*On the Completeness of Cartan Connections**

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A Cartan connection on a principal bundle is essentially a rule which enables us to develop every smooth curve of the base space onto a smooth curve in a fibre of an associated bundle. This fibre is regarded as tangent to the base space at the initial point of the curve, and the development is a curve starting at the point of tangency.

Special cases are affine and Euclidean connections. The tangent spaces are then affine and Euclidean spaces, respectively. A torsion-free Euclidean connection is called Riemannian. Torsion of a Cartan connection will be defined below.

A Cartan connection is complete (in the sense of Ehresmann, see [1]) if every smooth curve in any of the tangent spaces, starting at the point of tangency, is the development of some curve of the base space. It is well known ([2], Theorem 4.1, p. 172) that a Riemannian connection is complete if and only if the natural metric on the base space is complete. We shall extend this result to Euclidean connections.

In general, there is no natural metric on the base space of a Cartan connection, but there is always a set of natural metrics on the total space of the principal bundle. These metrics are complete if and only if any one of them is. The main purpose of this note is to prove that a Cartan connection is complete if and only if the natural metrics on the total space of the principal bundle are complete.

Consider the following conditions on a Cartan connection ω :

- 1) ω is complete,
- 2) every regular curve of a tangent space, starting at the point of tangency, is the development of some curve of the base space,
- 3) (supposing ω to be an affine connection) every line through the origin of a tangent space is the development of a curve in the base space.

Condition 3) is called geodesic completeness.

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It is clear that 1) implies 2), and that, in the affine case, 2) implies 3). It has long been an open question whether, in the affine case, 3) implies 1). We shall give two affine examples to show that 2) does not imply 1), and 3) does not imply 2).

1. Definition of Cartan connection. Let us first define *elementary geometry* to be a pair (G, G_0) , where G is a Lie group, and G_0 is a closed subgroup of G having the property that no subgroup of G_0 , other than the identity, is normal in G . Then G operates effectively to the right on the set $F = G/G_0$ of right cosets mod G_0 . Let $\varphi : G \rightarrow F$ be the quotient map, and let us denote the map $F \times G \rightarrow F$ multiplicatively.

If H is any closed subgroup of G which acts transitively on F , then we say that $(H, H \cap G_0)$ is a *restriction* of (G, G_0) , and (G, G_0) is an *extension* of $(H, H \cap G_0)$. Then we regard G/G_0 and $H/H \cap G_0$ as identical.

The coset $\varphi(1)$ is called the *origin* of F .

Examples. (1) Let A^n be the group of linear transformations of $\mathbf{R}^{n+1}$ which leave the hyperplane $\{1\} \times \mathbf{R}^n$ invariant. Let L^n be the subgroup of A^n which leaves the point $(1, 0, 0, \dots, 0)$ fixed. Then (A^n, L^n) is called affine geometry of dimension n . We denote A^n/L^n by S^n .

(2) Let E^n be the subgroup of A^n which acts isometrically on $\{1\} \times \mathbf{R}^n$. Let $O^n = E^n \cap L^n$. Then (E^n, O^n) is called Euclidean geometry of dimension n . (E^n, O^n) is a restriction of (A^n, L^n) , and $E^n/O^n = S^n$.

(3) There is a natural one-to-one correspondence between S^n and the group of translations of S^n . If we agree to identify S^n with this group we find that S^n is a closed subgroup of E^n . Then $(S^n, \{1\})$ is a restriction of Euclidean geometry which we may call *translation geometry*.

Now let (G, G_0) be an elementary geometry. To simplify our language, we assume that G is a linear group. Specifically, G is a C^∞ manifold imbedded in $\text{Hom}(\mathbf{R}^N, \mathbf{R}^N)$ (for some positive integer N), and also a closed subgroup of the linear group of $\mathbf{R}^N$. Let $\mathfrak{G}, \mathfrak{G}_0$ be the Lie algebras, respectively, of G, G_0 ; and let $\psi : \mathfrak{G} \rightarrow \mathfrak{G}/\mathfrak{G}_0$ be the projection. $\mathfrak{G}$ and $\mathfrak{G}_0$ are vector subspaces of $\text{Hom}(\mathbf{R}^N, \mathbf{R}^N)$.

A C^∞ function between manifolds will be called a *map*. A C^1 function will simply be called a *smooth* function.

Let $\pi : B \rightarrow M$ be a C^∞ principal bundle, having structure group G_0 , the total space B being a C^∞ manifold having the dimension of G . Then G_0 operates to the left on B , and we shall denote the map $G_0 \times B \rightarrow B$ multiplicatively. Let $p : T \rightarrow M$ be the associated bundle with fibre $F = G/G_0$ and structure group G_0 . The map $F \times B \rightarrow T$ will also be denoted multiplicatively.

There is a section $s : M \rightarrow T$ defined by the condition $s(\pi(b)) = \varphi(1) \cdot b$ ($b \in B$). $s(x)$ will be called the *origin* of the fibre $p^{-1}(x)$ ($x \in M$).

Definition. Given a bundle π which satisfies the above conditions, a (G, G_0) -connection on π (in the sense of Cartan) is a C^∞ vector-valued Pfaffian ω on taking values in $\mathfrak{G}$, and subject to the conditions:

- 1) Given any basis $u^i : \mathfrak{G} \rightarrow \mathbb{R}$ ($1 \leq i \leq \dim G$) of $\mathfrak{G}^*$, the real-valued Pfaffians $u^i \circ \omega$ are linearly independent at every point of B .
- 2) Let $\alpha, \beta : G_0 \times B \rightarrow \text{Hom}(\mathbb{R}^N, \mathbb{R}^N)$ be given by $\alpha(g, b) = g$ and $\beta(g, b) = g^{-1}$ ($g \in G_0 ; b \in B$); and let $\gamma, \delta : G_0 \times B \rightarrow B$ be given by $\gamma(g, b) = gb$ and $\delta(g, b) = b$. Then

$$\gamma^*\omega = d\alpha \cdot \beta + \alpha \cdot (\delta^*\omega) \cdot \beta,$$

the multiplications on the right being exterior products of matrix-valued forms (of degrees 0 and 1).

Condition 2) implies that, if $\sigma : U \rightarrow B$ is any local section (U an open subset of M), the values of ω on $\pi^{-1}(U)$ are determined by the values of $\sigma^*\omega$ on U . Hence the connection ω may be defined on B by giving the $\sigma^*\omega$ for all σ of a family of sections covering M . Condition 2) then gives the compatibility relation in the overlaps, and condition 1) will be fulfilled if and only if the forms $v^i \circ \psi \circ (\sigma^*\omega)$ are linearly independent ($1 \leq i \leq \dim G - \dim G_0$), where (v^i) is any basis of $(\mathfrak{G}/\mathfrak{G}_0)^*$.

If $(\tilde{G}, \tilde{G}_0)$ is any extension of the geometry (G, G_0) , the bundle $\pi : B \rightarrow M$ may be extended to the principal bundle $\tilde{\pi} : \tilde{B} \rightarrow M$ with group $\tilde{G}_0$, and $B \subset \tilde{B}$. Then condition 2) tells us that ω is uniquely extendable to a connection $\tilde{\omega}$ on $\tilde{B}$. Thus a translation connection generates a Euclidean connection, which in turn generates an affine connection.

Definition. Given a connection ω on B , the 2-form $\Omega = d\omega - \omega \cdot \omega$ is the *curvature* of ω . The 2-form $\Lambda = \psi \circ \Omega$ is called the *torsion*. A Euclidean connection is called *Riemannian* if its torsion is zero.

2. Development of a curve. Let $f : [0, 1] \rightarrow M$ be a smooth (C^1) curve in M . Let $g : [0, 1] \rightarrow B$ be any smooth curve such that $f = \pi \circ g$. There exists a continuous function $h : [0, 1] \rightarrow \mathfrak{G}$ such that $g^*\omega = h(t) dt$; and there is a unique smooth curve $k : [0, 1] \rightarrow G$ such that $k'(t) = h(t)k(t)$ ($0 \leq t \leq 1$) and $k(0) = 1$. Let $\hat{f} : [0, 1] \rightarrow T$ be defined by: $\hat{f}(t) = \varphi(k(t)) \cdot g(0)$. $\hat{f}$ is called the *development* of f (with respect to the connection ω). Let us prove that $\hat{f}$ is independent of the lifting g .

Let $\bar{g} : [0, 1] \rightarrow B$ be another smooth curve such that $\pi \circ \bar{g} = f$. Define $\bar{h}, \bar{k}, \bar{\hat{f}}$ as before. There exists a unique smooth curve $l : [0, 1] \rightarrow G_0$ such that $\bar{g}(t) = l(t) \cdot g(t)$ ($0 \neq t \neq 1$). Then, by 2), $\bar{h} = l' \cdot l^{-1} + l \cdot h \cdot l^{-1}$. Define $\bar{k} : [0, 1] \rightarrow G$ by the formula $\bar{k}(t) = l(t)k(t)l^{-1}(0)$. Then $\bar{k}' = \bar{h}\bar{k}$ and $\bar{k}(0) = 1$. Hence $\bar{k} = k$. Then

$$\hat{\bar{f}}(t) = \varphi(l(t)k(t)l^{-1}(0)) \cdot \bar{g}(0) = (k(t)) \cdot g(0) = \hat{f}(t).$$

This proves that the development $\hat{f}$ depends only on f and not on the lifting g .

The curve $\hat{f}$ takes values in the fibre $p^{-1}(f(0))$, and $\hat{f}(0)$ is the origin of that fibre.

Now let f be any smooth curve in T such that $p(f(t)) = p(f(0))$ ($0 \leq t \leq 1$) and $f(0) = s(p(f(0)))$. Let $b \in B$ be such that $\pi(b) = p(f(0))$, and let $g : [0, 1] \rightarrow G$ be any smooth curve such that $\varphi(g(t)) \cdot b = f(t)$ ($0 \leq t \leq 1$) and $g(0) = 1$. Then $h(t) = g'(t) \cdot g^{-1}(t) \in \mathfrak{G}$ ($0 \leq t \leq 1$). Let us extend h to a continuous function $k : \mathbb{R} \rightarrow \mathfrak{G}$, with $k(t) = h(t)$ ($0 \leq t \leq 1$). Let $\alpha : B \times \mathbb{R} \rightarrow B$ and $t : B \times \mathbb{R} \rightarrow \mathbb{R}$ be the projections, and let $u^i : \mathfrak{G} \rightarrow \mathbb{R}$ ($1 \leq i \leq \dim G$) be any basis of $\mathfrak{G}^*$. Then the Pfaffian system $u^i \circ (\alpha^* \omega - k(t) dt) = 0$ foliates $B \times \mathbb{R}$ by lines. This gives us, for sufficiently small $\epsilon > 0$, a unique smooth curve $l : [0, \epsilon] \rightarrow B$ such that $l^* \omega = h(t) dt$ and $l(0) = b$. Clearly $f|_{[0, \epsilon]}$ is the development of $\pi \circ l$. In case ϵ may be taken to be 1, we say that f is *antidevelopable* and that $\pi \circ l$ is the *antidevelopment* of f .

Note that two smooth curves $f_1, f_2 : [0, 1] \rightarrow M$, such that $f_1(0) = f_2(0)$, have the same velocity vector at $t = 0$ if and only if their developments have. This gives an isomorphism between the tangent vector space of M at $f_1(0)$ and the tangent vector space of $p^{-1}(f_1(0))$ at $s(f_1(0))$. In this sense we say that $p^{-1}(x)$ is tangent to M at x , the point of tangency being $s(x)$ ($x \in M$).

Definition. The connection ω is *complete* if every smooth (C^1) curve in T , taking values in a single fibre and starting at the origin of that fibre, is anti-developable.

3. The canonical metrics on B . Let $u = (u^i)$ ($1 \leq i \leq \dim G$) be any basis of $\mathfrak{G}^*$. Let $f : [0, 1] \rightarrow B$ be a smooth curve. We define $|f|_u$ (the u -length of f) to be $\int_0^1 (\sum_i [g^i(t)]^2)^{1/2} dt$, where $g^i(t)$ is defined by $f^*(u^i \circ \omega) = g^i(t) dt$. Given b_1, b_2 , members of a component of B , we define $d_u(b_1, b_2)$ to be the greatest lower bound of the u -lengths of curves connecting b_1 and b_2 . d_u is a Riemannian metric on every component of B . If each component of B is a complete metric space when endowed with d_u , we say that d_u endows B with a structure of complete Riemannian metric. It is clear that for any two bases u, v of $\mathfrak{G}^*$, there is a constant L such that $d_u \leq Ld_v$ and $d_v \leq Ld_u$. Hence one of these Riemannian metrics on B is complete if and only if all of them are. Each d_u is called a *canonical metric* on B .

Theorem 1. *The connection ω is complete if and only if the canonical metrics on B are complete.*

Proof. Suppose the canonical metrics complete. Pick a particular basis u of $\mathfrak{G}^*$. Let $f : [0, 1] \rightarrow T$ be a smooth curve such that $p(f(t)) = p(f(0))$ ($0 \leq t \leq 1$) and $f(0) = s(p(f(0)))$. Pick a smooth $g : [0, 1] \rightarrow G$ such that $g(0) = 1$, and $\varphi(g(t)) \cdot b = f(t)$ ($0 \leq t \leq 1$). Let $\epsilon \in (0, 1]$ be the largest number such that there exists $h : [0, \epsilon] \rightarrow B$ with $h(0) = b$ and $h^* \omega = g'(t)g^{-1}(t) dt$. If $\eta : G \rightarrow \text{Hom}(\mathbb{R}^N, \mathbb{R}^N)$ is the inclusion map, we find that the forms $\alpha^i = u^i \circ (d\eta \cdot \eta^{-1})$ ($1 \leq i \leq \dim G$) endow G with a Riemannian metric $ds^2 =$

$\sum_i (\alpha^i)^2$, which we may also call d_u . Moreover, the α^i are invariant under right translation; hence the metric is homogeneous, and, by Theorem 4.5 on p. 176 of [2], complete.

Since $h^*(u^i \circ \omega) = g^*\alpha^i$, we have $|h|_u = |g/[0, \epsilon]|_u$, which means h has finite length and, by the completeness of B , $\lim_{t \rightarrow 1} h(t)$ must exist. But since the graph of h is the integral curve of a foliation, $\lim_{t \rightarrow 1} h(t)$ cannot exist unless $\epsilon = 1$. Hence h may be extended to a rectifiable curve, defined on the interval $[0, 1]$, and smooth on the interval $[0, 1)$. But, since $h^*\omega$ has a continuous limit at $t = 1$, and ω spans B , hence h is smooth on the interval $[0, 1]$. Then $\pi \circ h$ is the antidevelopment of f .

Now suppose every smooth curve f in T , subject to the usual conditions, has an antidevelopment. We pick u as before, and suppose B is an incomplete Riemannian space. By Theorem 4.1, p. 172 of [2], there is a geodesic $g : [0, l) \rightarrow B$ (geodesic with respect to the Riemannian metric d_u), parametrized by arc length, such that $\lim_{t \rightarrow l} g(t)$ diverges. Let $h : [0, l) \rightarrow G$ be the unique curve such that $g^*\omega = dh \cdot h^{-1}$ and $h(0) = 1$. Then h is also parametrized by arc length, and, since the metric on G is complete, h can be extended to a Lipschitzian function $\bar{h} : [0, l] \rightarrow G$, with $\bar{h}(t) = h(t)$ ($0 \leq t < l$). Now define $\alpha : [0, 1] \rightarrow [0, l]$ by $\alpha(t) = 2lt - lt^2$ ($0 \leq t \leq 1$). Then $\bar{h} \circ \alpha$ is smooth. Let $f : [0, 1] \rightarrow T$ be given by $f(t) = \varphi(h(\alpha(t))) \cdot g(0)$. f admits an antidevelopment $k : [0, 1] \rightarrow M$.

Let $m : [0, 1] \rightarrow B$ be any smooth function such that $\pi \circ m = k$. Then there is a smooth function $n : [0, 1) \rightarrow G_0$ such that $g(\alpha(t)) = n(t)m(t)$ ($0 \leq t < 1$). We wish to prove the existence of $\lim_{t \rightarrow 1} n(t)$.

Let $M : [0, 1] \rightarrow G$ be determined by the conditions $M'(t)M^{-1}(t) dt = m^*\omega$ and $n(0)M(0) = h(\alpha(0)) = 1$. Then there is a smooth function $L : [0, 1] \rightarrow G_0$ such that $L(t)M(t) = \bar{h}(\alpha(t))$ ($0 \leq t \leq 1$). But then $L(t) = n(t)$ ($0 \leq t < 1$). Hence $L(1) = \lim_{t \rightarrow 1} n(t)$. Therefore

$$\lim_{t \rightarrow 1} g(t) = \lim_{t \rightarrow 1} g(\alpha(t)) = \lim_{t \rightarrow 1} n(t) \cdot m(t) = L(1)m(1),$$

which contradicts the hypothesis that $\lim_{t \rightarrow l} g(t)$ diverges. Therefore the metric on B is complete.

Theorem 2. *A Euclidean connection is complete if and only if the base space, endowed with its natural metric, is complete.*

Proof. This is the case where $G = E^n$ (the Euclidean group) and $G_0 = O^n$ (the orthogonal group). $N = n + 1$, and so the form ω is a matrix $(\omega^i_j)_{0 \leq i, j \leq n}$ of real-valued Pfaffians. Since ω takes values in $\mathfrak{G}$, we have $\omega^0_i = 0$ ($1 \leq i \leq n$) and $\omega^i_j + \omega^j_i = 0$ ($1 \leq i, j \leq n$). Hence we may define a canonical metric on B by

$$ds^2 = \sum_{1 \leq i \leq n} (\omega^0_i)^2 + \sum_{1 \leq i < j \leq n} (\omega^i_j)^2.$$

If $\sigma : U \rightarrow B$ is any local section map (U an open subset of M), the formula

$$ds^2 = \sum_{1 \leq i \leq n} (\sigma^*\omega^0_i)^2$$

defines the natural metric on U , this metric being independent of section. Therefore, if $f : [0, 1] \rightarrow B$ is any smooth curve, the formula

$$|f| = \int_0^1 \left(\sum_{1 \leq i \leq n} [g^i(t)]^2 + \sum_{1 \leq i < j \leq n} [g_i^j(t)]^2 \right)^{1/2} dt$$

(where $f^* \omega^i = g^i(t) dt$ and $f^* \omega_j^i = g_j^i(t) dt$) gives the length of f ; and the formula

$$|\pi \circ f| = \int_0^1 \left(\sum_i [g^i(t)]^2 \right)^{1/2} dt$$

gives the length of $\pi \circ f$. We conclude that $|\pi \circ f| \leq |f|$. On the other hand, by parallel translating a frame along the curve $\pi \circ f$ we obtain a smooth curve $h : [0, 1] \rightarrow B$ such that $\pi \circ h = \pi \circ f$ and $h^* \omega_j^i = 0$ ($1 \leq i, j \leq 1$). Then $|h| = |\pi \circ f|$, which proves that every smooth curve in M admits a lifting which has the same length.

The proof of the theorem is now immediate. If B were complete but M incomplete, there would exist a smooth function $f : [0, l) \rightarrow M$, parametrized by arc length, with $\lim_{t \rightarrow l} f(t)$ divergent. But there is a lifting $g : [0, l) \rightarrow B$, also parametrized by arc length. Then $\lim_{t \rightarrow l} g(t)$ must exist, and hence $\lim_{t \rightarrow l} f(t) = \pi(\lim_{t \rightarrow l} g(t))$. The proof of the converse is similar.

4. Two counterexamples. In proving Theorem 1, in order to show that B was a complete metric space, we made use of the fact that a certain curve in T was antidevelopable; and the curve was smooth but had a critical point. Supposing it were possible to antidevelop all the smooth *regular* curves which lie in a fibre of T and start at the origin of the fibre, one naturally wonders if the completeness of B could be proved. Here is an example to show that it could not.

Our example will be a translation connection. This means $G = F = S^n$ (the group of translations of Euclidean n -space). $G_0 = \{1\}$. We may then set $B = M$, with π the identity map. We shall let $B = C_{\mathbf{R}^2} \setminus \{0\}$, i.e., the punctured plane. In this case $\mathfrak{G} = \mathbf{R}^2$. Let $\omega = (\omega^1, \omega^2)$, where $\omega^1 = \cos(1/r) dr + r \sin(1/r) d\theta$ and $\omega^2 = -\sin(1/r) dr + r \cos(1/r) d\theta$ (r, θ being polar coordinates). Then $(\omega^1)^2 + (\omega^2)^2 = dr^2 + r^2 d\theta^2$, so that the canonical metric on B is the usual Euclidean metric on the plane. Since the origin is missing, B is an incomplete metric space; hence, by Theorem 1, the connection is incomplete.

Now let $f : [0, 1] \rightarrow T$ be a C^1 function such that df is never zero, f takes values in a single fibre, and $f(0)$ is the origin of that fibre. Now $T = B \times \mathbf{R}^2$ and p is the projection onto the first factor. Hence, for some $b \in B$, $f(t) = (b, g(t))$ ($0 \leq t \leq 1$), with $g(0) = 0$ and $g'(t)$ never zero. Hence $g'(t) = (u(t) \cos v(t), u(t) \sin v(t))$, $u, v : [0, 1] \rightarrow \mathbf{R}$ being continuous, and u positive.

Let $\epsilon \in (0, 1]$ be the largest number such that there exists a function $h : [0, \epsilon) \rightarrow B$, with $h(0) = b$ and $h^* \omega = dg$. h is the antidevelopment of $f|_{[0, \epsilon)}$, and f is antidevelopable if and only if $\epsilon = 1$ and $\lim_{t \rightarrow 1} h(t)$ exists in B . Therefore,

f will fail to be antidevelopable if and only if $\lim_{t \rightarrow \infty} h(t) = 0$. Let us suppose this last condition is true.

Now $h(t) = (r(t) \cos \theta(t), r(t) \sin \theta(t))$, $r, \theta : [0, \epsilon) \rightarrow \mathbf{R}$ being smooth and $r(t) > 0$. Then $h^*\omega = dg$ means

$$\cos [1/r(t)]r'(t) + r(t) \sin [1/r(t)]\theta'(t) = u(t) \cos v(t)$$

and

$$-\sin [1/r(t)]r'(t) + r(t) \cos [1/r(t)]\theta'(t) = u(t) \sin v(t),$$

which means

$$r'(t) = u(t) \cos v(t) + [1/r(t)].$$

There exists $t_1 \in (0, \epsilon)$ such that $|v(t) - v(\epsilon)| < \pi/2$ for all $t \in [t_1, \epsilon]$.

There exists $r_2 \in (0, r(t_1))$ such that $\cos (v(t) + 1/r_2) > 0$ ($t_1 \leq t \leq \epsilon$). Let t_2 be the smallest number in the interval $[t_1, \epsilon]$ such that $r(t_2) = r_2$. Then $r'(t_2) > 0$, which means that $r(t) < r(t_2)$ for some $t \in (t_1, t_2)$, whereas $r(t_1) > r(t_2)$. This is impossible.

We conclude that every smooth regular curve is antidevelopable, whereas some smooth curves with critical points are not. In particular, the lines are developable, which means that the connection is geodesically complete. ω can be extended to a Euclidean connection $\tilde{\omega}$ and an affine connection $\tilde{\tilde{\omega}}$. The associated bundle of an extended connection is unchanged (except for structure group), as is the rule for development of curves. Hence $\tilde{\omega}$ and $\tilde{\tilde{\omega}}$ are again examples of incomplete connections having the property that smooth regular curves can be antideveloped.

We now give an example of an affine connection which is geodesically complete, but which has a parabola incapable of being antideveloped.

Our connection will be affine, with $n = 2$. We set $M = \mathbf{R}^2$, and $B = \mathbf{R}^2 \times L^2$ (product bundle). Let $\sigma : M \rightarrow B$ be the identity cross-section ($\sigma(x) = (x, 1)$). ω will be defined on B if we give $\sigma^*\omega$ on M . We set $\sigma^*\omega_0^1 = dx^1$, $\sigma^*\omega_0^2 = dx^2$, $\sigma^*\omega_1^1 = 0$, $\sigma^*\omega_1^2 = dx^2$, $\sigma^*\omega_2^1 = -dx^2$, $\sigma^*\omega_2^2 = dx^1$. This connection is invariant under the translation group. The geodesics are the translates of the two curves $f(t) = (t, 0)$ ($t \in \mathbf{R}$), and $g(t) = (\frac{1}{2} \log \sec (t\sqrt{2}), t)$ ($-\pi/2\sqrt{2} < t < \pi/2\sqrt{2}$). Developing these curves in the fibre at $(0, 0)$, and noting that $P = M \times \mathbf{R}^2$, we get $\hat{f}(t) = (0, 0, t, 0)$ and

$$\hat{g}(t) = (0, 0, 0, (1/\sqrt{2}) \log (\sec (t\sqrt{2}) + \tan (t\sqrt{2})).$$

This means that every line through the origin in any fibre is antidevelopable, i.e., ω is geodesically complete. Now define $h : (0, \infty) \rightarrow M$ by $h(t) = (3 \log t, \sqrt{2} \log t)$. Developing h into the fibre at $(0, 0)$, we get

$$\hat{h}(t) = (0, 0, 8t - \frac{5}{2}t^2 - \frac{1}{2}, \frac{5}{2}t^2 - 4t + \frac{3}{2}).$$

$\hat{h}$ is a part of a parabola.

$$\lim_{t \rightarrow 0} \hat{h}(t) = (0, 0, -\frac{1}{2}, \frac{3}{2}).$$

But h is not extendable; hence the entire parabola is not antidevelopable.

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