Interpolation, Ergodicity, and Martingales*

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1. Introduction. A bounded linear operator T on a Lebesgue space $L^{p_1}(\Omega_1, \Sigma_1, \mu_1)$ into another such space $L^{p_2}(\Omega_2, \Sigma_2, \mu_2)$ is said to be of type (p_1, p_2) ; i.e., $T \in (L^{p_1}, L^{p_2})$. The celebrated convexity theorem of M. Riesz [19] states that if a linear operator is simultaneously of types (p_1, p_2) and (q_1, q_2) with bounds M_1 and M_2 respectively, then it is also of type (r_t, s_t) , and with bound $M_t \leq M_1^{1-t} M_2^t$, where

$$r_t^{-1} = (1-t)p_1^{-1} + tq_1^{-1}, \quad s_t^{-1} = (1-t)p_2^{-1} + tq_2^{-1}, \quad 0 \leq t \leq 1.$$

This result plays a crucial role in many parts of analysis and particularly its importance in (abstract) ergodic theory is well-known ([5], Ch. VIII). The special case of interest here is that $M_1 = M_2 = 1$ so that $M_t \leq 1$, $0 \leq t \leq 1$; thus T is a contraction on all L^p spaces if it is so on L^1 and L^∞ .

Recently the result on contractions was extended in [1] to a class of positive operators: if T is a contraction on L^1 and L^p , for some $p > 1$, then T can be extended to all L^p , $1 \leq p \leq \infty$ such that it remains a contraction. An immediate question is whether the $L^p(p > 1)$ space requirement may be replaced by a weaker condition such as the space of functions f for which $(|f| \log^+ |f|)$ is integrable. The natural setting then is to replace L^1 and L^p here by the Orlicz spaces L^{Φ_1} and L^{Φ_2} with $\Phi_1 \prec \Phi_2$ (Φ_1 weaker than Φ_2). Precise definitions are given below.) However, the convexity theorem of Riesz is no longer available here and since the latter plays a key role in the proof of [1], and in all such extensions, a fresh attack on the problem is necessary—involving a generalization of the convexity theorem itself. Looking at the many extensions and the influence of this result in analysis (see references at the end of the paper), it is clear that the above mentioned generalization will have useful implications in both the theory and applications.

The purpose of this paper is first to prove some interpolation theorems for a class of Orlicz spaces L^Φ which include the L^p theory (so that Riesz' convexity theorem is properly extended), and then, using the former, to obtain some general ergodic and martingale theorems. A brief outline of the results is as follows. After preliminaries in the next section, the interpolation theory of

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linear operators will be considered in Section 3, together with some extensions to multilinear forms and to operators on spaces with change of measures, (cf. [22] for the L^p case). Then applications to ergodic theory and related questions occupy Section 4; and the convergence of generalized martingales (first introduced in [20]) is considered in Section 5. Finally, some discussion on problems of interest from the present point of view is included.

It should be noted, at this point, that there exist several extensions, of the Riesz convexity theorem, in the literature (cf. [25]). However, in all these cases the L^p -space structure was retained (with certain extensions to the Hardy spaces H^p) and the manipulation of the conjugate exponents (p, p') plays a critical role here. Even in the abstract treatments this influence is apparent. Such an advantage is not available for the L^Φ -spaces, and many of the interesting methods, used in the L^p case, are not always applicable. Moreover, the L^p space, for $1 < p < \infty$, is reflexive and the corresponding case for L^Φ is that Φ is strictly convex; but L^Φ need *not* be reflexive so that most extensions to L^Φ involve real difficulties. In what follows, therefore, a subclass of L^Φ -spaces, recently isolated by Calderón [4], will be considered and, thanks to the elegant complex method (cf. [5], [25]), an extension of the convexity theorem (with a *proper choice of norms* in the L^Φ -spaces) will be obtained with precise bounds. The theory of intermediate spaces ([4], [11]) will be clarified further in the treatment of L^Φ -spaces which, in turn, may lead to further developments in abstract theory [3]. Hereafter, operator means linear operator. Some of the following work can be extended to sublinear operators.

2. Preliminaries. Let (Φ, Ψ) be complementary Young's functions, *i.e.*, symmetric (non-trivial) convex functions on the line, vanishing at the origin and satisfying the Young's inequality ([10], pp. 12–13),

$$(2.1) \quad xy \leq \Phi(x) + \Psi(y), \quad x, y \geq 0; \quad \Psi(y) = \sup_{x \geq 0} \{x|y| - \Phi(x)\}.$$

Then $L^\Phi(\Omega, \Sigma, \mu)$, or L^Φ for short, denotes the Orlicz space of (equivalence classes of) measurable scalar functions on a measure space (Ω, Σ, μ) , such that $f \in L^\Phi$ iff (= if and only if) $N_\Phi(f) < \infty$ where

$$(2.2) \quad N_\Phi(f) = \inf \left\{ k > 0: \int_\Omega \Phi\left(\frac{|f|}{k}\right) d\mu \leq 1 \right\}.$$

[Here and below, $f \in L^\Phi$ means f is any member of its equivalence class.] L^Ψ and $N_\Psi(\cdot)$ are defined similarly. It is known that L^Φ and L^Ψ are Banach (or B-) spaces (cf. [23]) with $N_\Phi(\cdot)$ and $N_\Psi(\cdot)$ as norms. There is another norm in these spaces:

$$(2.3) \quad \|f\|_\Phi = \sup \left\{ \int_\Omega |fg| d\mu: N_\Psi(g) \leq 1 \right\}.$$

Hereafter any measure space, (Ω, Σ, μ) appearing, will be assumed, for convenience, to have the finite subset property. This means, roughly, that every

set of positive μ -measure contains a subset of positive finite μ -measure (cf. [16], Sec. 2, for a discussion on this point). Then the norms (2.2) and (2.3) are equivalent in that (cf. [25], p. 174)

$$(2.4) \quad N_{\Phi}(f) \leq \|f\|_{\Phi} \leq 2N_{\Phi}(f), \quad f \in L^{\Phi}.$$

Now some properties of Young's functions will be stated as they are needed below. If Φ_1, Φ_2 are two Young's functions then (i) $\Phi_1 \prec \Phi_2$ [or $\Phi_2 \succ \Phi_1$] iff there exists a $k > 0$ such that

$$\overline{\lim_{x \rightarrow \infty}} [\Phi_1(kx)/\Phi_2(x)] < \infty,$$

and (ii) $\Phi_1 < \Phi_2$, or $\Phi_2 > \Phi_1$, iff

$$\lim_{x \rightarrow \infty} [\Phi_1(kx)/\Phi_2(x)] = 0, \quad \text{for all } k > 0.$$

These are clearly partial orderings among all Young's functions. [If $\Phi_i(x) = |x|^p$, $i = 1, 2$, then the familiar orderings for L^p -spaces are obtained.] The relations between $\prec$ and $<$ are clarified from the following:

Lemma 1. Let Φ_1, Φ_2 be two continuous Young's functions and L^{Φ_i} , $i = 1, 2$, be the corresponding Orlicz spaces on a finite measure space (Ω, Σ, μ) . Then (a) $L^{\Phi_1} \supset L^{\Phi_2}$ iff $\Phi_1 \prec \Phi_2$; (b) $\Phi_1 < \Phi_2$ iff there exists a Young's function Φ such that $\Phi(\Phi_1(\cdot)) \prec \Phi_2(\cdot)$.

The proof of the "if" part is immediate, and the "only if" is proved in ([10], p. 112 ff.). (b) is a consequence of a result of T. Andô. If the measure space is arbitrary, then Lemma 1(a) holds only if the ordering is strengthened. In particular, the statement holds if $\Phi_1(x) \leq \Phi_2(kx)$ for all $x \geq 0$.

Definition 1. If Φ is a Young's function, let $\mathfrak{M}^{\Phi}$ be the closed subspace of L^{Φ} determined by the μ -step functions of L^{Φ} . If Φ is continuous and $L^{\Phi} = \mathfrak{M}^{\Phi}$, Φ is said to satisfy the *growth condition*. If $\Phi(2x) \leq c\Phi(x)$ for $0 < c < \infty$, and $x \geq 0$, then Φ is said to satisfy the Δ_2 -condition.

It follows that $L^{\Phi} = \mathfrak{M}^{\Phi}$ whenever Φ satisfies a Δ_2 -condition and if the measure space is finite then it is sufficient to assume for Δ_2 , $\Phi(2x) \leq c\Phi(x)$, for $x \geq x_0$, for some $x_0 > 0$, (cf. [10], [24]). Suppose Φ_1, Φ, Φ_2 are Young's functions such that $\Phi_1 \prec \Phi \prec \Phi_2$. Unlike in the L^p -spaces, it is not necessarily true that Φ satisfies a growth condition (say Δ_2) whenever Φ_1 and Φ_2 satisfy one. A counterexample is obtained from an example given in ([10], pp. 28–29). Thus the $\mathfrak{M}^{\Phi}$ spaces, rather than the L^{Φ} , are the proper generalizations of the L^p -spaces. [These are the M^{Φ} spaces, for continuous Φ , of [15], and are discussed in some detail in [15_{II}].] For the interpolation theory certain subclasses of Φ 's, admitting detailed analysis, will be considered.

Definition 2. A Young's function Φ is said to satisfy the Δ' -condition if $\Phi(xy) \leq c\Phi(x)\Phi(y)$ for all $x, y \geq 0$ and some $0 < c < \infty$. If Φ_1, Φ_2 are two

Young's functions let Φ_s be the inverse of $\Phi_s^{-1} = (\Phi_1^{-1})^{1-s}(\Phi_2^{-1})^s$ for $0 \leq s \leq 1$ where Φ^{-1} is the (unique) inverse of the Young's function Φ .

It is immediate from definition that a Φ satisfying the Δ' -condition automatically satisfies the Δ_2 -condition. If the measure space is finite, it suffices to assume for L^Φ , the Δ' as $\Phi(xy) \leq c\Phi(x)\Phi(y)$ for $x, y \geq x_0$ for some $x_0 > 0$. Since Φ is convex iff Φ^{-1} is concave, it can be seen that for $0 \leq s \leq 1$, Φ_s is also a convex function with properties similar to those of Φ , (cf. [4], p. 165). This class of Young's functions was isolated in [4], and that is appropriate here. Some properties of Φ_s are given in the following:

Lemma 2. *Let Φ_1, Φ_2 be two Young's functions with Ψ_1, Ψ_2 as their complementary functions, in the sense of Young. Suppose for each $0 \leq s \leq 1$, Φ_s and Ψ_s are the convex functions defined from the Φ_i and the Ψ_i , $i = 1, 2$ respectively (as in Definition 2). Then for each s , the pair (Φ_s, Ψ_s) satisfies the Young's inequality (cf. (2.1)). Moreover, if Φ_1, Φ_2 are ordered (by $\prec$ or $\succ$) then Φ_s lies between Φ_1 and Φ_2 in the same sense, and if Φ_1, Φ_2 both satisfy a Δ' or Δ_2 -condition (a growth condition) so does Φ_s for each $0 \leq s \leq 1$.*

Proof. It is to be shown that (Φ_s, Ψ_s) satisfies the inequality (2.1) for each $0 < s < 1$. Since these are convex functions Φ_s^{-1} and Ψ_s^{-1} are concave and uniquely defined, so that (2.1) follows if the corresponding inequality is established for these concave functions; i.e., for all $t, u \geq 0$, $\Phi_s^{-1}(t)\Psi_s^{-1}(u) \leq t + u$, (cf. [10], p. 8 and 13, and [25_I], p. 25).

So if $t_1, t_2 \geq 0$, are arbitrary, then

$$\begin{aligned}\Phi_s^{-1}(t_1) \cdot \Psi_s^{-1}(t_2) &= \{[\Phi_1^{-1}(t_1)]^{1-s}[\Phi_2^{-1}(t_1)]^s\} \cdot \{[\Psi_1^{-1}(t_2)]^{1-s}[\Psi_2^{-1}(t_2)]^s\} \\ &= \{[\Phi_1^{-1}(t_1)\Psi_1^{-1}(t_2)]^{1-s}[\Phi_2^{-1}(t_1)\Psi_2^{-1}(t_2)]^s\} \\ &\leq (t_1 + t_2)^{1-s} \cdot (t_1 + t_2)^s = t_1 + t_2,\end{aligned}$$

where the fact that (Φ_i, Ψ_i) , $i = 1, 2$ are complementary is used. Since t_1, t_2 are arbitrary it follows that, converting the above inequality to Φ_s, Ψ_s , the inequality of (2.1) holds. The remaining statements are similarly proved. For instance Φ satisfies a Δ' condition iff $\Phi^{-1}(cuw) \geq \Phi^{-1}(u)\Phi^{-1}(v)$, for all $u, v \geq 0$ and that for Δ_2 one must have $2\Phi^{-1}(y) \leq \Phi^{-1}(cy)$, $y \geq 0$, $c > 0$. Thus the last statement of the Lemma is proved from the definition of Φ_s^{-1} using a simple computation as above. The ordering relations are immediate.

The following statement on $\mathfrak{M}^\Phi$ is also useful later on, and it was proved in [15_{II}]; it can also be deduced from ([15], Lemma 2).

Lemma 3. *Let Φ be a Young's function and $L^\Phi(\Omega, \Sigma, \mu)$ be the Orlicz space. Let*

$$M^\Phi = \left\{ f: \int_\Omega \Phi(k|f|) d\mu < \infty, \text{ all } k > 0 \right\}.$$

Then $M^\Phi = \mathfrak{M}^\Phi$ if Φ is continuous; and $\mathfrak{M}^\Phi = L^\Phi$ if Φ is discontinuous and $\mu(\Omega) < \infty$ (in which case $M^\Phi = \{\theta\}$). In any case $M^\Phi \subset \mathfrak{M}^\Phi \subset L^\Phi$ holds.

Remark. In Lemma 2, it is *not* claimed that Φ_* and Ψ_* are complementary to each other in the sense of Young, *i.e.*, it is true that the Young's inequality of (2.1) holds, but the second part of (2.1) is not claimed to hold. So if Ψ_*^+ is the Young's complementary function to Φ_* , then, by Lemma 2, it follows that $\Psi_*^+(x) \leq \Psi_*(x)$, for all $x \geq 0$. Hence $N_{\Psi_*^+}(f) \leq N_{\Psi_*}(f)$. This observation will be needed later.

In spite of the above remark, Ψ_*^+ and Ψ_* are closely related as seen from the following:

Lemma 4. *In the notations introduced above, the Orlicz spaces L^{Ψ_*} and $L^{\Psi_*^+}$ contain the same elements and have equivalent norms, *i.e.*,*

$$(2.5) \quad N_{\Psi_*^+}(f) \leq N_{\Psi_*}(f) \leq K_s N_{\Psi_*^+}(f), \quad f \in L^{\Psi_*} = L^{\Psi_*^+},$$

and $1 \leq K_s \leq K < \infty$; $K_s = 1$ if $s = 0, 1$.

Proof. It was shown above that

$$N_{\Psi_*^+}(f) \leq N_{\Psi_*}(f),$$

so that $L^{\Psi_*} \subset L^{\Psi_*^+}$. To establish the opposite inclusion, let $f \in L^{\Psi_*^+}$. For non-triviality, it may be assumed that $|f| > \delta$ for some $\delta > 0$ (since a bounded function in $L^{\Psi_*^+}$ belongs to L^{Ψ_*}), and $0 < s < 1$. It may also be assumed $\|f\|_{\Psi_*^+} \leq 1$. By definition, Φ_* is a complementary Young's function to Ψ_*^+ , and if $\Psi_*^{+'}$ is the derivative of Ψ_*^+ , which exists a.e. and which may be taken to exist everywhere for this proof, one has $\Psi_*^{+'}(|f|) \in L^{\Phi_*}$, $N_{\Phi_*}[\Psi_*^{+'}(|f|)] \leq 1$, by ([10], p. 73). [Though the result of [10] was stated only for a finite non-atomic measure, the proof shows that the result holds also in the general case.] But then, by ([4], p. 122), there exists an $h \in L^1$, $\|h\|_1 \leq 1$ such that

$$(2.6) \quad \Psi_*^{+'}(|f|) \leq \Phi_*^{-1}(|h|).$$

In fact this is just an alternate form of the definition of the norm (2.2). However, $\Psi_*^{+'}$ and Φ_*' , the derivatives, are inverse functions to each other by the definition of Young's complementary functions (*cf.* [10], p. 11 or [25_I], p. 25). So (2.6) is equivalent to

$$(2.7) \quad |f| \leq \Phi_*'(\Phi_*^{-1}(|h|)).$$

It will now be shown that the inequality (2.7) implies $f \in L^{\Psi_*}$. Let $\alpha_s(\cdot) = \Phi_*^{-1}(\cdot)$. From the identity $\Phi_*(\alpha_s(x)) = x$, one obtains on (implicit) differentiation, after an elementary but somewhat tedious computation, the following result for $x > 0$.

$$(2.8) \quad \Phi_*'(\alpha_s(x)) = P(x) \cdot [\Phi_1'(\alpha_1(x))]^{1-s} [\Phi_2'(\alpha_2(x))]^s, \quad 0 < s < 1,$$

where

$$P(x) = \left\{ (1-s) \left[\frac{\alpha_2 \cdot \Phi_2'(\alpha_2)}{\alpha_1 \cdot \Phi_1'(\alpha_1)} \right](x) + s \left[\frac{\alpha_1 \cdot \Phi_1'(\alpha_1)}{\alpha_2 \cdot \Phi_2'(\alpha_2)} \right](x) \right\}^{-1}.$$

Since the minimum of the quantity in $\{ \}$ is seen to be one for $x > 0$, it follows that $0 \leq P(x) \leq 1$ for all $x > 0$. So (2.7) and (2.8) yield

$$(2.9) \quad |f| \leq [\Phi_1(\alpha_1(|h|))]^{1-s} [\Phi_2'(\alpha_2(|h|))]^s, \quad 0 < s < 1.$$

However $\alpha_i(|h|) (\in L^{\Phi_i})$ is of $\|\cdot\|_{\Phi_i}$ norm at most one, so that by the above quoted result of [10], $\Phi_i'(\alpha_i(|h|)) \in L^{\Psi_i}$, $i = 1, 2$, and is of $N_{\Psi_i}(\cdot)$ norm at most one. Thus (2.9) implies

$$(2.10) \quad |f| \leq (|g_1|)^{1-s} (|g_2|)^s, \quad 0 < s < 1,$$

where $N_{\Psi_i}(g_i) \leq 1$. This means, by ([4], p. 123), $f \in L^{\Psi_s}$ (and has $N_{\Psi_s}(\cdot)$ -norm at most one). Thus $L^{\Psi_s+} = L^{\Psi_s}$.

Finally, since L^{Ψ_s+} and L^{Ψ_s} are B-spaces and $N_{\Psi_s+}(f) \leq N_{\Psi_s}(f)$, it follows, by a form of the Closed Graph Theorem (cf. [6], Theorem 2.12.2), that there exists a $K_s \geq 1$ such that (2.5) holds. Since, as s varies in the unit interval Ψ_s varies from Ψ_1 to Ψ_2 continuously, K_s is a continuous function of s on $[0, 1]$ and $K_s = 1$ for $s = 0, 1$ (recall that $N_{\Psi_s+} = N_{\Psi_s}$ then). Actually the various estimates on the norms above shows, if $\|f\|_{\Psi_s+} = 1$ then $N_{\Psi_s}(f) \leq 1 = \|f\|_{\Psi_s+} \leq 2N_{\Psi_s+}(f)$, by (2.4), so that $1 \leq K_s \leq 2$ in (2.5). This completes the proof of the lemma.

The following special case of the lemma can be deduced from some results of [3], and because of its independent interest a sketch of this alternate proof will now be given.

Corollary. *Let Ψ_s, Ψ_s^+ be as above and the complementary functions $\Phi_i, \text{ of } \Psi_i, \text{ be such that } L^{\Phi_i} = \mathfrak{M}^{\Phi_i}, i = 1, 2$. Then (2.5) holds.*

Sketch of proof. The additional assumption implies that $L^{\Phi_1} \cap L^{\Phi_2}$ is dense in both L^{Φ_1} and L^{Φ_2} , and by construction (cf. [4]) L^{Φ_s} is an interpolation space (in the sense of [3], p. 72) between L^{Φ_1} and L^{Φ_2} . Moreover $[L^{\Phi_1}, L^{\Phi_2}]$ and $[L^{\Psi_1}, L^{\Psi_2}]$ are compatible couples, [3]. So by ([3], p. 81) $(L^{\Phi_s})^*$ or L^{Ψ_s+} is an intermediate space between L^{Ψ_1} and L^{Ψ_2} . But $L^{\Psi_s} (\subset L^{\Psi_s+})$ is, by construction an interpolation space and $L^{\Psi_1} \cap L^{\Psi_2} \subset L^{\Psi_s}$. It can be checked that L^{Ψ_s} satisfies the conditions of ([3], Theorem 7.IV, 4° and Theorem 7.VII) so that one concludes $L^{\Psi_s} = L^{\Psi_s+}$. The inequalities (2.5) follows from an application of the Closed Graph Theorem as before.

Notation. The following notation will be used hereafter without comment. If $\Phi_i, i = 1, 2, \Phi_s$ are the Young's functions, of Definition 2, then Ψ_s^+ stands for the Young's complementary function of Φ_s and Ψ_s is the function given by Definition 2 for the complementary functions Ψ_i of $\Phi_i, i = 1, 2$. This "plus" notation will be used in this sense. Since $L^{\Psi_s} (= L^{\Psi_s+})$ has two (equivalent) norms, one has for L^{Φ_s} another norm, in addition to (2.3), given by

$$(2.3)' \quad \|f\|_{\Phi_s} = \sup \left\{ \int_{\Omega} |fg| \, d\mu : N_{\Psi_s}(g) \leq 1 \right\}.$$

Thus if N_{Ψ} is replaced by $N_{\Psi,+}$ in (2.3)' it reduces to (2.3), and for $s = 0, 1$, both are always identical. The equivalence of the norms (2.3) and (2.3)' is an easy computation, or [clearly $\|\cdot\|'_{\Phi} \leq \|\cdot\|_{\Phi}$] is again a consequence of the Closed Graph Theorem. If $T : L^{\Phi\bullet} \rightarrow L^{\Phi\bullet}$, is a bounded operator (in any one, and hence all, of the norms) let $\|T\|'_{\Phi}$ and $\|T\|_{\Phi}$ be:

(2.11)

$$\|T\|'_{\Phi} = \sup \{ \|Tf\|'_{\Phi} : \|f\|_{\Phi} \leq 1 \}; \quad \|T\|_{\Phi} = \sup \{ \|Tf\|_{\Phi} : \|f\|_{\Phi} \leq 1 \}.$$

Clearly $\|T\|'_{\Phi} \leq \|T\|_{\Phi}$. Theorem 1 below shows that if T is a contraction on L^{Φ_1} and L^{Φ_2} , in either norm, then it is a contraction on L^{Φ} in $\|T\|'_{\Phi}$ norm. In Sections 4 and 5 the operator norm is $\|T\|'_{\Phi}$ and, for convenience, the prime will be dropped. Hereafter if Φ, Ψ are continuous, it will be assumed, for convenience, that Φ', Ψ' are continuous (cf. [16] Sec. 2).

3. Interpolation theory. In this section first a generalization of the interpolation theorem of M. Riesz to the (complex) L^{Φ} -spaces will be obtained. With this certain extensions to multilinear forms and to spaces with change of measures will be considered later. Hereafter, exceptional null sets will not be stated everytime when no confusion results.

Theorem 1. Let (Ω, Σ, μ) and $(S, \mathcal{S}, \nu)$ be measure spaces and $\Phi_i, Q_i, i = 1, 2$ be two pairs of Young's functions whose complementary functions are denoted by $\Psi_i, R_i, i = 1, 2$ respectively. Suppose that $L^{\Phi_1}(\mu), L^{\Psi_1}(\mu)$ and $L^{Q_1}(\nu), L^{R_1}(\nu)$ are the Orlicz spaces on the first and second measure spaces respectively, and assume that Φ_i satisfies a growth condition: $L^{\Phi_i}(\mu) = \mathfrak{M}^{\Phi_i}, i = 1, 2$. Let Φ_s and $Q_s, 0 \leq s \leq 1$, be the Young's functions of Definition 2, and suppose that the spaces L^{Φ_s} and L^{Q_s} are endowed with norm (2.3)[(2.3)'] and the complementary spaces L^{Ψ_s} and L^{R_s} are given the norm (2.2). If the operator $T : L^{\Phi_i}(\mu) \rightarrow L^{Q_i}(\nu), i = 1, 2$ (i.e., T is simultaneously defined on both $L^{\Phi_i}(\mu)$ to $L^{Q_i}(\nu)$) with bounds M_1 and M_2 , then T is also defined on $L^{\Phi_s}(\mu)$ to $L^{Q_s}(\nu)$ for each $0 \leq s \leq 1$, i.e., $T : L^{\Phi_s}(\mu) \rightarrow L^{Q_s}(\nu)$ and moreover,

$$(3.1) \quad \|Tf\|_{Q_s,+} \leq M_1^{1-s} M_2^s \|f\|_{\Phi_s},$$

and

$$0 \leq s \leq 1, \quad f \in L^{\Phi_s}(\mu).$$

$$(3.1)' \quad \|Tf\|'_{Q_s} \leq M_1^{1-s} M_2^s \|f\|_{\Phi_s}.$$

Remark. It is of interest to note that the norms in the original spaces and their complementary ones are (2.3), (2.3)', and (2.2) respectively. If this is not so, the constants in (3.1), (3.1)' will be different and the more precise bound need not hold. This will be clear from the proof. The general plan of the proof is similar to that of [25] but with some essential differences; and because of the importance of the result complete details are presented below, keeping the argument as close to ([25]_{II}, p. 95) as possible.

Proof. The boundary cases, that one of Φ_i , Ψ_i or of Q_i , R_i , $i = 1, 2$ is discontinuous, are essentially L^1 or L^∞ and these cases follow as in the classical situation once the theorem is proved when all these convex functions are continuous. Consequently only the latter case will be proved.

Let $\alpha(t) = \Phi^{-1}(t)$, so that $\alpha_s(t) = [\alpha_1(t)]^{1-s}[\alpha_2(t)]^s$, $0 \leq s \leq 1$ and $\alpha_s(\cdot)$ is a concave function satisfying $\alpha_s(0) = 0$. [Similarly let $\delta_s(\cdot)$ be the concave function inverse to Q_s .] If $z = x + iy$ is a complex number, consider $\alpha_s(t) = [\alpha_1(t)]^{1-z}[\alpha_2(t)]^z$, in the strip $0 \leq x \leq 1$, of the complex plane. Similarly let $\beta_s(\cdot)$ be the function associated with $R_s(\cdot)$ and let $R_s^+(\cdot)$ be the complementary function to Q_s . Let $\mathfrak{L}$ be the set of (complex) integrable step functions on (Ω, Σ, μ) , so that by hypothesis, and Lemma 2, the set $\mathfrak{L}$ is a dense subset of $L^{\Phi_s}(\mu)$, $0 \leq s \leq 1$. Since a bounded (linear) operator is uniquely determined if it is defined on a dense subset, it suffices to consider the domain of T as $\mathfrak{L}$ and prove the theorem for all $f \in \mathfrak{L}$, in the respective spaces, for $0 < s < 1$.

By hypothesis, for $f \in \mathfrak{L}(\subset L^{\Phi_s})$, one has

$$(3.2) \quad \|Tf\|_{Q_s} \leq M_s \|f\|_{\Phi_s}, \quad i = 1, 2.$$

For $f \in \mathfrak{L}$, Tf is well-defined, so that, for a fixed s ,

$$(3.3) \quad \|Tf\|_{Q_{s+}} = \sup \left\{ \left| \int_s (Tf) \cdot g \, d\nu \right| : N_{R_s}(g) \leq 1, g \text{ a step function} \right\}.$$

This follows, for instance, from ([14], Lemma 1 which clearly holds for arbitrary measures and complex functions though it was stated for the nonatomic σ -finite case). In what follows, it is no restriction to assume $\|f\|_{\Phi_s} = 1$. Since Φ_s is continuous and f is a step function it follows, as in ([10], p. 88), with $2 \geq k^* \geq 1$, that $k^* - 1 = \int_\Omega \Phi_s(k^* f) \, d\mu$. (Really $k^* = k_s^*$ is a continuous function of $0 \leq s \leq 1$. Fix this f and a g satisfying (3.3) and consider the integral

$$(3.4) \quad I = \int_s (Tf) \cdot g \, d\nu.$$

This integral can be extended to the strip $0 \leq x \leq 1$ as follows. Writing $f(\cdot) = |f|(\cdot)e^{i\theta(\cdot)}$ and $g(\cdot) = |g|(\cdot)e^{i\theta'(\cdot)}$, define the step functions F_s and G_s depending on the complex parameter z as:

$$(3.5) \quad F_s = \frac{1}{k^*} \alpha_s(\Phi_s(k^* |f|)) \cdot e^{i\theta}; \quad G_s = \beta_s(R_s(|g|)) \cdot e^{i\theta'},$$

where α_s , β_s are defined at the beginning of the proof, and where $\alpha_s(0) = \beta_s(0) = 0$. Since the functions of (3.5) are step functions the following extension of (3.4) is defined,

$$(3.6) \quad I(z) = \int_s (TF_s) \cdot G_s \, d\nu.$$

The facts that $\alpha_s(\Phi_s)$ and $\beta_s(R_s)$ are identity functions imply $|I| = |I(s)|$. It only remains to show that $I(z)$ is continuous and bounded on $0 \leq x \leq 1$

and regular in its interior so that (3.1) can be obtained as a consequence of the Three-Line Theorem (cf. [5], [25_{II}]).

To simplify $I(z)$, let

$$f = \sum_{i=1}^k a_i \chi_{E_i}, \quad g = \sum_{i=1}^k b_i \chi_{F_i}$$

be a typical representation, and set $a_i = |a_i| e^{i\theta_i}$, $b_i = |b_i| e^{i\theta'_i}$. Substituting these in (3.5) and then in (3.6) one obtains

$$I(z) = \frac{1}{k^*} \sum \exp [i(\theta_i + \theta'_i)] \alpha_z(\Phi_s(k^* |a_i|) \beta_z(R_s(|b_i|)) \cdot \int_S (T\chi_{E_i}) \cdot \chi_{F_i} d\nu,$$

which is linear in exponentials of the form d^z with $d > 0$. In fact a typical d^z is:

$$\alpha_1(\Phi_s(k^* |a_i|)^{1-z} \alpha_2(\Phi_s(k^* |a_i|))^z \cdot \beta_1(R_s(|b_i|))^{1-z} \beta_2(R_s(|b_i|))^z.$$

Thus $I(z)$ has all the stated requirements. So it remains to find the bounds of $I(z)$ on the boundary of the strip, i.e., for $x = 0$ and $x = 1$. Note that, if $x = 0$, $TF_z \in L^{Q_1}(\nu)$ and $G_z \in L^{R_1}(\nu)$, so that, in that case, by the Hölder inequality ([15], p. 80), (3.6) gives with (3.2)

$$(3.7) \quad |I(z)| \leq \|TF_z\|_{Q_1} \cdot N_{R_1}(G_z) \leq M_1 \|F_z\|_{\Phi_1} \cdot N_{R_1}(G_z).$$

The right side can be evaluated as

$$(3.8) \quad \int_{\Omega} \Phi_1(k^* |F_z|) d\mu = \int_{\Omega} \Phi_1(|\alpha_v(\Phi_s(k^* |f|))|) d\mu = \int_{\Omega} \Phi_s(k^* |f|) d\mu = k^* - 1.$$

Since k_s^* is continuous, it follows *via* Fatou's Lemma, that:

$$\int_{\Omega} \Phi_1(k_0^* |F_z|) d\mu \leq k_0^* - 1, \quad \text{and} \quad \|F_z\|_{\Phi_1} = \frac{1}{k_0^*} \left[1 + \int_{\Omega} \Phi_1(k_0^* |F_z|) d\mu \right] \leq 1,$$

where $k_0^* = \lim_{s \rightarrow 0} k_s^*$, the value corresponding to the $\|\cdot\|_{\Phi_1}$ - norm. Thus $\|F_z\|_{\Phi_1} \leq 1$. Clearly,

$$(3.9) \quad \int_S R_1(|G_z|) d\nu = \int_S R_s(|g|) d\nu \leq 1,$$

so that $N_{R_1}(G_z) \leq 1$, and $|I(z)| \leq M_1$. A similar computation for $x = 1$ shows that $|I(z)| \leq M_2$.

It now follows, from the Three-Line Theorem, that for all $f \in \mathfrak{L}$, and hence for all $f \in L^{\Phi}$,

$$|I| = |I(s)| \leq M_1^{1-s} M_2^s \|f\|_{\Phi}.$$

This is equivalent to (3.1). Using norm (2.3)' in (3.3), an identical argument proves (3.1)' because of Lemma 4. This completes the proof of the theorem.

It is noted that the equality in (3.1) can be discussed by considering the conditions for equality, in Hölder inequality (3.7), in the usual manner, but the corresponding discussion for (3.1)' is more complicated. The above proof

implies the following apparently more general result, since in (2.3) g 's may be simple ([10], p. 87).

Theorem 2. Let Φ_i , Q_i , $i = 1, 2$ be arbitrary Young's functions and let Φ_* , Q_* be the functions, for $0 \leq s \leq 1$, given by Definition 2. If $L^{\Phi_i}(\mu)$, $L^{Q_i}(\nu)$ are the Orlicz spaces and if the operator $T : \mathfrak{M}^{\Phi_i}(\mu) \rightarrow L^{Q_i}(\nu)$ $i = 1, 2$, is simultaneously defined on both spaces with bounds M_1 , M_2 , then $T : \mathfrak{M}^{\Phi_*} \rightarrow L^{\Phi_*}$ is also defined, for each $0 \leq s \leq 1$, and, with (2.3) or (2.3)' as norm for L^{Φ} and L^Q , and (2.2) for L^{Ψ} and L^R , one has

$$(3.10) \quad \|Tf\|_{Q_*} \leq M_1^{1-s} M_2^s \|f\|_{\Phi_*}, \quad f \in \mathfrak{M}^{\Phi_*}, \quad 0 \leq s \leq 1.$$

$$(3.10)' \quad \|Tf\|_{Q_*} \leq M_1^{1-s} M_2^s \|f\|_{\Phi_*},$$

Remark. If the same $N_{\Phi}(\cdot)$ (and $N_Q(\cdot)$)-type norm is used in both the $L^{\Phi}(L^Q)$ and the $L^{\Psi}(L^R)$, then instead of (3.1) [or (3.10)] one has $N_{Q_*}(Tf) \leq KM_1^{1-s} M_2^s N_{\Phi_*}(f)$, $f \in L^{\Phi_*}$. This is seen from the work of ([3] and [4]). The above result is comparable with ([4], p. 115, Theorem 4), but the norms used in the latter are different. Since the precise bounds are needed in the following work, it becomes important to use the norms that are closer to the standard ones, as in Theorems 1 and 2. Also if

$$\Phi_i(x) = |x|^{p_i}, \quad i = 1, 2 (p_1 \leq p_2) \quad \text{and} \quad Q_i(x) = |x|^{q_i}, \quad q_1 \leq q_2$$

the above result reduces to Riesz' theorem, [19]. (In this case, the same $N_{\Phi}(\cdot)$ -type norm can be used in all the spaces.)

According to ([9], p. 221) the L^{Φ_*} , $0 \leq s \leq 1$, of Theorem 1 is an intermediate space (espaces de traces of [11]). This is not necessarily so if L^{Φ} is considered merely for $\Phi_1 \prec \Phi \prec \Phi_2$ even on finite measure spaces. In this connection, the following result [in the L^{Φ} -terminology] of Lions ([11], p. 154) is striking and is stated for comparison in the present context. (See also [9], p. 223, and the statement below follows the latter.)

Theorem 3. (Lions) If Φ_1 , Φ_2 are Young's functions such that the corresponding Orlicz spaces satisfy $L^{\Phi_i} = \mathfrak{M}^{\Phi_i}$, $i = 1, 2$, and if the operator $T : L^{\Phi_i} \rightarrow L^{\Phi_i}$, $i = 1, 2$ is simultaneously defined and bounded by M_1 and M_2 , then for each $0 < s < 1$ there exists a Banach (function) space $\mathfrak{X}_s$ with step functions dense and a norm defined in such a way that $\mathfrak{X}_s$ is an intermediate space between L^{Φ_1} and L^{Φ_2} ; thus

$$(3.11) \quad \|Tf\|_{\mathfrak{X}_s} \leq M_1^{1-s} M_2^s \|f\|_{\mathfrak{X}_s}, \quad 0 < s < 1, \quad f \in \mathfrak{X}_s.$$

In his proof of this result, Lions [11] has constructed $\mathfrak{X}_s$ as an isomorphic image of a certain quotient space, and, in the present context, it is not clear whether $\mathfrak{X}_s$ is also an Orlicz space (except as in the special case of Theorem 1 where $\mathfrak{X}_s$ was L^{Φ_*} , obtained directly). It will be of interest to characterize all the intermediate Orlicz spaces of a given pair, and this will lead to an extension of some of Lions' work [11]. It should also be pointed out that Calderón ([4], p.

124) has shown that an (arbitrary) L^{Φ} , $0 \leq s \leq 1$, and an intermediate B-space $\mathfrak{X}$ of functions f on Ω are equivalent (*i.e.*, have equivalent norms); the norm in $\mathfrak{X}$ is defined as: $f \in \mathfrak{X}$ iff $|f(x)| \leq \lambda |g(x)|^{1-s} |h(x)|^s$, for some $\lambda > 0$, and some $\|g\|_{\Phi_1} \leq 1$, $\|h\|_{\Phi_2} \leq 1$. The norm of f is the infimum of all $\lambda > 0$ satisfying the above inequality. [This corresponds to the norm (2.2).] However, the more precise inequality (3.1) (or (3.10)) is not implied by this result. But the recent abstract study ([3], p. 114) indicates that there exists a different (perhaps unconventional) norm in the space relative to which the method can be "normalized", *i.e.*, inequalities of the type (3.1) are possible.

3.1 An extension. Here an extension of the preceding results to multilinear forms will be stated, as it is of interest in the interpolation theory of H^{Φ} spaces, introduced in ([23], p. 46), which are natural generalizations of the Hardy spaces H^p .

Recalling that a map $T: \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_n \rightarrow \mathfrak{Y}$, where $\mathfrak{X}_i$ and $\mathfrak{Y}$ are (normed) linear spaces, is called a multilinear operator if $T(x_1, \dots, x_n)$ is linear in each $x_i \in \mathfrak{X}_i$, the following extension of Theorem 1 can be stated. (The norm on the product space is taken as the sum of the norms in the component spaces.)

Theorem 4. Let $(\Omega_0, \Sigma_0, \mu_0)$ and $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, \dots, n$ be measure spaces and Φ_i^j, Q^j , $i = 1, \dots, n$, $j = 1, 2$ be Young's functions. Suppose $L^{\Phi_i^j}(\mu_i) = \mathfrak{M}^{\Phi_i^j}(\mu_i)$, $i = 1, \dots, n$, $j = 1, 2$. Let T be a multilinear operator

$$(3.12) \quad T: L^{\Phi_1^j}(\mu_1) \times \cdots \times L^{\Phi_n^j}(\mu_n) \rightarrow L^{Q^j}(\mu_0), \quad j = 1, 2,$$

simultaneously on the spaces such that

$$(3.13) \quad \|T(f_1, \dots, f_n)\|_{Q^j} \leq M_j \prod_{i=1}^n \|f_i\|_{\Phi_i^j}, \quad j = 1, 2, \quad f_i \in L^{\Phi_i^1} \cap L^{\Phi_i^2}.$$

If Φ_i^s and Q^s , $0 \leq s \leq 1$, are defined from (Φ_i^1, Φ_i^2) and (Q^1, Q^2) , as in Definition 2, then T is also defined on $L^{\Phi_1^s}(\mu_1) \times \cdots \times L^{\Phi_n^s}(\mu_n) \rightarrow L^{Q^s}(\mu_0)$ for each s , and if the $L^{\Phi_i^j}$ and L^{Q^j} are given the norms (2.3) and the complementary spaces $L^{\Psi_i^j}$ and L^{R^j} are given the norms (2.2), then one has

$$(3.14) \quad \|T(f_1, \dots, f_n)\|_{Q^s} \leq M_1^{1-s} M_2^s \prod_{i=1}^n \|f_i\|_{\Phi_i^s}, \quad f_i \in L^{\Phi_i^s}(\mu_i).$$

The proof proceeds along the same lines as that of Theorem 1, once it is noted that the step functions are dense in $L^{\Phi_i^j}(\mu_i) \times \cdots \times L^{\Phi_n^j}(\mu_n)$, $j = 1, 2$, since they are dense in each of the component spaces. Since the argument is similar to ([25_{II}], p. 106), and the modifications needed are straight forward, the details (which are lengthy) will be omitted.

Another interesting extension, which is not considered here, but which is possible, is to consider the interpolation of operators on $L^{\Phi}(\mathfrak{X}_i) \rightarrow L^{Q}(\mathfrak{Y}_i)$, where $L^{\Phi}(\mathfrak{X})$ stands for the $\mathfrak{X}$ -valued (Orlicz) function space with $\mathfrak{X}$ as a B-space, to $L^{\Phi}(\mathfrak{X}_s) \rightarrow L^{Q}(\mathfrak{Y}_s)$, with $\mathfrak{X}_s$ and $\mathfrak{Y}_s$ as the intermediate spaces of the interpolation pairs, $(\mathfrak{X}_1, \mathfrak{X}_2)$ and $(\mathfrak{Y}_1, \mathfrak{Y}_2)$, (they exist by [11], [4]).

3.2. Interpolation with change of measures. In this subsection the interpolation problem concerns spaces with change of measures. In the preceding results an operator is defined on $L^{\Phi_i}(\mu)$ to $L^{Q_i}(\nu)$, $i = 1, 2$ where the measure μ is assumed to be the same in L^{Φ_1} and L^{Φ_2} and similarly the same one for L^{Q_1} and L^{Q_2} . This is evidently an unnatural restriction. This will now be relaxed, but the problem becomes difficult. The first generalization for the L^p -spaces was considered in [22], and the dependence of this problem on the exponents will become clear in the L^Φ -extension. In the latter case, the result gets much more involved and a subclass of L^Φ spaces, in which Φ satisfies the Δ' -condition, will be considered. This extends the result of [22], but the following result is presented more in the hope of clarifying the structure of the problem and its usefulness in applications, than in the generality itself. It will be noted that the problem is intimately related to the type of space involved, and this should be useful in the future abstract studies.

Let $(\Omega, \Sigma, \tilde{\mu}_i)$ and $(S, \mathcal{S}, \tilde{\nu}_i)$ be measure spaces and $L^{\Phi_i}(\tilde{\mu}_i)$ and $L^{Q_i}(\tilde{\nu}_i)$, $i = 1, 2$ be the Orlicz spaces on the first and respectively the second measure spaces, where Φ_i and Q_i , $i = 1, 2$ both will be assumed to satisfy the Δ' -conditions (and hence Δ_2 -conditions), i.e., $\Phi(xy) \leq C\Phi(x)\Phi(y)$, $x, y \geq 0$. It will also be assumed (which is only for convenience) that $\Phi_i(x) > 0$, $Q_i(x) > 0$ for $x > 0$, and that $\Phi_i(1) = 1$, $i = 1, 2$, so that $\Phi_s(1) = 1$ for $0 \leq s \leq 1$. Also in the Δ' -condition one may assume the constant $C = 1$, i.e., $\Phi(xy) \leq \Phi(x)\Phi(y)$. Suppose there exist positive functions $a_i(\cdot)$, $b_i(\cdot)$ and measures μ, ν such that

$$(3.15) \quad \tilde{\mu}_i(E) = \int_E a_i d\mu, \quad \tilde{\nu}_i(F) = \int_F b_i d\nu, \quad i = 1, 2, \quad E \in \Sigma, \quad F \in \mathcal{S}.$$

Note that if $\tilde{\mu}_i, \tilde{\nu}_i$ are both σ -finite then $\mu = \tilde{\mu}_1 + \tilde{\mu}_2, \nu = \tilde{\nu}_1 + \tilde{\nu}_2$ will suffice in the above and a_i, b_i exist by the Radon-Nikodým Theorem. If further $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are equivalent measures then $a_i > 0$ and a similar assumption on $\tilde{\nu}_i$ implies $b_i > 0$. The assumption of (3.15) is an abstraction of this fact.

It is now necessary to introduce a family of measures μ_s and ν_s on (Ω, Σ) and $(S, \mathcal{S})$ so that $L^{\Phi_s}(\mu_s)$ and $L^{Q_s}(\nu_s)$ can be defined. Let $\Phi_i(x) = \max(1, \Phi_i(x))$, $i = 1, 2$ and similarly $\bar{Q}_i$. Recalling the definitions of $\Phi_s, Q_s, \alpha_s, \delta_s$ introduced earlier, define the measures on Σ and $\mathcal{S}$ as

$$(3.16) \quad \begin{aligned} \mu_s(E) &= \int_E \frac{\Phi_s[\alpha_1(a_1)] \cdot \Phi_s[\alpha_2(a_2)]}{\bar{\Phi}_1[\alpha_2(a_2)] \cdot \bar{\Phi}_2[\alpha_1(a_1)]} d\mu, \\ \zeta(E) &= \int_E \frac{d\mu}{\bar{\Phi}_1[\alpha_2(a_2)] \cdot \bar{\Phi}_2[\alpha_1(a_1)]}, \\ \nu_s(F) &= \int_F \frac{Q_s[\delta_1(b_1)] \cdot Q_s[\delta_2(b_2)]}{\bar{Q}_1[\delta_2(b_2)] \cdot \bar{Q}_2[\delta_1(b_1)]} d\nu, \\ \eta(F) &= \int_F \frac{d\nu}{\bar{Q}_1[\delta_2(b_2)] \cdot \bar{Q}_2[\delta_1(b_1)]}, \end{aligned}$$

for $E \in \Sigma$, $F \in \mathcal{S}$. It is clear that $\mu_{i-1}(E) \leq \tilde{\mu}_i(E)$, $\nu_{i-1}(F) \leq \tilde{\nu}_i(F)$, $i = 1, 2$, and, moreover, these measures are equivalent. With these measures let $L^{\Phi_s}(\mu_s)$, $L^{Q_s}(\nu_s)$, $0 \leq s \leq 1$, be the Orlicz spaces with change of measures. The following result holds.

Theorem 5. Suppose, in the notations introduced above, that T is a linear operator defined simultaneously on $L^{\Phi_i}(\mu_i)$ to $L^{Q_i}(\nu_i)$, with Φ_i and Q_i satisfying the Δ' -condition, as above. If there are numbers M_i such that

$$(3.17) \quad \|Tf\|_{Q_i, \nu_i} \leq M_i \|f\|_{\Phi_i, \mu_i}, \quad f \in L^{\Phi_i}(\mu_i), \quad i = 0, 1,$$

where $\|\cdot\|_{\Phi_s, \mu_s}$ etc. is the norm (2.3), on $L^{\Phi_s}(\mu_s)$, with (2.2) on the complementary spaces, then for each $0 \leq s \leq 1$, $T: L^{\Phi_s}(\mu_s) \rightarrow L^{Q_s}(\nu_s)$ is defined and

$$(3.18) \quad \|Tf\|_{Q_s, \nu_s} \leq K_s M_1^{1-s} M_2^s \|f\|_{\Phi_s, \mu_s}, \quad f \in L^{\Phi_s}(\mu_s),$$

holds with $0 < K_s < \infty$ depending only on μ_i , ν_i , Φ_i , $i = 1, 2$, and s , but not on f . If $\tilde{\mu}_1 = \tilde{\mu}_2$ and $\tilde{\nu}_1 = \tilde{\nu}_2$, then $K_s = 1$ with $\|Tf\|_{Q_s, \nu_s}$ in (3.18) (even if Q_i is arbitrary and only Φ_i satisfies a growth condition).

Note. The result extends the L^p -case ([22], Theorem 2.11), and the idea of the proof is, as in [22], to reduce the result to that of Theorem 1 after mapping $L^{\Phi_s}(\mu_s)$ into $L^{\Phi_s}(\zeta)$ and $L^{Q_s}(\nu_s)$ into $L^{Q_s}(\eta)$ where ζ , η are the measures defined in (3.16). Details follow in a series of steps. Unlike in the L^p -case, the K_s in (3.18) may be greater than unity.

Proof. (1) By definition μ_i and $\tilde{\mu}_{i+1}$ are equivalent measures, and similarly ν_i and $\tilde{\nu}_{i+1}$, $i = 0, 1$. Since Φ_i , Q_i satisfy a Δ' -(hence Δ_2 -) condition, it follows that step functions are dense in $L^{\Phi_i}(\mu_i)$, $L^{\Phi_i}(\tilde{\mu}_i)$ and $L^{Q_i}(\nu_i)$, $L^{Q_i}(\tilde{\nu}_i)$ as well as $L^{\Phi_s}(\mu_s)$, $L^{Q_s}(\nu_s)$.

(2) If $\mathfrak{M}(\Omega, \Sigma)$ is the set of all (scalar) measurable functions on Ω , then there exists a linear map $A: \mathfrak{M}(\Omega, \Sigma) \rightarrow \mathfrak{M}(\Omega, \Sigma)$ such that $A \upharpoonright L^{\Phi_s}(\mu_s)$, the restriction, is a bounded invertible map into $L^{\Phi_s}(\zeta)$ satisfying

$$(3.19) \quad k_s \|f\|_{\Phi_s, \mu_s} \leq \|Af\|_{\Phi_s, \zeta} \leq \|f\|_{\Phi_s, \mu_s}, \quad f \in L^{\Phi_s}(\mu_s)$$

for some $0 < k_s < \infty$, independent of f , and $0 \leq s \leq 1$.

For, let A be defined by $Af = \alpha_1(a_1)\alpha_2(a_2)f$, $f \in \mathfrak{M}(\Omega, \Sigma)$. Then A is linear and $Af = 0$ iff $f = 0$ so that it is invertible. It will be shown that this A satisfies (3.19) above. If $f \in L^{\Phi_s}(\mu_s)$, then with the Δ' -condition on Φ_i (so Φ_s also satisfies the same by Lemma 2),

$$\begin{aligned} \int_{\Omega} \Phi_s[Af] d\zeta &= \int_{\Omega} \Phi_s[\alpha_1(a_1) \cdot \alpha_2(a_2)f] d\zeta \\ &\leq \int_{\Omega} \Phi_s(f) d\mu_s, \quad \text{by (3.16).} \end{aligned}$$

This implies $\|Af\|_{\Phi_s, \zeta} \leq \|f\|_{\Phi_s, \mu_s}$ and the right inequality of (3.19) follows.

To obtain the other bound, suppose the assertion is false. Then there exists

a sequence $\{f_n\}$, $f_n \geq 0$, such that $\|f_n\|_{\Phi_s, \mu_s} = 1$, and $\lim_{n \rightarrow \infty} \|Af_n\|_{\Phi_s, \zeta} = 0$. Since $\{f_n\}$ is countable, by an easy extension of ([5], III. 8.5) to the present case, it may be assumed that μ_s is sigma finite for the proof of this step. Then the Radon-Nikodým derivative $g_s = d\zeta/d\mu_s$ exists and is positive since ζ and μ_s are equivalent measures. Hence one has, with the Δ' -condition and a simple explicit calculation of g_s ,

$$\begin{aligned}
 (3.20) \quad 0 &\leq \int_{\Omega} \frac{\Phi_s(f_n) d\mu_s}{\Phi_s(A^{-1}) \cdot \Phi_s[\alpha_1(a_1)] \cdot \Phi_s[\alpha_2(a_2)]} \leq \int_{\Omega} \frac{\Phi_s(Af_n)}{\Phi_s[\alpha_1(a_1)] \cdot \Phi_s[\alpha_2(a_2)]} d\mu_s \\
 &= \int_{\Omega} \Phi_s(Af_n) g_s d\mu_s . \\
 &= \int_{\Omega} \Phi_s(Af_n) d\zeta \rightarrow 0,
 \end{aligned}$$

as $n \rightarrow \infty$ since $\|Af_n\|_{\Phi_s, \zeta} \rightarrow 0$.

Let $B = \{\omega : V_s = \Phi_s(A^{-1}) \Phi_s[\alpha_1(a_1)] \cdot \Phi_s[\alpha_2(a_2)] \leq 1\}$. Then one has

$$(3.21) \quad \int_B \Phi_s(f_n) d\mu_s \leq \int_B \frac{1}{V_s} \Phi_s(f_n) d\mu_s \leq \int_{\Omega} \Phi_s(Af_n) d\zeta \rightarrow 0$$

by (3.20). On $B' (= \Omega - B)$, $V_s > 1$, so that, by (3.20) and convexity of Φ_s ,

$$(3.22) \quad \int_{B'} \Phi_s\left(\frac{f_n}{V_s}\right) d\mu_s \leq \int_{\Omega} \frac{1}{V_s} \Phi_s(f_n) d\mu_s \rightarrow 0.$$

If $\mathcal{S}_{B'}$ is the closed subspace of $L^{\Phi_s}(\mu_s)$, generated by $\{f_n\}$ restricted (or truncated) to B' , and $\mathcal{S}_{B'}^+$ denotes a similar space relative to $\{f_n/V_s\}$, then the bounded map $U: f_n \rightarrow f_n/V_s$ maps $\mathcal{S}_{B'}$ onto $\mathcal{S}_{B'}^+$ in a one-to-one manner (thus U^{-1} exists), so that by the inverse boundedness theorem ([5], II. 2.2), U^{-1} is bounded. Hence there is a $0 < C < \infty$ such that

$$\|f_n\|_{\mathcal{S}_{B'}} = \left\| U^{-1}\left(\frac{1}{V_s} f_n\right) \right\|_{\mathcal{S}_{B'}} \leq C \left\| \frac{1}{V_s} f_n \right\|_{\mathcal{S}_{B'}^+} \rightarrow 0$$

by (3.22). This implies $\int_{B'} \Phi_s(f_n) d\mu_s \rightarrow 0$ as $n \rightarrow \infty$. This and (3.21) imply that $\int_{\Omega} \Phi_s(f_n) d\mu_s \rightarrow 0$ so that $\|f_n\|_{\Phi_s, \mu_s} \rightarrow 0$. But this contradicts the fact that $\|f_n\|_{\Phi_s, \mu_s} = 1$. So the supposition is not true and (3.19) holds as stated. A similar statement holds for $L^{Q_s}(\nu_s)$ and $L^{Q_s}(\eta)$.

(3) The proof of the theorem can now be completed. Since step functions are dense in $L^{\Phi_s}(\mu_s)$, $0 \leq s \leq 1$, T is defined densely on all these spaces. If A and $\tilde{A}$ are the operators of step 2, let $\mathfrak{N}_{\zeta, s}$ and $\mathfrak{N}_{\eta, s}$ be the closures of $A[L^{\Phi_s}(\mu_s)]$ and $\tilde{A}[L^{Q_s}(\nu_s)]$ in $L^{\Phi_s}(\zeta)$ and $L^{Q_s}(\eta)$ respectively. Then the operator

$$W = \tilde{A}TA^{-1}: \mathfrak{N}_{\zeta, s} \rightarrow \mathfrak{N}_{\eta, s},$$

is linear and by the properties of step 2,

$$k_i^s \|A^{-1}f\|_{\Phi_s, \mu_s} \leq \|f\|_{\Phi_s, \zeta} \leq \|A^{-1}f\|_{\Phi_s, \mu_s}$$

for some $0 < k_i^s < \infty$, $i = 0, 1$. Consequently for $f \in \mathfrak{N}_{\mathfrak{f}, i}$, $i = 0, 1$

$$\|Wf\|_{Q_{i, \nu}} \leq \|TA^{-1}f\|_{Q_{i, \nu}} \leq M_i \|A^{-1}f\|_{\Phi_{i, \mu_i}} \leq \frac{M_i}{k_i^s} \|f\|_{\Phi_{i, \mathfrak{f}}} < \infty.$$

Hence by Theorem 2, and Lemma 4,

$$\|Wf\|_{Q_{i, \nu}} \leq 2 \left(\frac{M_1}{k_1^s} \right)^{1-s} \left(\frac{M_2}{k_2^s} \right)^s \cdot \|f\|_{\Phi_{i, \mathfrak{f}}}, \quad f \in \mathfrak{N}_{\mathfrak{f}, s}.$$

But again by step 2, for $f \in L^{\Phi_s}(\mu_s)$, there is a $d_s > 0$ such that

$$d_s \|Tf\|_{Q_{i, \nu}} \leq \|\tilde{A}Tf\|_{Q_{i, \nu}} = \|W Af\|_{Q_{i, \nu}} \leq 2 \left(\frac{M_1}{k_1^s} \right)^{1-s} \left(\frac{M_2}{k_2^s} \right)^s \|f\|_{\Phi_{i, \mu_s}},$$

using the preceding estimate. If $K_s = 2[d_s(k_1^s)^{1-s}(k_2^s)^s]^{-1}$, then one has

$$\|Tf\|_{Q_{i, \nu}} \leq K_s M_1^{1-s} M_2^s \|f\|_{\Phi_{i, \mu_s}}, \quad f \in L^{\Phi_s}(\mu_s),$$

which is (3.18). Since $\Phi_i(1) = \Phi_s(1) = Q_s(1) = Q_i(1) = 1$, by normalization it follows from the above estimates that $d_s = 1 = k_1^s = k_2^s$ so that [with (3.1)] $K_s = 1$ whenever $\tilde{\mu}_1 = \tilde{\mu}_2$ and $\tilde{\nu}_1 = \tilde{\nu}_2$; so $\mu_1 = \mu_2$, $\nu_1 = \nu_2$. This completes the proof of the theorem.

Remark. The above proof also holds if (3.17) is only given for $L^{\Phi_i}(\tilde{\mu}_i)$ and $L^{\Phi_i}(\tilde{\nu}_i)$ since then one can suitably contract the domain of T to these spaces. Also by analogous arguments, E. Stein's result on the interpolation of a family of operators depending on an analytic parameter (cf. [25_{II}], p. 100) can be extended.

4. Ergodic theorems. In this and the following sections some applications of the foregoing results on interpolation to the (abstract) pointwise ergodic theorem, and related questions, will be considered. The following result is a slight extension of ([1], Theorem 1) and will be useful in the later proofs.

Proposition 1. Let Φ_i , $i = 1, 2, 3$ be Young's functions such that

$$\Phi_1(x) \leq c_1|x|; \quad \Phi_3(x) \leq c_2|x|, \quad \text{for } |x| < x_0 < \infty$$

and

$$\Phi_3(x) = +\infty, \quad \text{for } |x| > x_0 (c_1 > 0, c_2 \geq 0),$$

Φ_2 strictly convex, and L^{Φ_i} be the Orlicz spaces on a σ -finite measure space (Ω, Σ, μ) , satisfying $L^{\Phi_s} = \mathfrak{N}^{\Phi_s}$. If $T: L^{\Phi_i} \rightarrow L^{\Phi_i}$, $i = 1, 2$, is a positive contraction operator with $(\chi_A$ is the indicator of A),

$$\sum_{k=0}^{\infty} T^k \chi_A = \infty,$$

for some A of positive finite measure (i.e., T is "regular"), then T may be extended such that $T: \mathfrak{N}^{\Phi_s} \rightarrow \mathfrak{N}^{\Phi_s}$, for each $0 \leq s \leq 1$, is also a positive contraction, where

Φ_s is the Young's function whose inverse is given by $\Phi_s^{-1} = [\Phi_2^{-1}]^{1-s}[\Phi_3^{-1}]^s$. (Note that $\Phi_1 < \Phi_2 < \Phi_3$.)

Proof. To begin with, note that T is a contraction on L^{Φ_s} where Φ_s is given by $\Phi_s^{-1} = [\Phi_1^{-1}]^{1-s}[\Phi_2^{-1}]^s$, $0 \leq s \leq 1$, by Theorem 1. The conclusion is seen to hold as a consequence of Theorem 2. Indeed, since L^{Φ_1} is here a subspace of L^1 , by ([24], p. 82), the proof of [1] applies and the operator T has an extension, which will also be denoted by T , to the space L^∞ of bounded measurable functions. Since Φ_3 is discontinuous, $\mathfrak{M}^{\Phi_s}$ is a subspace of L^∞ ([24], p. 82) so that T has the stated extension, by the above, to $\mathfrak{M}^{\Phi_s}$. The statement now is a consequence of Theorem 2, as claimed. (Or, if $\mu(\Omega) < \infty$, $L^{\Phi_1} = L^1$, $\mathfrak{M}^{\Phi_s} = L^\infty$ and the result holds by Theorem 2, and the extension to σ -finite μ is standard. See [1].)

Remark. The result coincides with [1], if $\Phi_2(x) = |x|^p$, $p > 1$. It must be pointed out that the "regularity" of T is stated here only for comparison with [1]. Actually this is redundant since $T: L^1 \rightarrow L^1$ and is a nontrivial positive contraction implies $\sum_{k=0}^\infty T^k f = \infty$, a.e., for any $f > 0$, $f \in L^1$; and f lives on the conservative part of Ω , relative to T . The existence of such a decomposition of Ω into the conservative and dissipative parts is a classical result of E. Hopf.

The following result complements the preceding and is also of interest.

Proposition 2. Let Φ be a strictly convex Young's function such that $L^\Phi = \mathfrak{M}^\Phi$, L^Φ being the Orlicz space on a σ -finite measure space (Ω, Σ, μ) . Suppose $T: L^\Phi \rightarrow L^\Phi$ is a positive contraction operator such that $T_A = T|_{L^\Phi(A)}$, the restriction of T to $L^\Phi(A)$ the latter being the closed subspace of L^Φ of functions that live on A , satisfies $T_A(f\chi_A) = \chi_A(Tf)$, for all $f \in L^\Phi$, $A \in \Sigma$. Then T may be extended such that $T: \mathfrak{M}^{\Phi_s} \rightarrow \mathfrak{M}^{\Phi_s}$, for each $0 \leq s \leq 1$, is also a positive contraction where Φ_s has the same meaning as in Proposition 1. (i.e., $[\Phi^{-1}]^{1-s}[\Phi_3^{-1}]^s$, where Φ_3 is a discontinuous Young's function.)

Proof. The proof is similar to the preceding proposition. Let Σ_1 be the ring of sets in Σ of finite μ -measure. Then $L^\Phi = \mathfrak{M}^\Phi$ implies the set

$$\bigcup_{A \in \Sigma_1} L^\Phi(A)$$

is dense in L^Φ . Consequently one may consider the action of T on $L^\Phi(A)$, $A \in \Sigma_1$, and piece together the result on each space $L^\Phi(A)$ to all of L^Φ , as follows. First it will be shown that $\|T_A f\|_\infty \leq \|f\|_\infty$ for any bounded function in $L^\Phi(A)$. Since $A \in \Sigma_1$ is arbitrary, it follows from this that $\|Tf\|_\infty \leq \|f\|_\infty$, for all $f \in L^\Phi \cap L^\infty$.

Since $A \in \Sigma_1$, $1 \in L^\Phi(A)$, and it suffices to show $T_A 1 \leq 1$, a.e., from which it immediately follows that $\|T_A f\|_\infty \leq \|f\|_\infty$, $f \in L^\Phi(A) \cap L^\infty$. Let $(T_A 1)(\omega) = 1 + h(\omega)$ so that $h(\omega) \geq -1$ (recall that T_A is a positive operator). Suppose $h(\omega) \leq 0$, a.e., is false. Then there is a set $B \subset A$, of positive measure,

such that $h(\omega) > 0$ on B . Since Φ is strictly convex, $\Phi'(\cdot)$ (the derivative) exists, is positive, and is continuous so that

$$(4.1) \quad \begin{aligned} \Phi(1+x) &= \Phi(1) + x\Phi'(1) + x[\Phi'(1+\theta x) - \Phi'(x)] \\ &> \Phi(1) + x\Phi'(1), \end{aligned}$$

for some $0 < \theta < 1$, and $x > -1$; $x \neq 0$ (by the Mean-Value Theorem). Thus

$$\int_B \Phi(T_A 1) d\mu = \int_B \Phi(1+h) d\mu > \int_B \Phi(1) d\mu + \Phi'(1) \int_B h d\mu > \int_B \Phi(1) d\mu,$$

since $h > 0$ on B . But by the property of T , one may consider $T_A \mid L^\Phi(B)$, as T_B , so that the above equation becomes,

$$\int_B \Phi(T_B \chi_B) d\mu > \int_B \Phi(\chi_B) d\mu.$$

This, however, implies $\|T_B \chi_B\|_\Phi > \|\chi_B\|_\Phi$ which contradicts the fact that T and hence T_A and T_B are contractions. Thus $\|Tf\|_\infty \leq \|f\|_\infty$, for $f \in L^\Phi \cap L^\infty$.

Now if $\Omega_n \in \Sigma_1$ are such that $\Omega_n \uparrow \Omega$ (by σ -finiteness), and $\Phi_0 \succ \Phi$ is any Young's function then $0 \leq f \in L^{\Phi_0}$ and $f_n = f\chi_{\Omega_n}$, imply $f_n \nearrow f$ and $f_n \in L^\Phi(A) \subset L^\Phi$. If $\tilde{T}f = \lim_{n \rightarrow \infty} Tf_n$, then $\tilde{T}$ is well defined since Tf_n is a pointwise convergent sequence of scalar functions having a unique limit. Since on L^Φ , $\tilde{T} = T$, it follows that, by defining $\tilde{T}f$ as $\tilde{T}(f^+) - \tilde{T}(f^-)$ for arbitrary function in L^{Φ_0} , $\|\tilde{T}f\|_\infty \leq \|f\|_\infty$, for $f \in \mathfrak{M}^{\Phi_0}$. The rest of the argument is identical with the previous proposition, and the result follows.

Remark. It may not be possible to extend T "backwards" to L^1 , and the two propositions complement in some cases. By considering Φ_i appropriately, so that $|x|^p$, $1 \leq p < \infty$, can be "bounded" above and below by such Φ_i 's, it can be seen that the L^p case is subsumed by Propositions 1 and 2 in the corresponding cases.

The following pointwise ergodic theorem can now be proved.

Theorem 6. Let Φ_1 , Φ_2 and Φ_3 be Young's functions such that $\Phi_1(x) \leq c_1 |x|$;

$$\Phi_3(x) \leq c_2 |x|, \quad \text{for } |x| < x_0 (0 < x_0 < \infty),$$

$$\Phi_3(x) = \infty, \quad \text{if } |x| > x_0, \quad (c_1 > 0, c_2 \geq 0),$$

and Φ_2 is strictly convex and satisfies a growth condition. If L^{Φ_i} is an Orlicz space on a measure space (Ω, Σ, μ) , [and $L^{\Phi_i} = \mathfrak{M}^{\Phi_i}$, $i = 1, 2$, holds], then (a) for a positive contraction operator $T: L^{\Phi_i} \rightarrow L^{\Phi_i}$, $i = 1, 2$, simultaneously,

$$(4.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} T^i f, \quad f \in L^{\Phi_i}, \quad 0 \leq s \leq 1 \quad \text{or} \quad f \in \mathfrak{M}^{\Phi_i}, \quad 0 \leq s < 1,$$

exists a.e., and is finite, where Φ_i^0 is the inverse of $(\Phi_i^0)^{-1} = [\Phi_1^{-1}]^{1-s} [\Phi_2^{-1}]^s$ and Φ_i is the inverse of $\Phi_i^{-1} = [\Phi_2^{-1}]^{1-s} [\Phi_3^{-1}]^s$; and (b) for a positive linear contraction

operator $T: L^{\Phi^*} \rightarrow L^{\Phi^*}$ satisfying the conditions of Proposition 2 for $T_A = T \mid L^{\Phi^*}(A)$, one has:

$$(4.4) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} T^i f, \quad f \in \mathfrak{M}^{\Phi^*}, \quad 0 \leq s < 1$$

exists a.e. and is finite, where Φ_s is defined in (a) above.

The proof of this result is facilitated by the following auxiliary result, a mean ergodic theorem, which also has some independent interest.

Proposition 3. Let $\Phi_1, \Phi_2, \Phi_3, \Phi_s$ and Φ_s^0 be as in Theorem 6. If T is a positive contraction either (i) on both $L^{\Phi_i} \rightarrow L^{\Phi_i}$, $i = 1, 2$ or (ii) only on $L^{\Phi^*} \rightarrow L^{\Phi^*}$ and satisfies the conditions of Proposition 2 for $T_A = T \mid L^{\Phi^*}(A)$, then the averages

$$A_n f = \frac{1}{n} \sum_{i=0}^{n-1} T^i f$$

converge strongly for $f \in L^{\Phi^*}$, $0 \leq s \leq 1$ and $f \in \mathfrak{M}^{\Phi^*}$, $0 \leq s < 1$ in case (i); and for $f \in \mathfrak{M}^{\Phi^*}$, $0 \leq s < 1$ in case (ii), respectively.

Proof. Let $S = \{A_n f, n = 0, 1, \dots\}$. Since $\|A_n f\|_{\Phi} \leq \|f\|_{\Phi}$, and f is an arbitrarily fixed element of L^{Φ} , the set S is bounded. Then by the Yosida-Kakutani Theorem ([5], VIII. 5.3) $A_n f$ converges strongly iff $T^n f/n \rightarrow 0$ (which obviously holds here) and that the bounded set S is weakly sequentially compact, for f in a fundamental set. So only the latter condition is to be established. Since Φ satisfies a growth condition, step functions—hence bounded functions—form a fundamental set in the space. So f may be taken to be bounded.

Now to prove the weak sequential compactness of S , first recall that, for every continuous Φ , $\mathfrak{M}^{\Phi}$ is weakly complete (implied by [10], Theorem 14.4; or [13], Theorem 3.2); so it suffices to show that S is conditionally weakly sequentially compact and the latter is proved, with ([13], Theorem 5.1), as follows. For this, one has only to verify the conditions, for each $g \in L^{\Psi}$,

$$(a) \quad \rho(g) = \sup_n \int_{\Omega} |(A_n f)g| d\mu < \infty,$$

$$(b) \quad \rho(g\chi_{E_n}) \rightarrow 0 \quad \text{as} \quad E_n \searrow \phi.$$

It is clear that (a) holds, by the Hölder inequality, since $\|A_n\| \leq 1$. To show (b), consider the two cases of the hypothesis separately. Since S is countable, all its functions vanish outside a σ -finite set, so for the rest of the proof μ may be assumed σ -finite itself.

Case (i). The hypothesis here is the same as that of Proposition 1, so that $\|Tf\| \leq \|f\|_{\infty}$ for $f \in \mathfrak{M}^{\Phi} \cap L^{\infty}$, where $\Phi = \Phi_s^0$, $0 \leq s \leq 1$, or Φ_s , $0 \leq s < 1$. It may be assumed that $f \geq 0$, replacing f by $|f|$ if necessary. So it follows that $\chi_E \cdot T^n f \leq \chi_E(T^{n-1}f) \leq \dots \leq \chi_E f$, a.e. Hence

$$(4.5) \quad \rho(g\chi_{E_n}) = \sup_n \frac{1}{n} \sum_{i=0}^{n-1} \int_{E_n} (T^i f) |g| d\mu$$

$$\begin{aligned}
&= \sup_n \frac{1}{n} \sum_{i=0}^{n-1} \int_{\Omega} \chi_{E_n}(T^i f) \cdot |g| \, d\mu \leq \int_{\Omega} (f \chi_{E_n}) |g| \, d\mu \\
&\leq \|f \chi_{E_n}\|_{\Phi} N_{\Psi}(g) \rightarrow 0 \quad \text{as } E_n \searrow \phi,
\end{aligned}$$

since $f \in \mathfrak{M}^{\Phi}$, and Φ is continuous, and hence the norm in $\mathfrak{M}^{\Phi}$ is absolutely continuous ([10], p. 87). Thus the result holds in this case.

Case (ii). Since the operator T here satisfies the same hypothesis as in Proposition 2 so that $\|Tf\|_{\infty} \leq \|f\|_{\infty}$ for $f \in \mathfrak{M}^{\Phi} \cap L^{\infty}$, where $\Phi = \Phi_s$, $0 \leq s < 1$. Consequently the argument of the above case can be repeated. Thus the result holds in this case also, and with it the proof is complete.

For the pointwise convergence proof, the following two modified lemmas of E. Hopf ([5], VIII. 6.2 and 5) are also needed.

Lemma 5. *Let Φ be a strictly convex Young's function such that $L^{\Phi} = \mathfrak{M}^{\Phi}$, on a σ -finite measure space, and $T: L^{\Phi} \rightarrow L^{\Phi}$ be a positive contraction operator satisfying the hypothesis of Proposition 2. Then for every $f \in L^{\Phi} \cap L^{\infty}$, and every n , $\int_E f \, d\mu \geq 0$ where*

$$E = \{u: \sup_{1 \leq k \leq n} A_k f \geq 0\}, \quad \text{and} \quad A_n = \frac{1}{n} \sum_{i=0}^{n-1} T^i,$$

as before.

Proof. If $\Phi(x) = |x|$, and T is a positive contraction then this is Hopf's result, without further hypothesis, as given in [5]. The fact that $T: L^1 \rightarrow L^1$ implies that the adjoint $T^*: L^{\infty} \rightarrow L^{\infty}$ is also a positive contraction was used in the proof. However, the additional hypothesis on T , of the lemma, compensates this as follows. Since $T: L^{\Phi} \rightarrow L^{\Phi}$ is a positive contraction with an extra property on $T_A = T|_{L^{\Psi}(A)}$, it follows that its adjoint $T^*: L^{\Psi} \rightarrow L^{\Psi}$ is a positive contraction and $T_A^*: L^{\Psi}(A) \rightarrow L^{\Psi}(A)$ satisfies the same hypothesis. However, Ψ need not satisfy any growth condition. So one has to consider the subspace $\mathfrak{M}^{\Psi}$ instead, (which, however, is sufficient here). Then by Proposition 2, T^* can be extended to L^{∞} so as to remain a positive contraction on it (since μ is σ -finite). Using this extended T^* on L^{∞} , the proof of ([5], VIII. 6.2) can be repeated without change to the present extension.

Lemma 6. *Let $T: L^{\Phi} \rightarrow L^{\Phi}$ and A_n be as in Lemma 5. Then for each $f \in L^{\Phi}$, $0 \leq s < 1$ one has*

$$\sup_{1 \leq n \leq \infty} |(A_n f)(u)| < \infty$$

for almost all u , where Φ_s is defined for $\Phi (= \Phi_2)$ and Φ_3 in Theorem 6.

The proof of the result is the same as in ([4], p. 675) where the modified result (Lemma 5 above) is used in place of the original Hopf's result, and the argument is unaltered.

It is now possible to quickly complete the proof of the pointwise convergence,

and, with the preceding results, the general idea of the proof of Hopf–Dunford–Schwartz works in the present case.

Proof of Theorem 6. Let $S = \{A_n f, n \geq 1\}$. As in the proof of Proposition 3, for the present proof, (Ω, Σ, μ) may be assumed σ -finite. In parts (a) and (b) of the theorem, using Propositions 1 and 2 respectively, the operator T may be extended to L^∞ so as to remain a positive contraction. Consequently by Proposition 3, $A_n f$ converges strongly for each bounded f in L^Φ ($\Phi = \Phi_s^0$ or Φ_s , as the case may be). Hence if $f, g \in \mathfrak{M}^\Phi$, with g bounded, then the decomposition $h = f + (I - T)g$ where $Tf = f$, (T is the extended operator) gives the set $\{h\}$ to be dense in $\mathfrak{M}^\Phi$ by ([5], VIII. 5.2). Also,

$$|A_n f - f| \leq \frac{2|g|}{n} \leq \frac{c}{n} \rightarrow 0,$$

since T maps $L^\infty \rightarrow L^\infty$. Thus $A_n h$ converges a.e. for a dense set of elements in L^Φ and so by Lemma 6 above, $\sup_n |A_n h|$ is finite a.e. It therefore follows, by a theorem of Banach ([5], p. 332), that $A_n f$ converges for all $f \in \mathfrak{M}^\Phi$. This completes the proof of the theorem.

Remark. The preceding result admits ([7], Theorems 1 and 2) to be extended to non-finite measure spaces. [In the L^1 case, the assumption (for Theorem 1 only), is to be replaced by the uniform integrability of $A_n f$, for f in a fundamental set. Then Proposition 3 holds and the result follows.] The multiparameter case of ([5], p. 679) also admits a similar extension with somewhat longer arguments. They will not be considered here.

5. Martingales. The main result of this section concerns the (mean) convergence of certain generalized martingales—the latter concept was introduced by Rota [20] in connection with his unified study of (mean) ergodic and martingale theorems. First a simple result on the (Abel) mean convergence of certain semigroups [of class (E), cf. [6], p. 509] will be stated to use it in the proof of mean convergence later.

Definition 3. A strongly continuous semigroup $\{T(t), t \geq 0\}$, on a B-space $\mathfrak{X}$, of type ω_0 (i.e., $\lim_{t \rightarrow \infty} (1/t) \log \|T(t)\| = \omega_0$ exists, $\omega_0 < \infty$) is said to be of class (E) if (i) $\mathfrak{X}_0 = [x : \int_0^1 \|T(t)x\| dt < \infty]$ is dense in $\mathfrak{X}$ and (ii) the linear operator $R(\lambda)$, given by $R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x dt$, on $\mathfrak{X}_0$ is defined and bounded for all $\lambda > \omega_0$. [The integral is a Bochner integral.]

The $R(\lambda)$ given in the above definition satisfies the resolvent equation $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$, for $\lambda, \mu > \omega_0$, (cf. [6], p. 510). The following result is an immediate consequence of a theorem of E. Hille (cf. [6], p. 520) and is needed in the form given below.

Lemma 7. Let Φ_1, Φ_2, Φ_3 be Young's functions such that $\Phi_1(x) \leq c_1 |x|$, for all $x \geq 0$, $c_1 > 0$; Φ_2 is strictly convex and Φ_3 is discontinuous and such that

if L^{Φ^*} is the corresponding Orlicz space on (Ω, Σ, μ) , a σ -finite measure space, $L^{\Phi^*} = \mathfrak{M}^{\Phi^*}$, $i = 1, 2$. If $\{T(t), t \geq 0\}$ is a strongly continuous positive contraction semi-group of class (E) on either (i) both L^{Φ^*} , $i = 1, 2$ or (ii) only L^{Φ^*} such that $T_A(t) = T(t) \mid L^{\Phi^*}(A)$, for A of finite measure and every $t \geq 0$, satisfies the conditions of Proposition 2, then it is strongly Abel ergodic at infinity, i.e., $\lim_{\lambda \rightarrow 0^+} \lambda R(\lambda)f$ exists in norm for each $f \in L^{\Phi^*}$, $0 \leq s \leq 1$ and $f \in L^{\Phi^*}$, $0 \leq s < 1$ in case (i); and $f \in L^{\Phi^*}$, $0 \leq s < 1$ in case (ii). Here Φ_* and Φ_*^0 have the same meanings as before.

Proof. The result follows, as in Proposition 3, from the above mentioned theorem of Hille, if it is shown that $\{\lambda R(\lambda)f; 0 < \lambda < \omega\}$ is conditionally weakly sequentially compact, for $f \in L^{\Phi^*}$, $[\Phi = \Phi_*^0 \text{ or } \Phi_*]$, and some $\omega > 0$. (Note that $\omega_0 = 0$ here.) For this, due to strong continuity of the semi-group, it is sufficient to show, for an arbitrary but fixed sequence $\lambda_n \searrow 0$, that $\{\lambda_n R(\lambda_n)f\}$ is weakly sequentially compact and the set is bounded, (cf. also [5], p. 430; [12], p. 505). The latter is immediate, since $\|\lambda R(\lambda)f\|_{\Phi} \leq \|f\|_{\Phi} < \infty$. The proof of the weak sequential compactness is similar to that of Proposition 3.

Briefly, let $f_n = \lambda_n R(\lambda_n)f$, and $\rho(g) = \sup_n \int_{\Omega} |f_n g| d\mu$. For each $g \in L^{\Psi}$, one has $\rho(g) \leq \sup_n \|f_n\|_{\Phi} N_{\Psi}(g) \leq \|f\|_{\Phi} N_{\Psi}(g) < \infty$, and

$$\begin{aligned} \rho(g \chi_{E_p}) &= \sup_n \int_{\Omega} \left| \chi_{E_p} g \lambda_n \int_0^{\infty} e^{-\lambda_n t} T(t)f dt \right| d\mu \\ &= \sup_n \int_0^{\infty} \lambda_n e^{-\lambda_n t} \left[\int_{\Omega} |g \chi_{E_p} T(t)f| d\mu \right] dt, \end{aligned}$$

by Fubini's Theorem ([5], [6]),

$$\leq \sup_n \int_0^{\infty} \lambda_n e^{-\lambda_n t} \int_{\Omega} |g| (\chi_{E_p} |f|) d\mu dt,$$

as in (4.5), since $T(t)$ is a positive contraction,

$$\leq \|f \chi_{E_p}\|_{\Phi} N_{\Psi}(g) \cdot 1 \rightarrow 0, \quad E_p \searrow \phi,$$

since $\|\cdot\|_{\Phi}$ is absolutely continuous. Hence ([13], Theorem 5.1) applies, and $\{\lambda_n R(\lambda_n)f\}$ is weakly sequentially compact. This completes the proof of the lemma.

For the remaining work, the concept of a generalized martingale, introduced by Rota, is recalled in the following

Definition 4. (a) A bounded operator R on L^{Φ} is said to be a *Reynold's operator* if it satisfies the algebraic (Reynold's) identity:

$$(5.1) \quad R(fg) = (Rf)(Rg) + R[(f - Rf)(g - Rg)], \quad f, g \in L^{\Phi} \cap L^{\infty}.$$

If R is a bounded projection on L^{Φ} satisfying the (averaging) identity

$$(5.2) \quad R(Rg) = (Rf)(Rg), \quad f, g \in L^{\Phi} \cap L^{\infty},$$

then R is called an *averaging operator* in L^Φ . [So every averaging is a Reynold's operator.] (b) If $\{R_t, t > 0\}$ is a family of pairwise commuting Reynold's operators on L^Φ , it is called a *generalized martingale* (cf. [20], p. 2065) if the following algebraic identity holds:

$$(5.3) \quad (sR_t - tR_s)R_s f = (s - t)R_t R_s^2 f, \quad 0 < t < s, \quad f \in L^\Phi \cap L^\infty.$$

The above concept is indeed a proper generalization of the classical martingale concept (cf. [12]). In fact, if $\{P_t, t > 0\}$ is a family of projection operators in L^Φ such that $P_s P_t = P_t P_s = P_t$ for $0 < s < t$ then it satisfies (5.3) and if further $\{P_t, t > 0\}$ are contractions, $L^\Phi = \mathfrak{M}^\Phi$, and the measure space is a probability space, then by ([17], Theorem 1) P_t coincides with a conditional expectation operator relative to a sub σ -field Σ_t of Σ so that $\{P_t f, t > 0\}$ is a martingale in L^Φ . Thus the terminology is justified.

The following important result on the characterization of Reynold's operators on L^Φ , also due to Rota ([21], Theorem 2), is stated here for ready reference in the proof below.

Theorem 7 (Rota). *Let R be a Reynold's operator in $L^\Phi (= \mathfrak{M}^\Phi)$ on a finite measure space with $\Phi \prec \Phi_0 (\Phi_0(x) = x^2)$ such that it is a contraction on L^Φ and L^∞ , and $R1 = 1$. Then there exist uniquely (i) a sub σ -field Σ_1 of Σ , (ii) a conditional expectation P in L^Φ onto $L^\Phi \cap L^\infty(\Omega, \Sigma, \mu)$, (iii) a semi-group $\{V(t), t \geq 0\}$ of measure preserving transformations on $L^\Phi(\Omega, \Sigma, \mu)$ into itself such that: (a) $L^\infty(\Omega, \Sigma_1, \mu)$ is invariant under R , (b) R and P commute, the range of P being the closure of the range of R in L^Φ , and (c) R can be represented as*

$$(5.4) \quad Rf = \int_0^\infty e^{-t} V(t) P f dt, \quad f \in L^\infty(\Omega, \Sigma, \mu),$$

the integral being a Bochner integral.

Remark. This result was actually stated in [21] for the L^2 -spaces. However, the proof can be modified slightly to show that the above statement holds. Note also that, as shown in [21], the assumptions of the theorem imply that R is a positive contraction. On the other hand, a positive contraction on L^{Φ_1} and L^{Φ_2} , by Propositions 1 and 2, is also defined on L^∞ ; so the latter formulation is used below.

The result of this section can now be stated as follows:

Theorem 8. *Let Φ_1, Φ_2, Φ_3 be Young's functions and L^{Φ_i} be the corresponding Orlicz space such that $L^{\Phi_i} = \mathfrak{M}^{\Phi_i}$, $i = 1, 2, 3$, on a finite measure space. Suppose either (i) $\Phi_1(x) \leq c|x|$, $c > 0$, $x \geq 0$, and Φ_2 is strictly convex and Φ_3 discontinuous or (ii) Φ_2 is strictly convex and $\Phi_2 \prec \Phi_0 (\Phi_0(x) = x^2)$. If $\{R_t f, t > 0\}$ is a generalized martingale sequence, with $R_t 1 = 1$, in both L^{Φ_1} and L^{Φ_2} in the first case and in L^{Φ_1} and L^{Φ_3} in the second case, and that R_t is a positive contraction in either case, then $\lim_{t \rightarrow 0+} R_t f$ exists in the mean (i.e., in norm) of L^Φ , $\Phi = \Phi_1^0$, $0 \leq s \leq 1$, and $\Phi = \Phi_2$, $0 \leq s < 1$ in the first case and $\Phi = \Phi_3$, $0 \leq s < 1$*

in the second case, where Φ_s^0 and Φ_s are defined before (in Propositions 1 and 2). [In the first case one can state $\Phi = \Phi_s$ where $(\Phi_s)^{-1} = [\Phi_1^{-1}]^{1-s}[\Phi_3^{-1}]^s$, $0 \leq s < 1$, but this is less general than the former.]

Remark. In the second case with $\Phi_2(x) = |x|^p$, $p > 1$, the corresponding result on pointwise convergence was given by Rota ([20], p. 2065). It seems that using the method of ([18], Sec. 3; and [2], pp. 116–117), together with the above theorem, Rota's theorem on pointwise convergence ([20]), can be extended to some non-reflexive spaces, including the L^p -spaces, $1 \leq p < \infty$. However, the above theorem does not include the "closed martingales" of [18] (or prediction sequences of [2]) since the operators there, while satisfying (5.3), are not linear and possibly unbounded. Also certain other martingales arising out of the Cesàro-means (cf. [8], p. 534) are not covered here, since the measure space there is non-finite but an extra condition *can* be given to cover the latter. Note that ([8], Theorem 5.3) is subsumed by the above result.

Proof. Since R_t is a generalized martingale, it satisfies (5.3) for $0 < t < 1$ (taking $s = 1$ there) for all $f \in L^\Phi \cap L^\infty$ where Φ is a fixed Young's function ($= \Phi_s^0$ or Φ_s appropriately). If $\mathfrak{X}$ is the closure of $\mathfrak{X}_0 = L^\Phi \cap L^\infty$, in L^Φ , then (5.3) can be written as

$$(5.5) \quad \left(\frac{R_t}{t} - \frac{R_s}{s} \right) = (s - t) \frac{R_t}{t} \cdot \frac{R_s}{s}, \quad 0 < t < s \leq 1,$$

on $\mathfrak{X}$. Taking $s = 1$ hereafter, one notes that, as in step 4 of Rota's proof ([21], p. 79), R_1^{-1} exists (as an unbounded operator) on $\mathfrak{X}$. This statement is justified since by Proposition 1 in the first case (R_t is a positive contraction on all L^p , in particular), and R_t is defined on $L^{\Phi^0} = L^2$ in the second case, the hypothesis of the step is satisfied. But (5.5) is just a resolvent equation, so that ([6], Theorem 5.8.3) is applicable, and one may conclude from that theorem that R_t/t is the resolvent of the closed operator $D = I - R_1^{-1}$ (also see [22a], p. 259, Ex. 3), whose spectrum lies in the left half plane. So by Hille–Yosida theorem (in the form [5], VIII. 1.14) R_t may be expressed as (5.4), i.e., more particularly as (cf. also [21]),

$$(5.6) \quad R_t f = t \int_0^\infty e^{-tu} V(u) f \, du, \quad f \in \mathfrak{X}_t = \text{cl} \{R_t f; f \in L^\infty\},$$

where $\{V(u), u \geq 0\}$ is a strongly continuous positive contraction semi-group on $\mathfrak{X}_t$, (cl = closure in L^Φ -norm). Moreover, by ([6], p. 357) this is unique.

Now, in both cases, Rota's representation theorem applies and there is a conditional expectation P_t on L^Φ onto $\mathfrak{X}_t$, and that $\mathfrak{X}_{t_1} \supset \mathfrak{X}_{t_2}$ for $0 \leq t_1 \leq t_2 < 1$, so that (5.6) takes the form

$$(5.7) \quad R_t f = t \int_0^\infty e^{-tu} V(u) P_t f \, du, \quad f \in L^\infty.$$

As a consequence of steps (8) and (9) ([21], p. 80), one deduces the fact that $\{V(t), t \geq 0\}$ is actually a semi-group of measure preserving operators on L^∞ . By the definition of P_t one has $\mathfrak{X}_t = \text{cl } [L^\infty(\Omega, \Sigma_t, \mu)]$ as described in the theorem with $\Sigma_{t_1} \supset \Sigma_{t_2}$, $0 < t_1 < t_2 \leq 1$. $V(t)$ has all the properties in the first case and it has also those properties in the second case (i.e., $V(t)$ has an extension onto $\bigcup_{t>0} L^\infty(\Omega, \Sigma_t, \mu)$), by Proposition 2, since the measure preserving semi-group is seen to satisfy the hypothesis of Proposition 2.

It therefore follows that R_t here satisfies the hypothesis of $\lambda R(\lambda)$ of Lemma 7, since $\mathfrak{X} = \text{cl } [\bigcup_{n=1}^\infty \mathfrak{X}_{t_n}]$ is the required subspace of L^Φ there, and so the mean convergence is a consequence of that lemma. This completes the proof.

It is natural to consider the convergence problem as $t \rightarrow \infty$ in the above instead of $t \rightarrow 0^+$. In fact, an exactly similar procedure works where the unions will be replaced by intersections of the various spaces, and instead of Lemma 6 an analogous result, obtained from ([6], Theorem 18.7.1), will be used, replacing the $t_n \searrow 0$ by $t_n \nearrow \infty$. So the result will be stated without proof here.

Theorem 9. *Let Φ_i , and $L^{\Phi_i} = \mathfrak{M}^{\Phi_i}$, $i = 1, 2, 3$, on a finite measure space, be as in Theorem 8. If $\{R_t f, t \geq 1\}$ is a generalized martingale, with $R_1 1 = 1$, in both L^{Φ_1} and L^{Φ_2} (or in L^{Φ_2} and L^{Φ_3}) and that R_t is a positive contraction (in both cases), then $\lim_{t \rightarrow \infty} R_t f$ exists in the norm of L^Φ where $\Phi = \Phi_s^0$, $0 \leq s \leq 1$ and $\Phi = \Phi_s$, $0 \leq s < 1$ (or, $\Phi = \Phi_s$, $0 \leq s < 1$), where Φ_s , Φ_s^0 are the same as before.*

6. Final remarks. The following problems are related to the material of this paper and have yet to be worked out.

1. It will be of interest to extend the interpolation theory of Section 3, in the spirit of [11], to the L^Φ and H^Φ spaces (see [23], for the definition of the latter). This will enable a clarification and extension of the abstract treatments ([4] and [3]). The theory of integration (of the Lebesgue–Bochner type), generalizing [9], is also possible with the methods of this paper.

2. The work of Section 4 has some beneficial influence on some aspects of ergodic theory as well as the theory of Markov process in lifting the latter from the L^1 to the L^Φ -spaces.

3. It will be of considerable interest to extend the real variable method of M. Riesz, in his convexity theorem ([19], pp. 466–470), to the L^Φ -spaces where $\Phi_1 \prec \Phi \prec \Phi_2$, $Q_1 \prec Q \prec Q_2$ with a growth condition. This will give a new interpolation theorem not covered by this paper. In the notation of Section 4, it is possible to show that part of M. Riesz's work (pp. 466–469) extends and one can obtain the inequalities (all Young's functions being continuous),

$$MN_{\Psi_1}(h) \leq M_1 N_{Q_1}(g),$$

$$MN_{R_s}(\tilde{h}) \leq M_2 N_{\Phi_s}(f),$$

with $h = Tf$, $\tilde{h} = Tg$, $f \in L^{\Phi_1}$, $g \in L^{\Phi_2}$ and $T: L^{\Phi_1} \rightarrow L^{\Phi_2}$, with bounds M_i , $i = 1, 2$, f, g simple, and M is the bound of $T: L^\Phi \rightarrow L^Q$. Thus far, no

ordering relations between the Φ 's, Q 's, *etc.* are needed. Now suitable restrictions are needed to obtain the relations between M and M_1 and M_2 . These are yet to be found.

REFERENCES

- [1] M. A. AKCOGLU & R. V. CHACÓN, A convexity theorem for positive operators, *Z. Wahrscheinlichkeitstheorie*, **3** (1965) 328–332.
- [2] T. ANDÔ & I. AMEMIYA, Almost everywhere convergence of prediction sequence in $L^p(1 < p < \infty)$, *Z. Wahrscheinlichkeitstheorie*, **4** (1965) 113–120.
- [3] N. ARONSZAJN & E. GAGLIARDO, Interpolation spaces and interpolation methods, *Ann. Math. Pura ed Applicata* (IV), **68** (1965) 51–118.
- [4] A. P. CALDERÓN, Intermediate spaces and interpolation, the complex method, *Studia Math.*, **24** (1964) 113–190.
- [5] N. DUNFORD & J. T. SCHWARTZ, *Linear Operators: Part I*, Interscience, New York, 1958.
- [6] E. HILLE & R. S. PHILLIPS, *Functional Analysis and Semi-Groups*, Am. Math. Soc. Colloq. Publ. 31, Providence, 1957.
- [7] Y. ITO, Uniform integrability and the pointwise ergodic theorem, *Proc. Am. Math. Soc.*, **16** (1965) 222–227.
- [8] M. JERISON, Martingale formulation of ergodic theorems, *Proc. Am. Math. Soc.*, **10** (1959) 531–539.
- [9] G. L. KRABBE, Generalized measures whose values are operators into an intermediate space, *Math. Annalen*, **151** (1963) 219–238.
- [10] M. A. KRASNOSELSKII & YA. B. RUTICKII, *Convex Functions and Orlicz Spaces*, (Translation), P. Noordhoff Ltd., Groningen, 1961.
- [11] J. L. LIONS, Sur les espaces d'interpolation; dualité, *Math. Skand.*, **9** (1961) 147–177.
- [12] M. LOÈVE, *Probability Theory*, (3rd ed.), D. Van Nostrand Company, New York, 1963.
- [13] W. A. J. LUXEMBURG & A. C. ZAAENEN, Compactness of integral operators in Banach function spaces, *Math. Annalen*, **149** (1963) 150–180.
- [14] H. W. MILNES, Convexity of Orlicz spaces, *Pacific J. Math.*, **7** (1957) 1451–1483.
- [15] M. M. RAO, Linear functionals on Orlicz spaces, *Nieuw Arch. Wisk.*, **12** (1964) 77–98, and Part II, (to appear).
- [16] M. M. RAO, Smoothness of Orlicz spaces, *Indag. Math.*, **27** (1965) 670–690.
- [17] M. M. RAO, Conditional expectations and closed projections, *ibid*, **27** (1965) 100–112.
- [18] M. M. RAO, Notes on pointwise convergence of closed martingales, *ibid*, (to appear).
- [19] M. RIESZ, Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires, *Acta. Math.*, **49** (1926) 465–497.
- [20] G.-C. ROTA, Une théorie unifiée des martingales et des moyennes ergodiques, *Comptes Rendus*, **252** (1961) 2064–2066.
- [21] G.-C. ROTA, Reynolds operators, *Proc. Symp. Appl. Math. Am. Math. Soc.*, **16** (1964) 70–83.
- [22] E. M. STEIN & G. WEISS, Interpolation of operators with change of measures, *Trans. Am. Math. Soc.*, **87** (1958) 159–172.
- [22a] A. E. TAYLOR, *Introduction to Functional Analysis*, Wiley, New York, 1958.
- [23] G. WEISS, A note on Orlicz spaces, *Portuguese Mat.*, **15** (1956) 35–47.
- [24] A. C. ZAAENEN, *Linear Analysis*, North Holland Publishing Co., Amsterdam, 1953.
- [25] A. ZYGMUND, *Trigonometric Series, Vols. I, II.*, Cambridge University Press, 1959.

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