On the Fredholm Eigenvalues of Plane Domains

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1. Introduction. Linear integral equations play an important role in potential theory and in conformal mapping. They lead to constructive methods using Neumann series for solving the classical Dirichlet and Neumann problems in the plane and for finding conformal maps of simply connected domains. In many cases the Neumann series converges geometrically, and its rate depends on the least positive non-trivial Fredholm eigenvalue λ_2 of the corresponding homogeneous integral equation [1, 9, 7]. It is, therefore, of both theoretical and practical importance to estimate λ_2 and to consider the Fredholm eigenvalues as functionals of the boundary of a domain.

We first present a new lower bound for λ_2 in a very general setting, depending only on a bounded arc length—chord length ratio of the boundary curve (Theorem 1). We then give criteria concerning “convergence” of boundary curves which imply convergence of eigenvalues (Theorems 2 and 3). As an application we show with a mild condition on the boundary curve that the eigenvalue λ_2 is the decreasing limit of the corresponding eigenvalues for the level curves of a domain (Theorem 4).

2. The Fredholm eigenvalues. The Fredholm eigenvalues for a domain bounded by a smooth curve are values λ which satisfy the equation

$$(1) \quad \varphi(s) = \lambda \int_0^L \varphi(t) K(s, t) dt$$

for some continuous (eigen)function $\varphi(s) \not\equiv 0$. In this context the boundary of the domain is assumed to be a (closed) Jordan curve C of class C'_α (i.e., C may be represented parametrically in terms of its arc length s by the equation $z = \zeta(s)$, $0 \leq s \leq L$, where $\zeta'(s)$ satisfies a Hölder condition of order α). Then the Neumann–Poincaré kernel associated with C is

$$(2) \quad K(s, t) = -\frac{1}{\pi} \frac{\partial}{\partial n_t} \log |\zeta(s) - \zeta(t)|,$$

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where $\partial/\partial n_i$ indicates differentiation with respect to the inner normal from C at $\zeta(t)$. It is well known that the eigenvalues λ are all real and isolated and, except for the trivial eigenvalue $\lambda_1 = 1$, have modulus greater than one.

More generally, Fredholm eigenvalues are associated with a Jordan curve of bounded rotation (*i.e.*, a semi-tangent angle may be defined which is of bounded variation as a function of arc length) which is free of cusps. Then the Fredholm radius R is larger than one, and the eigenvalues are defined for $|\lambda| < R$ by the equation

$$(3) \quad \varphi(s) = \frac{\lambda}{\pi} \int_0^L \varphi(t) d\psi(s, t)$$

for continuous eigenfunctions $\varphi(s)$ (see [4]). The function

$$(4) \quad \psi(s, t) = \begin{cases} \arg [\zeta(t) - \zeta(s)], & \text{for } 0 \leq t < s, \\ \arg [\zeta(t) - \zeta(s)] + \pi, & \text{for } s < t \leq L, \end{cases}$$

is of bounded variation with respect to t . (Branches of the argument function are chosen so that $\psi(s, t)$ is continuous for $s \neq t$ and $\psi(s, t) = \psi(t, s)$.) Again the eigenvalues within the Fredholm radius are real and isolated and, except for the trivial eigenvalue $\lambda_1 = 1$, have modulus greater than one. If there are no eigenvalues except $\lambda_1 = 1$, we denote $\lambda_2 = R$. For smooth curves $R = \infty$ and equation (3) reduces to (1).

For a general rectifiable Jordan curve C one can define

$$(5) \quad 1/\lambda_2 = \sup \frac{|D_I(V) - D_E(V)|}{D_I(V) + D_E(V)}$$

where D_I and D_E are Dirichlet integrals over the interior and exterior of C , respectively, and the supremum is over all logarithmic potentials

$$V(z) = -\int_0^L g(t) \log |\zeta(t) - z| dt$$

with continuous density $g(t)$ for which $\int_0^L g(t) dt = 0$. For smooth curves this definition of λ_2 agrees with the least positive non-trivial eigenvalue of equation (1) (see [1, 8, 9]). It is in this generality that Theorem 1 is valid.

3. Lower bound for λ_2 . Denote by σ_{PQ} the shorter arc length and r_{PQ} the euclidean distance between points $P, Q \in C$. The following theorem gives a lower bound for λ_2 of equations (1), (3), and (5) with their respective conditions concerning the curve C .

Theorem 1. *Let C be a rectifiable Jordan curve with bounded arc length—chord length ratio; that is,*

$$(6) \quad \sigma_{PQ} \leq br_{PQ} \quad (1 < b < \infty)$$

uniformly for all $P, Q \in C$. Then the least positive non-trivial Fredholm eigenvalue

$$(7) \quad \lambda_2 \geq \frac{8e^{[\pi(1+b^{10}e^{2\pi})^2-1]/(.2284)} + 1}{8e^{[\pi(1+b^{10}e^{2\pi})^2-1]/(.2284)} - 1} > 1.$$

In the proof of Theorem 1 we use a simple geometric lemma:

Lemma 1. *Let C be a rectifiable Jordan curve. Suppose P, Q, R , and S are any four points of C which follow each other in that order. Then*

$$(8) \quad \sigma_{PQ} \cdot \sigma_{RS} \leq \sigma_{PR} \cdot \sigma_{QS}.$$

Proof. (i) Assume first that

$$(9) \quad \sigma_{PR} = \sigma_{PQ} + \sigma_{QR}.$$

Necessarily, either

$$(10) \quad \sigma_{QS} = \sigma_{QR} + \sigma_{RS}$$

or

$$(11) \quad \sigma_{QS} = \sigma_{PQ} + \sigma_{PS}.$$

If (10) is the case, then

$$\sigma_{PR} \cdot \sigma_{QS} = (\sigma_{PQ} + \sigma_{QR})(\sigma_{QR} + \sigma_{RS}) \geq \sigma_{PQ} \cdot \sigma_{RS}.$$

If (11) is the case, then

$$\sigma_{PR} \cdot \sigma_{QS} = (\sigma_{PQ} + \sigma_{QR})(\sigma_{PQ} + \sigma_{PS}) \geq \sigma_{PQ}(\sigma_{PQ} + \sigma_{PS} + \sigma_{QR}) \geq \sigma_{PQ} \cdot \sigma_{RS}$$

since $\sigma_{PQ} + \sigma_{PS} + \sigma_{QR}$ is the length of one of the arcs (not necessarily the shorter) between R and S . Then the lemma is proved assuming (9).

(ii) If the assumption (9) does not hold, then $\sigma_{PR} = \sigma_{PS} + \sigma_{RS}$. A similar analysis of the possibilities (10) and (11) yields the result.

Proof of Theorem 1. Ahlfors [1] proves that $1/\lambda_2 \leq k$ where k is the maximal eccentricity of a quasiconformal reflection with respect to C if one exists. In [2] he characterizes geometrically the curves which admit quasiconformal reflections. The condition is that

$$(12) \quad (r_{PQ} \cdot r_{RS}) / (r_{PR} \cdot r_{QS}) \leq A < \infty$$

for some constant A and any four points P, Q, R, S lying on the curve in that order. Condition (6), the fact that $r_{PQ} \leq \sigma_{PQ}$, and Lemma 1 imply that

$$\frac{r_{PQ} \cdot r_{RS}}{r_{PR} \cdot r_{QS}} \leq b^2 \frac{r_{PQ} \cdot r_{RS}}{\sigma_{PR} \cdot \sigma_{QS}} \leq b^2 \frac{\sigma_{PQ} \cdot \sigma_{RS}}{\sigma_{PR} \cdot \sigma_{QS}} \leq b^2.$$

Therefore the bound $A = b^2$ holds uniformly for the ratio (12) for the curve C . Since condition (12) is satisfied, a quasiconformal reflection exists and by [2, 3] (see also [8, p. 139]) may be chosen with maximal dilatation $K \leq [P^{-1}(B)]^2$ where $B = \pi(1 + A^6 e^{2\pi})^2 = \pi(1 + b^{10} e^{2\pi})^2$ and $P(\rho) = 1 + \theta(\rho) \log \rho$. The

function $\theta(\rho)$ increases from $\theta(1) = .2284$ to $\theta(\infty) = 1/\pi$. For our purposes we obtain a better bound using a result of Reed [5], namely, $K \leq 8P^{-1}(B)$. (The result referred to is the following: If the function $\mu(x)$ satisfies a ρ -condition (in the sense of [3]) then there exists a quasiconformal mapping of half-planes with boundary correspondence $x \rightarrow \mu(x)$ whose maximal dilatation is at most 8ρ (vs. ρ^2 in [3]).) Then $K \leq 8 \exp \{(B - 1)/(.2284)\}$, and since the maximal eccentricity $k = (K - 1)/(K + 1)$,

$$\lambda_2 \geq \frac{1}{k} \geq \frac{8e^{[\pi(1+b^{10}e^{2\pi})^2-1]/(.2284)} + 1}{8e^{[\pi(1+b^{10}e^{2\pi})^2-1]/(.2284)} - 1}.$$

Remark. One can easily extend Theorem 1 to curve systems $C = \bigcup_{k=1}^n C_k$ satisfying (6) on each C_k using the work of Springer [8]. In this case

$$\frac{\lambda_2 + 1}{\lambda_2 - 1} \leq K \frac{\lambda_2^* + 1}{\lambda_2^* - 1} \leq 8 \frac{\lambda_2^* + 1}{\lambda_2^* - 1} e^{[\pi(1+b^{10}e^{2\pi})^2-1]/(.2284)},$$

where the resulting bound involves the least eigenvalue λ_2^* larger than 1 for a conformally equivalent system of circles. A lower bound for λ_2^* is given in [10, p. 368].

4. Convergence of eigenvalues. We first introduce a normalization of the kernel $K(s, t)$. Parametrize the arc length s by a continuously differentiable function $s = s(x)$ on $[0, 1]$ with $s'(x) > 0$ (e.g., a linear function). Then define the normalized kernel $\hat{K}(x, y) = s'(y)K(s(x), s(y))$. As a result

$$\int_0^L f(t)K(s, t) dt = \int_0^1 f(s(y))\hat{K}(x, y) dy.$$

Theorem 2. Let $\{\lambda^{(n)}\}$ be a sequence of positive non-trivial Fredholm eigenvalues (not necessarily $\lambda_2^{(n)}$) for Jordan curves C_n , $n = 1, 2, \dots$. Assume that

- (i) the curves C_n satisfy $\sigma_{PQ} \leq br_{PQ}$ ($1 < b < \infty$) uniformly for all $P, Q \in C_n$, where b is independent of n , and
- (ii) there exists a Jordan curve C of class C'_α , $0 < \alpha \leq 1$, such that for suitably parametrized kernels $\hat{K}_n(x, y)$ and $\hat{K}(x, y)$ corresponding to C_n and C ,

$$(13) \quad \lim_{n \rightarrow \infty} \int_0^1 |\hat{K}_n(x, y) - \hat{K}(x, y)|^p dy = 0$$

uniformly for $0 \leq x \leq 1$, for some $p > 1$.

Then

- (a) the eigenvalues $\lambda^{(n)}$ are bounded uniformly away from 1, and
- (b) every limiting point of $\{\lambda^{(n)}\}$ is a non-trivial positive eigenvalue of C .

Proof. Part (a) is a consequence of Theorem 1. To prove (b) let $\{\lambda^{(n)}\}$ be a subsequence of $\{\lambda^{(n)}\}$ with limit Λ ($\Lambda > 1$ by (a)), and let $\{\varphi_n\}$ be the corre-

spending eigenfunctions with $L^p[0, 1]$ norm $\|\varphi_\nu\|_p = 1$,

$$P = \max \left(\frac{p}{p-1}, \frac{2-\alpha}{\alpha}, 2 \right).$$

By the Hölder and Minkowski inequalities

$$\begin{aligned} |\varphi_\nu(x)| &= \left| \lambda^{(\nu)} \int_0^1 \varphi_\nu(y) \hat{K}_\nu(x, y) dy \right| \leq m \|\hat{K}_\nu(x, \cdot)\|_q \\ &\leq m \|\hat{K}(x, \cdot)\|_q + m \|\hat{K}_\nu(x, \cdot) - \hat{K}(x, \cdot)\|_q, \end{aligned}$$

where $m = \sup_\nu \lambda^{(\nu)}$ and $q = P/(P-1) = \min(p, (2-\alpha)/(2-2\alpha), 2)$. Since C is of class C'_α , $K(s, t) |s-t|^{1-\alpha}$ is uniformly bounded and it follows that $\|\hat{K}(x, \cdot)\|_q \leq M < \infty$. Together with (13) this implies that $\{\varphi_\nu\}$ is a uniformly bounded sequence.

By the Hölder and Minkowski inequalities

$$\begin{aligned} |\varphi_\nu(x_1) - \varphi_\nu(x_2)| &= \left| \lambda^{(\nu)} \int_0^1 \varphi_\nu(y) [\hat{K}_\nu(x_1, y) - \hat{K}_\nu(x_2, y)] dy \right| \\ &\leq m \|\hat{K}_\nu(x_1, \cdot) - \hat{K}_\nu(x_2, \cdot)\|_q \\ &\leq m \|\hat{K}_\nu(x_1, \cdot) - \hat{K}(x_1, \cdot)\|_q + m \|\hat{K}(x_1, \cdot) - \hat{K}(x_2, \cdot)\|_q \\ &\quad + m \|\hat{K}(x_2, \cdot) - \hat{K}_\nu(x_2, \cdot)\|_q. \end{aligned}$$

The norm $\|\hat{K}(x_1, \cdot) - \hat{K}(x_2, \cdot)\|_q < \epsilon$ whenever $|x_1 - x_2| < \delta(\epsilon)$, and it follows from (13) that the other two norms can be made uniformly small as $n \rightarrow \infty$. Therefore $\{\varphi_\nu\}$ is an equicontinuous sequence.

By the Arzelà-Ascoli theorem the bounded equicontinuous sequence $\{\varphi_\nu\}$ has a uniformly convergent subsequence $\{\varphi_\mu\}$. Moreover, $\varphi(x) = \lim_{\mu \rightarrow \infty} \varphi_\mu(x)$ is continuous and not identically zero since $\|\varphi\|_p = 1$ by uniform convergence. Now $\varphi_\mu(x) = \lambda^{(\mu)} \int_0^1 \varphi_\mu(y) \hat{K}_\mu(x, y) dy$ where the $\varphi_\mu(x)$ converge uniformly to $\varphi(x)$ and $\lim_{\mu \rightarrow \infty} \lambda^{(\mu)} = \Lambda$. Therefore in order to prove that Λ is an eigenvalue of C with eigenfunction φ it is sufficient to show that $\lim_{\mu \rightarrow \infty} \int_0^1 \varphi_\mu(y) \hat{K}_\mu(x, y) dy = \int_0^1 \varphi(y) \hat{K}(x, y) dy$. But this is a consequence of the inequalities

$$\begin{aligned} \|\varphi_\mu \hat{K}_\mu(x, \cdot) - \varphi \hat{K}(x, \cdot)\|_1 &\leq \|\varphi_\mu\|_p \|\hat{K}_\mu(x, \cdot) - \hat{K}(x, \cdot)\|_q + \|\varphi_\mu - \varphi\|_p \|\hat{K}(x, \cdot)\|_q \\ &\leq \|\hat{K}_\mu(x, \cdot) - \hat{K}(x, \cdot)\|_q + \|\varphi_\mu - \varphi\|_p M, \end{aligned}$$

the hypothesis (13), and the uniform convergence of φ_μ to φ .

It is of further interest to consider the relationship between the least positive non-trivial eigenvalues $\lambda_2^{(n)}$ and λ_2 for the curves C_n approaching C .

Theorem 3. Suppose that the Jordan curves C_n are of class C'_{α_n} , $0 < \alpha_n \leq 1$, and satisfy hypotheses (i) and (ii) of Theorem 2. Assume that the curves C_n are all in the interior of C or all in the exterior of C and that the area between them diminishes to zero as $n \rightarrow \infty$. If $\lambda_2^{(n)}$ is the least positive non-trivial eigenvalue of C_n , then $\lim_{n \rightarrow \infty} \lambda_2^{(n)} = \lambda_2$ is the least positive non-trivial Fredholm eigenvalue of C .

Proof. By Theorem 2, $\lim_{n \rightarrow \infty} \lambda_2^{(n)} = \Lambda$ is a positive non-trivial eigenvalue of C . Therefore $\lambda_2 \leq \Lambda$. Hence it is sufficient to prove also that $\lambda_2^{-1} \leq \Lambda^{-1}$. One can show (see [9, pp. 21–23]) that $\lambda_2^{-1} \leq (\lambda_2^{(n)})^{-1} + \pi^{-1} D_n(V)$, where $D_n(V)$ is the Dirichlet integral over the region between C_n and C of a certain logarithmic potential with density on C . Since the area between C_n and C approaches zero as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} D_n(V) = 0$. Therefore $\lambda_2^{-1} \leq \Lambda^{-1}$.

We apply Theorem 3 to level curves of a domain. Let $z = z(w)$ be a conformal mapping of $|w| < 1$ onto the interior of a Jordan curve C . Then the level curve C_r , $0 < r < 1$, of the interior of C is the image of $|w| = r$ under the mapping.

Theorem 4. *Let C be a Jordan curve of class C'_α . Then the family $\{\lambda_2^{(r)}\}$ ($0 < r < 1$) of least positive non-trivial Fredholm eigenvalues of level curves for the interior of C decreases monotonically as $r \rightarrow 1$ to λ_2 , the least positive non-trivial Fredholm eigenvalue of C .*

Proof. Schiffer [6, p. 1209] proved that the eigenvalues $\lambda_2^{(r)}$ decrease monotonically as r increases; hence $\lim_{r \rightarrow 1} \lambda_2^{(r)} = \Lambda$ exists. To infer that $\Lambda = \lambda_2$ it is sufficient to show that C and its level curves satisfy the hypothesis of Theorem 3.

To verify condition (13) we normalize the kernels to $[0, 1]$ using the mapping function $z(re^{2\pi i x}) = \xi(r, x) + i\eta(r, x)$ from the unit disk to the interior of C . Then

$$(14) \quad \hat{K}_r(x, y) = \frac{1}{|z(re^{2\pi i x}) - z(re^{2\pi i y})|^2} \begin{vmatrix} \xi(r, y) - \xi(r, x) & \frac{\partial}{\partial y} \xi(r, y) \\ \eta(r, y) - \eta(r, x) & \frac{\partial}{\partial y} \eta(r, y) \end{vmatrix}.$$

Since $C \in C'_\alpha$,

$$0 < m \leq \left| \frac{z(re^{2\pi i x}) - z(re^{2\pi i y})}{x - y} \right| \leq M$$

uniformly for $0 \leq r \leq 1$, and for $0 < |x - y| \leq \delta$

$$|\hat{K}_r(x, y)| \leq \frac{\tilde{M}}{m^2} |x - y|^{\alpha-1}$$

uniformly in r , by Kellogg's theorem. Therefore for fixed p , $1 < p < 1/(1 - \alpha)$,

$$\begin{aligned} \int_{x-\delta}^{x+\delta} |\hat{K}_r(x, y) - \hat{K}_1(x, y)|^p dy \\ \leq \frac{2^p \tilde{M}^p}{m^{2p}} \int_{x-\delta}^{x+\delta} |x - y|^{(\alpha-1)p} dy = \frac{2^{p+1} \tilde{M}^p \delta^{1+(\alpha-1)p}}{m^{2p}[1 + (\alpha-1)p]} \end{aligned}$$

which is independent of x . Given $\epsilon > 0$, this expression can be made less than $\epsilon/2$ by choosing δ sufficiently small. Then

$$(15) \quad \int_0^1 |\hat{K}_r(x, y) - \hat{K}_1(x, y)|^p dy \leq \int_0^{x-\delta} + \frac{1}{2} \epsilon + \int_{x+\delta}^1.$$

For $|x - y| \geq \delta > 0$ the determinant in (14) converges uniformly in x as $r \rightarrow 1$ by Kellogg's theorem and its coefficient converges uniformly by uniform continuity. Therefore the remaining integrals in (15) can be made less than $\epsilon/2$ uniformly in x by choosing r sufficiently close to 1. Since ϵ is arbitrary, condition (13) is verified. The remaining hypotheses of Theorem 3 are obviously satisfied.

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