\frac{s_4(\bar{0})}{4} + \left(\frac{5}{8} - \frac{\beta}{4} \right) s_2^2(\bar{0}) \right| \leq 1 - \beta.$$

Put $\alpha = 5/8 - \beta/4$, then

$$(14) \quad \left| -\frac{s_4(\bar{0})}{4} + \alpha s_2^2(\bar{0}) \right| \leq 4\alpha - \frac{3}{2} \quad \text{for } \alpha \geq \frac{3}{4},$$

the equality holds for Koebe's function.

Similarly we obtain

$$(14) \quad \left| -\frac{s_4(\bar{0})}{4} + \alpha s_2^2(\bar{0}) \right| \leq -4\alpha + \frac{3}{2} \quad \text{for } \alpha \leq \frac{1}{4}.$$

This inequality holds for all functions $f(z) \in S$, (see [1]).

For $\frac{1}{4} < \alpha < \frac{3}{4}$, $f(z) \in \bar{S}$,

$$(15) \quad \left| -\frac{s_4(\bar{0})}{4} + \alpha s_2^2(\bar{0}) \right| \leq \frac{1}{2} + e^{-2(4\alpha-3)/(1-4\alpha)}.$$

It seems that (15) holds for all functions $f(z) \in S$. In particular for $\alpha = 3/8$ follows

$$\left| -\frac{s_4(\bar{0})}{4} + \frac{3}{8} s_2^2(\bar{0}) \right| \leq \frac{1}{2} + e^{-6},$$

and this was proved in [3].

Remark 2. In [1] it was proved that $|a_5 - a_2^4| \leq 11$. Therefore

$$(16) \quad |a_5 - \alpha a_2^4| \leq 16\alpha - 5 \quad \text{for } \alpha \geq 1, \quad f(z) \in S.$$

The sharp estimation of $|a_5 - \alpha a_2^4|$ for $\alpha \in (0,1)$ is unknown. From (12)

$$(17) \quad |a_5 - (2a_2a_4 - a_2^4) - \alpha a_2^4| \leq -16\alpha + 5 \quad \text{for } \alpha \leq 0, \quad f(z) \in S.$$

In particular for $\alpha = 0$

$$|a_5 + a_2^4| \leq 5 + 2 |a_2a_4| \leq 21.$$

Therefore

$$(16) \quad |a_5 - \alpha a_2^4| \leq -16\alpha + 5 \quad \text{for } \alpha \leq -1, \quad f(z) \in S.$$

If

$$\left| (2a_2a_4 - a_2^4) + \left(\frac{a_2^4}{2} - \frac{a_3^2}{2} \right) \right| \leq \frac{7}{2} \quad \text{for } f \in S$$

then $|a_5| \leq 5, f \in S$.

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