Existence and Uniqueness Theorems for a Class of Non-Linear Singular Integral Equations with Application to a Hydroelastic Problem

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Introduction. In this paper we shall be concerned with showing the existence and uniqueness of a solution, in the class of complex-valued Hölder continuous functions, of non linear singular integral equations with Cauchy type singularity. More specifically but without all hypotheses, we shall be concerned with the integral equations

$$w(z) = B(z) + \lambda C(w) + \lambda F[z, w(z), \lambda] + \frac{\lambda}{2\pi i} \int_{\Gamma} \frac{N[t, z, w(t), \lambda]}{t - z} dt,$$

where λ is a complex parameter ($|\lambda|$ small), Γ is a smooth contour in the complex t -plane, z is a point of Γ (the Cauchy principal value of the integral is understood), and for $z \in \Gamma$ and $w \in$ some closed convex set B , F and N are complex-valued functions and C is a bounded functional satisfying a certain functional smoothness condition. A definite bound for $|\lambda|$ in terms of bounds and Hölder constants for B , C , F and N and a geometrical constant for Γ will be given. We shall prove the existence and uniqueness of a solution for $|\lambda|$ not greater than this bound.

We shall prove the existence and uniqueness of a solution by successive approximations. Due to the length of the proof by successive approximations, we have proved preliminary results in the form of four lemmas.

The four preliminary lemmas mentioned are concerned briefly with: lemma 1 the inheritance of constants in Hölder conditions, lemma 2 an extension of the Plemelj-Privalov theorem with an evaluation of constants, lemma 3 an inheritance of Hölder constants for the successive approximations, lemma 4 also an inheritance statement.

There has been much interest recently in non linear integral equations as

is seen by the conference [1] held at Wisconsin and by the extensive bibliography of Krasnoselski [11]. The study of non-linear singular integral equations (with strong singularity) has arisen in the study of conformal mapping, the non linear Riemann–Hilbert problem and differential equations. The two main authors in this field are A. I. Guseinof [3] → [10] and W. Pogorzelski [15], [16], and [17]. There are certain basic differences and similarities between these papers, one of the major differences is: Γ can be a closed contour or an arc and one of the major similarities is: there is a parameter involved which must be small. In this paper, though we consider only a single contour, the form of the integral equation is more general than any of those considered in the above mentioned papers. Also a peculiarity of this paper is the unusual restriction on the relative range of the solution function and $B(z)$.

The physical problem which we shall describe briefly here, and in detail later, was the motivating factor that prompted us to consider the class of non-linear singular integral equations mentioned. Consider an isotropic, homogeneous infinite circular cylindrical rod with given elastic properties whose axis is perpendicular to an incompressible irrotational fluid flow. What will be the change in shape of the rod due to the pressure of the fluid? Consider a plane perpendicular to the axis of the rod. Outside the closed contour, the harmonic equation governs the fluid flow, and thus the pressure on the boundary of the closed contour; whereas, inside the closed contour, the biharmonic equation governs the stress and strain and thus the displacement. We must solve the biharmonic equation, subject to the pressure on the contour, *i.e.* subject to the solution of the harmonic equation, as a function of the boundary. From the solution of the biharmonic equation we calculate the displacement, this evaluated on the boundary will give the distortion from the original shape, due to the pressure of the fluid, and thus the solution to our problem.

In order to formulate this problem mathematically, we used the method developed by Muskhelishvili [13]. We have a system of two coupled ordinary differential equations given on the boundary of the unstressed contour:

$$\begin{aligned}\varphi(\sigma) + \overline{\sigma\varphi'(\sigma)} + \overline{\Psi(\sigma)} &= f(\sigma), \\ \kappa\varphi(\sigma) - \overline{\sigma\varphi'(\sigma)} - \overline{\Psi(\sigma)} &= 2\mu g(\sigma), \quad \sigma = e^{i\theta},\end{aligned}$$

where κ and μ are elastic constants, φ and Ψ are unknown complex-valued functions analytic in the unit circle, to be determined, $f(\sigma)$ is a function determined by the force per unit arc length on the undeformed boundary and thus is given from the solution of the harmonic equation and $g(\sigma)$ is the complex displacement vector. Remarkably enough, we managed to reduce the above question to solving a single integral equation of the form:

$$g'(t) = \alpha(t) + C(g') + U^2 L[t, g'(t)] + U^2 \frac{1}{2\pi i} \int_{|\sigma|=1} \frac{H[\sigma, g'(\sigma)]}{t - \sigma} d\sigma,$$

where U is the velocity of the uniform stream. This equation is of the type we have considered. Observe that insisting on $|\lambda|$ being bounded by a certain

constant is, in this application, the same as saying the uniform stream (*i.e.* the velocity of the fluid beyond disturbance due to the rod) is bounded.

Notation. Let Ω be a closed convex set of the complex z -plane and let F_ϵ be a closed subset of Ω . Let Γ be a simple smooth closed contour. Let $F[z, w, \lambda]$ and $N[t, z, w, \lambda]$ be complex valued functions of the complex variables z, w, λ and t, z, w , and λ respectively, and defined for z on Γ , t in Γ , w on Ω , $|\lambda| \leq \lambda_0 = \text{constant}$. Let $z = x + iy$, $w = u + iv$, $t = r + is$, $\lambda = \lambda_1 + i\lambda_2$ and

$$\begin{aligned} F[z, w, \lambda] &= \operatorname{Re} F[z, w, \lambda] + i \operatorname{Im} F[z, w, \lambda] \\ &= F_1[x, y, u, v, \lambda_1, \lambda_2] + iF_2[\dots], \end{aligned}$$

and similarly:

$$N[t, z, w, \lambda] = N_1[r, s, x, y, u, v, \lambda_1, \lambda_2] + iN_2[\dots].$$

By $N[t, z, w, \lambda] \in H[\eta'_1, \mu'_1; \eta'_2, \mu'_2; \eta'_3, \mu'_3]$ we shall mean for t_i and z_i on Γ ($i = 1, 2$) and w_i in Ω , $i = 1, 2$,

$$\begin{aligned} |N[t_2, z_2, w_2, \lambda] - N[t_1, z_1, w_1, \lambda]| \\ < \eta'_1 |t_2 - t_1|^{\mu'_1} + \eta'_2 |z_2 - z_1|^{\mu'_2} + \eta'_3 |w_2 - w_1|^{\mu'_3} \end{aligned}$$

and by

$$\frac{\partial N[t, z, w, \lambda]}{\partial u} \in H[\eta_1, \mu_1; \eta_2, \mu_2; \eta_3, \mu_3]$$

we shall mean for $t_i = r_i + is_i$, $z_i = x_i + iy_i$, $w_i = u_i + iv_i$,

$$\begin{aligned} \left| \frac{\partial N_j[r_2, s_2, x_2, y_2, u_2, v_2, \lambda_1, \lambda_2]}{\partial u} - \frac{\partial N_j[r_1, s_1, x_1, y_1, u_1, v_1, \lambda_1, \lambda_2]}{\partial u} \right| \\ < \eta_1 |t_2 - t_1|^{\mu_1} + \eta_2 |z_2 - z_1|^{\mu_2} + \eta_3 |w_2 - w_1|^{\mu_3}, \end{aligned}$$

for $j = 1, 2$. Similarly for $F[z, w, \lambda]$.

Theorem 1. *Given the non-linear singular integral equation:*

$$(E) \quad w(z) = B(z) + \lambda C(w) + \lambda F[z, w(z), \lambda] + \frac{\lambda}{2\pi i} \int_{\Gamma} \frac{N[t, z, w(t), \lambda]}{t - z} dt,$$

where

$$(H.1) \quad \rho(F_\epsilon \bar{\Omega}^\circ) \equiv \inf_{\substack{z \text{ in } F_\epsilon \\ t \text{ in } \bar{\Omega}^\circ}} |z - t| \geq \epsilon > 0.$$

$$(H.2) \quad B(z) \text{ is a single-valued function of } z, \text{ and for } z, z_1, z_2 \text{ on } \Gamma. B(z) \in F_\epsilon \text{ and } |B(z_2) - B(z_1)| < \beta |z_2 - z_1|^\mu, 0 < \mu < 1.$$

(H.3) $F[z, w, \lambda]$ is a function defined and single valued for $|\lambda| \leq \lambda_0$, z on Γ , $w \in \Omega$, with $F[z, w, \lambda] \in H[\eta_2^{*'}, \mu; \eta_3^{*'}, 1]$, F has derivatives with respect to u and v which satisfy:

$$\frac{\partial F}{\partial u} [z, w, \lambda] \in H[\eta_2^*; \mu; \eta_3^*, 1] \quad \text{and} \quad \frac{\partial F}{\partial v} [z, w, \lambda] \in H[\eta_2^*; \mu; \eta_3^*, 1],$$

where η_2^* , η_3^* , $\eta_2^{*'}$, $\eta_3^{*'}$ are constants independent of λ .

$$(H.4) \quad |F[z, w, \lambda]| < \bar{F}_0$$

and

$$\left| \frac{\partial F_j [x, y, u, v, \lambda_1, \lambda_2]}{\partial R} \right| < \bar{F}, \quad (j = 1, 2) (R = u, v),$$

for z on Γ , $w \in \Omega$ and $|\lambda| \leq \lambda_0$ where $\bar{F}_0$ and $\bar{F}$ are numerical constants.

(H.5) $N[t, z, w, \lambda]$ is a function defined and single-valued for $|\lambda| \leq \lambda_0$, z on Γ , t on Γ , $w \in \Omega$; with $N[t, z, w, \lambda] \in H[\eta_2', \mu; \eta_1', \mu_1; \eta_3', 1]$, N has derivatives with respect to u and v which satisfy:

$$\frac{\partial N}{\partial u} [t, z, w, \lambda] \in H[\eta_2, \mu; \eta_1, \mu_1; \eta_3, 1]$$

and

$$\frac{\partial N}{\partial v} [t, z, w, \lambda] \in H[\eta_2, \mu; \eta_1, \mu_1; \eta_3, 1].$$

$$(H.6) \quad |N(t, z, w, \lambda)| < \bar{N}_0 > 0 \quad \text{and} \quad \left| \frac{\partial N_j [r, s, x, y, u, v, \lambda_1, \lambda_2]}{\partial R} \right| < \bar{N},$$

$j = 1, 2$, $R = u, v$, for t on Γ , z on Γ , $w \in \Omega$, and $|\lambda| \leq \lambda_0$, where $\bar{N}_0$ and $\bar{N}$ are numerical constants.

(H.7) $C(w)$ is defined for $w \in \Omega$ and $|C(w)| \leq C_0$ for $w \in \Omega$.

(H.8) $|C(w_2) - C(w_1)| \leq C_1 ||w_2 - w_1||_e \equiv \max_{t \in \Gamma} |w_2(t) - w_1(t)|$.

Then among the set of functions defined on Γ that satisfy a Hölder condition with exponent μ and constant $K > \beta$ there is a unique single valued solution to the above equation which belongs to Ω provided $|\lambda|$ is sufficiently small.

Note: $|\lambda| < \min \{ \lambda_0, \lambda_1, \frac{1}{2}\lambda_2, (2\bar{b})^{-1}, \epsilon\lambda_1/2(K - \beta) \}$, where

$$\lambda_1 = \frac{K - \beta}{\eta_2^{*'} + \eta_3^{*'} K + A^*(\tilde{a}_1, \tilde{a}_2)\bar{N}_0}, \quad \lambda_2 = \frac{\epsilon}{\bar{F}_0 + \bar{N}_0 A(1, \tilde{a}_1) + C_0},$$

$$b = C_1 + A(\eta_3', a_1) + \eta_3^{*'}, \quad \bar{b} = b + \bar{b}$$

$$\bar{b} = 4\bar{F} + 4\eta_2^* + 12\eta_3^* K + A^*(a_1, a_2),$$

$$\tilde{a}_1 = (\eta_2' + K\eta_3')\bar{N}_0^{-1}, \quad \tilde{a}_2 = \eta_1'\bar{N}_0^{-1},$$

$$\begin{aligned}
A^*(\tilde{a}_1, \tilde{a}_2) &= \tilde{a}_1 \left\{ (2\pi k_0 \mu)^{-1} [1 + 2^{1+\mu} + 3^\mu] \right. \\
&\quad \left. + [\pi(1 - \mu)]^{-1} k_0^{-2+\mu} + \frac{1}{2} \left(1 + \frac{1}{\pi} |\pi i - \log k_0| \right) \right\} \\
&\quad + \frac{1}{2} \tilde{a}_2 \left\{ \left[1 + \frac{1}{\pi} |\pi i - \log k_0| \right] |z_2 - z_1|^{\mu_1 - \mu} + 2[\pi k_0(\mu_1 - \mu)e]^{-1} (|\Gamma|/4)^{\mu_1 - \mu} \right\}, \\
A(g, a_1) &= \frac{1}{2\pi} \left[\frac{1}{2} g + k_0^{\mu-1} a_1 \left(\frac{|\Gamma|}{2} \right) \frac{2}{\mu} \right], \quad a_1 = 4\bar{N}\bar{g} + 4\eta_2 + 12\eta_3 K, \quad a_2 = 4\eta_1, \\
0 < k_0 &\leq |z(s_2) - z(s_1)|/|s_2 - s_1| \leq 1, \quad \Gamma : z = z(s), \quad s = \text{arc length}, \\
\epsilon &= \inf_{t \in F^*, z \in \bar{\Omega}^c} |t - z| > 0.
\end{aligned}$$

Though the assumptions are clearly not independent, we have stated them to get a definite bound on $|\lambda|$ in terms of them.

Before proving the theorem, we shall prove a sequence of four lemmas.

Lemma 1. *Let $w_{n-1}(t)$ and $w_n(t)$ be defined and in Ω for t on Γ and*

$$(1.1) \quad |w_{n-1}(t_2) - w_{n-1}(t_1)| < K |t_2 - t_1|^\mu, \quad \text{for } t_1 \text{ and } t_2 \text{ on } \Gamma$$

$$|w_n(t_2) - w_n(t_1)| < K |t_2 - t_1|^\mu,$$

$$(1.2) \quad \text{for } \Delta_n(t) \equiv w_n(t) - w_{n-1}(t),$$

$$(a) \quad |\Delta_n(t)| < K_n \quad \text{for } t \text{ on } \Gamma,$$

$$(b) \quad |\Delta_n(t_2) - \Delta_n(t_1)| \leq \bar{g} K_n |t_2 - t_1|^\mu,$$

$$(1.3) \quad (\text{H.5}) \text{ holds,}$$

$$(1.4) \quad (\text{H.6}) \text{ holds.}$$

Then for

$$\Delta_n^*(t, z, \lambda) \equiv N[t, z, w_n(t), \lambda] - N[t, z, w_{n-1}(t), \lambda],$$

we have

$$(1.5) \quad |\Delta_n^*(t_2, z_2, \lambda) - \Delta_n^*(t_1, z_1, \lambda)| < K_n [a_2 |z_2 - z_1|^{\mu_1} + a_1 |t_2 - t_1|^\mu],$$

where

$$(a) \quad a_1 = 4\bar{N}\bar{g} + 4\eta_2 + 12\eta_3 K,$$

$$(b) \quad a_2 = 4\eta_1,$$

$$(c) \quad a_1 \text{ and } a_2 \text{ are constants independent of } n \text{ and } \lambda.$$

Proof. Since w_{n-1} and w_n are in Ω and Ω is convex, $w_{n-1} + \nu[w_n - w_{n-1}] \in \Omega$ for $0 \leq \nu \leq 1$. Consider

$$(1.6) \quad N^*(\nu) = N\{t_2, z_2, w_{n-1}(t_2) + \nu[w_n(t_2) - w_{n-1}(t_2)], \lambda\}$$

$$- N\{t_1, z_1, w_{n-1}(t_1) + \nu[w_n(t_1) - w_{n-1}(t_1)], \lambda\},$$

where ν is a real variable, $0 \leq \nu \leq 1$. $N^*(\nu)$ is a complex-valued function of ν .

Let

$$\Delta \Delta_n^* \equiv \Delta_n^*(t_2, z_2, \lambda) - \Delta_n^*(t_1, z_1, \lambda);$$

then

$$\begin{aligned} \Delta \Delta_n^* &= N[t_2, z_2, w_n(t_2), \lambda] - N[t_2, z_2, w_{n-1}(t_2), \lambda] \\ &\quad - \{N[t_1, z_1, w_n(t_1), \lambda] - N[t_1, z_1, w_{n-1}(t_1), \lambda]\}. \end{aligned}$$

And thus

$$(1.7) \quad \Delta \Delta_n^* = N^*(1) - N^*(0).$$

Let

$$(1.8) \quad N_1^*(\nu) \equiv \operatorname{Re} N^*(\nu) \quad \text{and} \quad N_2^*(\nu) \equiv \operatorname{Im} N^*(\nu). \quad \text{Then since (H.5)}$$

holds, we can use the mean value theorem to conclude:

$$(1.9) \quad \begin{aligned} N_1^*(1) - N_1^*(0) &= dN_1^*(\nu)/d\nu|_{\nu=\nu_1}, \quad 0 < \nu_1 < 1, \\ N_2^*(1) - N_2^*(0) &= dN_2^*(\nu)/d\nu|_{\nu=\nu_2}, \quad 0 < \nu_2 < 1. \end{aligned}$$

Let

$$(1.10) \quad \begin{aligned} t &= r + is, \quad z = x + iy, \quad w_n(t_k) = u(r_k, s_k) + iv(r_k, s_k), \\ p_k(\nu) &\equiv w_n(t_k) - \nu[w_n(t_k) - w_{n-1}(t_k)], \quad 0 \leq \nu \leq 1, \\ m_k(\nu) &= \operatorname{Re} p_k(\nu), \quad n_k(\nu) = \operatorname{Im} p_k(\nu), \\ \tau_k &= r_k, s_k, x_k, y_k. \end{aligned}$$

Then from (1.6)

$$N_j^*(\nu) = N_i[\tau_2, m_2(\nu), n_2(\nu), \lambda] - N_i[\tau_1, m_1(\nu), n_1(\nu), \lambda], \quad j = 1, 2,$$

where we have written λ for λ_1, λ_2 , and

$$(1.11) \quad \begin{aligned} \frac{dN_j^*(\nu)}{d\nu} &= \left\{ \frac{dm_2(\nu)}{d\nu} \frac{\partial}{\partial m_2} + \frac{dn_2(\nu)}{d\nu} \frac{\partial}{\partial n_2} \right\} N_i[\tau_2, m_2(\nu), n_2(\nu), \lambda] \\ &\quad - \left\{ \frac{dm_1(\nu)}{d\nu} \frac{\partial}{\partial m_1} + \frac{dn_1(\nu)}{d\nu} \frac{\partial}{\partial n_1} \right\} N_i[\tau_1, m_1(\nu), n_1(\nu), \lambda], \quad (j = 1, 2). \end{aligned}$$

From (1.10) we see that

$$(1.12) \quad \begin{aligned} \frac{dm_k(\nu)}{d\nu} &= -[u_n(r_k, s_k) - u_{n-1}(r_k, s_k)] = -\operatorname{Re} \Delta_n(t_k), \quad k = 1, 2, \\ \frac{dn_k(\nu)}{d\nu} &= -[v_n(r_k, s_k) - v_{n-1}(r_k, s_k)] = -\operatorname{Im} \Delta_n(t_k), \quad k = 1, 2. \end{aligned}$$

Introducing (1.12) into (1.11) we get

$$(1.13) \quad -\frac{dN^*(\nu)}{d\nu} = \left\{ \operatorname{Re} \Delta_n(t_2) \frac{\partial}{\partial m_2} + \operatorname{Im} \Delta_n(t_2) \frac{\partial}{\partial n_2} \right\} N_i[\tau_2, m_2, n_2, \lambda] \\ - \left\{ \operatorname{Re} \Delta_n(t_1) \frac{\partial}{\partial m_1} + \operatorname{Im} \Delta_n(t_1) \frac{\partial}{\partial n_1} \right\} N_i[\tau_1, m_1, n_1, \lambda].$$

But from (1.8) and (1.9) we see that

$$(1.14) \quad N^*(1) - N^*(0) = N^*(1) + iN_2^*(1) - N_1^*(0) - iN_2^*(0) \\ = \frac{dN_1^*(\nu)}{d\nu} \Big|_{\nu=\nu_1} + i \frac{dN_2^*(\nu)}{d\nu} \Big|_{\nu=\nu_2}, \quad 0 < \nu_1, \nu_2 < 1.$$

Utilizing (1.13), (1.14) becomes

$$(1.15) \quad -[N^*(1) - N^*(0)] \\ = \left\{ \operatorname{Re} \Delta_n(t_2) \frac{\partial}{\partial m_2} + \operatorname{Im} \Delta_n(t_2) \frac{\partial}{\partial n_2} \right\} N_1[\tau_2, m_2(\nu_1), n_2(\nu_1), \lambda] \\ - \left\{ \operatorname{Re} \Delta_n(t_1) \frac{\partial}{\partial m_1} + \operatorname{Im} \Delta_n(t_1) \frac{\partial}{\partial n_1} \right\} N_1[\tau_1, m_1(\nu_1), n_1(\nu_1), \lambda] \\ + i \left\{ \operatorname{Re} \Delta_n(t_2) \frac{\partial}{\partial m_2} + \operatorname{Im} \Delta_n(t_2) \frac{\partial}{\partial n_2} \right\} N_2[\tau_2, m_2(\nu_2), n_2(\nu_2), \lambda] \\ - i \left\{ \operatorname{Re} \Delta_n(t_1) \frac{\partial}{\partial m_1} + \operatorname{Im} \Delta_n(t_1) \frac{\partial}{\partial n_1} \right\} N_2[\tau_1, m_1(\nu_2), n_1(\nu_2), \lambda].$$

From (1.15) we see on recombining terms that

$$(1.16) \quad -[N^*(1) - N^*(0)] = \{ \operatorname{Re} \Delta_n(t_2) - \operatorname{Re} \Delta_n(t_1) \} \\ \cdot \left\{ \frac{\partial N_1[\tau_2, m_2(\nu_1), n_2(\nu_1), \lambda]}{\partial m_2} + i \frac{\partial N_2[\tau_2, m_2(\nu_2), n_2(\nu_2), \lambda]}{\partial m_2} \right\} \\ + \operatorname{Re} \Delta_n(t_1) \left\{ \frac{\partial N_1[\tau_2, m_2(\nu_1), n_2(\nu_1), \lambda]}{\partial m_2} - \frac{\partial N_1[\tau_1, m_1(\nu_1), n_1(\nu_1), \lambda]}{\partial m_1} \right\} \\ + i \operatorname{Re} \Delta_n(t_1) \left\{ \frac{\partial N_2[\tau_2, m_2(\nu_2), n_2(\nu_2), \lambda]}{\partial m_2} - \frac{\partial N_2[\tau_1, m_1(\nu_2), n_1(\nu_2), \lambda]}{\partial m_1} \right\} \\ + \{ \operatorname{Im} \Delta_n(t_2) - \operatorname{Im} \Delta_n(t_1) \} \\ \cdot \left\{ \frac{\partial N_1[\tau_2, m_2(\nu_1), n_2(\nu_1), \lambda]}{\partial n_2} + i \frac{\partial N_2[\tau_2, m_2(\nu_2), n_2(\nu_2), \lambda]}{\partial n_2} \right\} \\ + \operatorname{Im} \Delta_n(t_1) \left\{ \frac{\partial N_1[\tau_2, m_2(\nu_1), n_2(\nu_1), \lambda]}{\partial n_2} - \frac{\partial N_1[\tau_1, m_1(\nu_1), n_1(\nu_1), \lambda]}{\partial n_1} \right\} \\ + i \operatorname{Im} \Delta_n(t_1) \left\{ \frac{\partial N_2[\tau_2, m_2(\nu_2), n_2(\nu_2), \lambda]}{\partial n_2} - \frac{\partial N_2[\tau_1, m_1(\nu_2), n_1(\nu_2), \lambda]}{\partial n_1} \right\}.$$

Utilizing (1.1), (1.2), (1.3), and (1.4) we get from (1.16),

$$|N^*(1) - N^*(0)| < K_n |t_2 - t_1|^\mu [4\bar{N}\bar{g} + K_n\{4\eta_2 |t_2 - t_1|^\mu + 4\eta_1 |z_2 - z_1|^{\mu_1} \\ + 2\eta_3 |p_2(\nu_1) - p_1(\nu_1)| + 2\eta_3 |p_2(\nu_2) - p_1(\nu_2)|\}].$$

Collecting terms using (1.10) and the fact that $0 < \nu_1 < 1$ and $0 < \nu_2 < 1$, we get

$$|N^*(1) - N^*(0)| < |t_2 - t_1|^\mu \{4\bar{N}\bar{g}K_n + 4\eta_2 K_n\} + |z_2 - z_1|^{\mu_1} 4\eta_1 K_n \\ + 2K_n\eta_3 |t_2 - t_1|^\mu \{K + 2\nu_1 K + K + 2\nu_2 K\} \\ < K_n |t_2 - t_1|^\mu [4\bar{N}\bar{g} + 4\eta_2 + 12K\eta_3] + K_n |z_2 - z_1|^{\mu_1} 4\eta_1,$$

which proves the lemma since by (1.7)

$$\Delta \Delta_n^* = N^*(1) - N^*(0).$$

Lemma 2. *Let:*

- (2.1) Γ be a simple smooth closed rectifiable contour,
- (2.2) $\Delta_n^*[t, z, \lambda]$ be a complex function defined and single valued for z and t on Γ , $|\lambda| \leq \lambda_0$,
- (2.3) $|\Delta_n^*[t, z, \lambda]| < K_n g$ for t and z on Γ and $|\lambda| \leq \lambda_0$, where K_n is a constant and g is a constant independent of n ,
- (2.4) $|\Delta_n^*[t_2, z_2, \lambda] - \Delta_n^*[t_1, z_1, \lambda]| < K_n[a_1 |t_2 - t_1|^\mu + a_2 |z_2 - z_1|^{\mu_1}],$
 $1 > \mu_1 > \mu > 0$ for t_i and z_i ($i = 1, 2$) on Γ and $|\lambda| \leq \lambda_0$.

Then for z on Γ we have for

$$\gamma(z, \lambda) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Delta_n^*[t, z, \lambda]}{t - z} dt,$$

$$(2.5) \quad |\gamma(z, \lambda)| < K_n A,$$

$$(2.6) \quad |\gamma(z_2, \lambda) - \gamma(z_1, \lambda)| < K_n A^* |z_2 - z_1|^\mu,$$

where

$$(2.6.1) \quad A = A(g, a_1) = \frac{1}{2\pi} \left[\frac{1}{2} g + \frac{1}{k_0^{1-\mu}} a_1 \left(\frac{|\Gamma|}{2} \right)^\mu \frac{2}{\mu} \right]$$

and

$$(2.6.2) \quad A^* = A^*(a_1, a_2) = \frac{a_1}{2\pi k_0 \mu} [1 + 2^{1+\mu} + 3^\mu] \\ + \frac{1}{2} \left[1 + \frac{1}{\pi} |-\log k_0 + \pi i| \right] [a_2 |z_2 - z_1|^{\mu_1-\mu} + a_1] \\ + \frac{a_1}{\pi(1-\mu)k_0^{2-\mu}} + \frac{a_2}{\pi k_0(\mu_1 - \mu)e} \left(\frac{|\Gamma|}{4} \right)^{\mu_1-\mu},$$

where $|\Gamma|$ is the length of Γ and k_0 is such that if $\Gamma : z = z(s)$, $s = \text{arc length}$ and $z(s_1) = z_1$ and $z(s_2) = z_2$ then

$$0 < k_0 \leq \left| \frac{z_2 - z_1}{s_2 - s_1} \right| \leq 1.$$

Proof. Let $t = t(s)$, $s = \text{arc length}$ of Γ , $z_1 = t(s_1)$, $z_2 = t(s_2)$, $h = z_2 - z_1$, $s_2 - s_1 = \sigma$ where $0 < 2\sigma < \frac{1}{2} |\Gamma|$, $|\Gamma| = \text{length of } \Gamma$. Then

$$\begin{aligned} \gamma(z_2, \lambda) &= \frac{1}{2\pi i} \left(\int_{\Gamma_1} + \int_{\Gamma_2} \right) \frac{\Delta_n^*[t, z_1 + h, \lambda] - \Delta_n^*[z_1 + h, z_1 + h, \lambda]}{t - (z_1 + h)} dt \\ &\quad + \frac{1}{2\pi i} \Delta_n^*[z_1 + h, z_1 + h, \lambda] \int_{\Gamma} \frac{dt}{t - z_2}, \\ \gamma(z_1, \lambda) &= \frac{1}{2\pi i} \left(\int_{\Gamma_1} + \int_{\Gamma_2} \right) \frac{\Delta_n^*[t, z_1, \lambda] - \Delta_n^*[z_1, z_1, \lambda]}{t - z_1} dt \\ &\quad + \frac{1}{2\pi i} \Delta_n^*[z_1, z_1, \lambda] \int_{\Gamma} \frac{dt}{t - z_1}, \end{aligned}$$

where Γ_1 is that part of Γ containing z_1 such that the end points are t' and t'' , where $t' = t(s_1 - 2\sigma)$, $t'' = t(s_1 + 2\sigma)$ and $\Gamma_2 = \Gamma - \Gamma_1$. Let us denote the six terms by J_k , $k = 1, 2, \dots, 6$ respectively. Then by the Plemelj formulas

$$(2.7) \quad J_3 = \frac{1}{2} \Delta_n^*[z_1 + h, z_1 + h, \lambda]$$

and

$$(2.8) \quad J_6 = \frac{1}{2} \Delta_n^*[z_1, z_1, \lambda].$$

As for J_1 and J_4 we have

$$(2.9) \quad |J_1| + |J_4| < \frac{K_n a_1}{2\pi} \int_{s_1-2\sigma}^{s_1+2\sigma} |t - z_1 - h|^{\mu-1} ds + \frac{K_n a_1}{2\pi} \int_{s_1-2\sigma}^{s_1+2\sigma} |t - z_1|^{\mu-1} ds.$$

But for all z_1 and z_2 on Γ there exists a constant k_0 [12] such that

$$(2.10) \quad 0 < k_0 \leq \left| \frac{z_2 - z_1}{s_2 - s_1} \right| \leq 1.$$

Thus since $z_2 = z(s_2) = z_1 + h$ and $s_2 = s_1 + \sigma$ and $\mu - 1 < 0$,

$$\begin{aligned} |J_1| + |J_4| &< \frac{K_n a_1}{2\pi} \left\{ k_0^{\mu-1} \int_{s_1-2\sigma}^{s_1+2\sigma} |s - s_1 - \sigma|^{\mu-1} ds + k_0^{\mu-1} \int_{s_1-2\sigma}^{s_1+2\sigma} |s - s_1|^{\mu-1} ds \right. \\ &< \frac{K_n a_1}{2\pi} \left\{ k_0^{\mu-1} \int_0^{3\sigma} s^{*\mu-1} ds^* + k_0^{\mu-1} \int_0^{\sigma} s^{*\mu-1} ds^* + 2k_0^{\mu-1} \int_0^{2\sigma} s^{*\mu-1} ds^* \right\} \\ &< \frac{K_n a_1}{2\pi} \left\{ \frac{k_0^{\mu-1}}{\mu} (3\sigma)^\mu + \frac{k_0^{\mu-1}}{\mu} \sigma^\mu + \frac{2k_0^{\mu-1}}{\mu} (2\sigma)^\mu \right\}, \end{aligned}$$

and since $1 \geq k_0 > 0$,

$$(2.11) \quad \begin{aligned} &< \frac{K_n a_1 k_0^\mu}{2\pi k_0 \mu} \{1 + 2 \cdot 2^\mu + 3^\mu\} \sigma^\mu, \\ &|J_1| + |J_4| < \frac{K_n a_1}{2\pi k_0 \mu} \{1 + 2^{1+\mu} + 3^\mu\} |h|^\mu. \end{aligned}$$

Finally we consider J_2 and J_5 .

$$(2.12) \quad \begin{aligned} J_2 - J_5 &= \frac{1}{2\pi i} \int_{\Gamma_s} \frac{\Delta_n^*[z_1, z_1, \lambda] - \Delta_n^*[z_1 + h, z_1 + h, \lambda]}{t - z_1} dt \\ &+ \frac{1}{2\pi i} \int_{\Gamma_s} \left[\frac{1}{t - z_1} - \frac{1}{t - z_1 - h} \right] \{ \Delta_n^*[z_1 + h, z_1 + h, \lambda] - \Delta_n^*[t, z_1 + h, \lambda] \} dt \\ &+ \frac{1}{2\pi i} \int_{\Gamma_s} \frac{\Delta_n^*[t, z_1 + h, \lambda] - \Delta_n^*[t, z_1, \lambda]}{t - z_1} dt. \end{aligned}$$

Let us write

$$(2.13) \quad J_2 - J_5 = I_1 + I_2 + I_3,$$

where the I_k are the integrals in (2.12). For I_1 we have

$$(2.14) \quad |I_1| \leq \frac{K_n}{2\pi} [a_1 |h|^\mu + a_2 |h|^{\mu_1}] \left| \int_{\Gamma_s} \frac{dt}{t - z_1} \right|.$$

But

$$\int_{\Gamma_s} \frac{dt}{t - z_1} = \log \frac{t' - z_1}{t'' - z_1} = \log \left| \frac{t' - z_1}{t'' - z_1} \right| + \pi i$$

for the appropriate choice of the branch of the logarithm. Since

$$\begin{aligned} 0 < k_0 &\leq \frac{|t' - z_1|}{2\sigma}, \quad \frac{|t'' - z_1|}{2\sigma} \leq 1, \\ 0 < \left| \frac{t' - z_1}{t'' - z_1} \right| &\leq \frac{2\sigma}{2\sigma k_0} = \frac{1}{k_0}. \end{aligned}$$

Thus

$$\left| \int_{\Gamma_s} \frac{dt}{t - z_1} \right| \leq |-\log k_0 + \pi i|$$

and (2.14) becomes:

$$(2.15) \quad |I_1| \leq \frac{K_n}{2\pi} [a_2 |h|^{\mu_1} + a_1 |h|^\mu] |-\log k_0 + \pi i|.$$

For I_2 we have

$$|I_2| \leq a_1 \frac{K_n}{2\pi} |h| \int_{\Gamma_s} \frac{ds}{|t - z_1| |t - z_1 - h|^{1-\mu}}.$$

Making use of (2.10) we get:

$$(2.16) \quad |I_2| \leq \frac{K_n}{2\pi} a_1 |h| k_0^{\mu-2} \int_{\Gamma_2} \frac{ds}{|s - s_1| |s - s_1 - \sigma|^{1-\mu}}.$$

Note that

$$(2.17) \quad \frac{1}{|s - s_1| |s - s_1 - \sigma|^{1-\mu}} = \frac{1}{|s - s_1|^{2-\mu} \left| 1 - \frac{\sigma}{s - s_1} \right|^{1-\mu}},$$

but $|\sigma/(s - s_1)| \leq \frac{1}{2}$ for s on Γ_2 ; thus (2.17) becomes

$$\frac{1}{|s - s_1| |s - s_1 - \sigma|^{1-\mu}} \leq \frac{2^{1-\mu}}{|s - s_1|^{2-\mu}}.$$

From which, since $4\sigma < |\Gamma|$, (2.16) becomes:

$$(2.18) \quad \begin{aligned} |I_2| &\leq \frac{K_n}{2\pi} a_1 k_0^{\mu-2} 2^{1-\mu} |h| \left(\int_{-|\Gamma|/2+s_1}^{s_1-2\sigma} + \int_{s_1+2\sigma}^{|\Gamma|/2+s_1} \right) \frac{ds}{|s - s_1|^{2-\mu}}, \\ |I_2| &\leq \frac{K_n}{2\pi} a_1 k_0^{\mu-2} 2^{1-\mu} \frac{2(2\sigma)^{\mu-1}}{1-\mu} |h| \left| 1 - \left(\frac{4\sigma}{|\Gamma|} \right)^{1-\mu} \right|. \end{aligned}$$

Again using the fact that $0 < 4\sigma/|\Gamma| \leq 1$ and $0 < 1 - \mu < 1$ we get:

$$(2.19) \quad |I_2| \leq \frac{K_n}{\pi(1-\mu)} a_1 k_0^{\mu-2} \sigma^{\mu-1} |h| \leq \frac{K_n a_1}{\pi(1-\mu) k_0^{2-\mu}} |h|^\mu,$$

where in the second inequality we have made use of $0 < k_0 \leq |h|/\sigma \leq 1$.

As for I_3 we get, using (2.4) and (2.10),

$$(2.20) \quad |I_3| \leq \frac{K_n}{2\pi} a_2 |h|^{\mu_1} \int_{\Gamma_2} \frac{ds}{|t - z_1|} \leq \frac{K_n}{2\pi k_0} a_2 |h|^{\mu_1} \int_{\Gamma_2} \frac{ds}{|s - s_1|}.$$

But

$$(2.21) \quad \begin{aligned} \int_{\Gamma_2} \frac{ds}{|s - s_1|} &= \left(\int_{-|\Gamma|/2+s_1}^{s_1-2\sigma} + \int_{s_1+2\sigma}^{|\Gamma|/2+s_1} \right) \frac{ds}{|s - s_1|} \\ &= 2 \left| \log 2\sigma - \log \frac{|\Gamma|}{2} \right| = 2 \left| \log \frac{4\sigma}{|\Gamma|} \right|. \end{aligned}$$

Also note that if $\alpha \equiv \mu_1 - \mu > 0$, $\sigma^\alpha \log |\Gamma|/4\sigma$ achieves its maximum, since $4\sigma < |\Gamma|$, when $\sigma = (|\Gamma|/4)e^{-1/\alpha}$ and thus

$$|\sigma^\alpha \log 4\sigma/|\Gamma|| \leq (|\Gamma|/4)^\alpha \frac{1}{\alpha e}.$$

From which it follows using (2.10), (2.20) (2.21) that

$$(2.22) \quad |I_3| \leq \frac{K_n}{\pi k_0} a_2 \left(\frac{|\Gamma|}{4} \right)^{\mu_1-\mu} \frac{1}{(\mu_1 - \mu)e} |h|^\mu.$$

Collecting terms we conclude finally with the aid of (2.11), (2.13), (2.15), (2.19), (2.22), (2.7), (2.8) and (2.4) that

$$\begin{aligned}
|\gamma(z_2, \lambda) - \gamma(z_1, \lambda)| &\leq |J_1| + |J_4| + |J_2 - J_5| + |J_3 - J_6| \\
&\leq |J_1| + |J_4| + |I_1| + |I_2| + |I_3| + |J_3 - J_6| \\
&\leq K_n |h|^\mu \frac{a_1}{2\pi k_0 \mu} [1 + 2^{1+\mu} + 3^\mu] \\
&\quad + \frac{K_n |h|^\mu}{2\pi} |-\log k_0 + \pi i| [a_2 |h|^{\mu_1-\mu} + a_1] \\
&\quad + K_n |h|^\mu \frac{a_1}{\pi(1-\mu)k_0^{2-\mu}} \\
&\quad + K_n |h|^\mu \frac{a_2}{\pi k_0(\mu_1 - \mu)e} (|\Gamma|/4)^{\mu_1-\mu} \\
&\quad + K_n |h|^\mu \frac{1}{2} [a_2 |h|^{\mu_1-\mu} + a_1],
\end{aligned}$$

which proves the result (2.6).

Next we prove (2.5). To this end note that

$$2\pi i \gamma(z, \lambda) = \int_{\Gamma} \frac{\Delta_n^*(t, z, \lambda) - \Delta_n^*(z, z, \lambda)}{t - z} dt + \Delta_n^*(z, z, \lambda) \int_{\Gamma} \frac{dt}{t - z}.$$

Thus using (2.3) and (2.4) we see that

$$2\pi |\gamma(z, \lambda)| \leq K_n a_1 \int_{\Gamma} \frac{ds}{|t - z|^{1-\mu}} + \frac{1}{2} K_n g.$$

Let $t(s_0) = z$ then since $0 < k_0 \leq |t - z|/|s - s_0| \leq 1$ and $1 - \mu > 0$

$$\begin{aligned}
2\pi |\gamma(z, \lambda)| &\leq \frac{K_n a_1}{k_0^{1-\mu}} \int_{s_0 - |\Gamma|^{1/2}}^{|\Gamma|^{1/2} + s_0} \frac{ds}{|s - s_0|^{1-\mu}} + \frac{1}{2} K_n g \\
&\leq \frac{K_n a_1}{k_0^{1-\mu}} \left(\frac{|\Gamma|}{2} \right)^\mu \frac{2}{\mu} + \frac{1}{2} K_n g = K_n \left[\frac{1}{k_0^{1-\mu}} a_1 \left(\frac{|\Gamma|}{2} \right)^\mu \frac{2}{\mu} + \frac{1}{2} g \right].
\end{aligned}$$

Lemma 3. Let:

(3.1) The hypotheses of the theorem be satisfied,

(3.2) $|w_0(t_2) - w_0(t_1)| < K |t_2 - t_1|^\mu$ for t_1 and t_2 on Γ , where $K > \beta$ (β of H.2),

(3.3) $|w_1(t_2) - w_1(t_1)| < K |t_2 - t_1|^\mu$ for t_1 and t_2 on Γ , where $K > \beta$ (β of H.2),

(3.4) $w_0(z)$ and $w_1(z)$ belong to Ω for z on Γ .

Then for

$$|\lambda| \leq \min \{ \lambda_0, \lambda_1, \lambda_2 \},$$

where

$$\lambda_1 = \frac{K - \beta}{\eta_2^{*'} + \eta_3^{*'} K + A^* \bar{N}_0}, \quad \text{and} \quad \lambda_2 = \frac{\epsilon}{\bar{F}_0 + \bar{N}_0 A + C_0},$$

A and A^* are the constants of lemma 2 with $K_n = \bar{N}_0$, $g = 1$, $a_2 = \eta_1' / \bar{N}_0 \equiv \tilde{a}_2$, and

$$a_1 = \frac{\eta_2' + K\eta_3'}{\bar{N}_0} \equiv \tilde{a}_1,$$

$$(3.5) \quad |w_k(t_2) - w_k(t_1)| < K |t_2 - t_1|^\mu \text{ for all } k = 0, 1, 2, 3, \dots,$$

$$(3.6) \quad w_k(z) \text{ belongs to } \Omega \text{ for } z \text{ on } \Gamma, \text{ for all } k = 0, 1, 2, 3, \dots \text{ where the } w_k(z) \text{ are the successive substitutions, i.e.,}$$

$$w_{k+1}(z) = B(z) + \lambda C(w_k) + \lambda F[z, w_k(z), \lambda] + \frac{\lambda}{2\pi i} \int_{\Gamma} \frac{N[t, z, w_k(t), \lambda]}{t - z} dt.$$

Proof. By induction on n . Assuming theorem true for $k \leq n$, we want to show it is true for $k = n + 1$. Since by assumption $w_n(z) \in \Omega$ for z on Γ , we have for z_1 and z_2 on Γ

$$(3.7) \quad |w_{n+1}(z_2) - w_{n+1}(z_1)| \leq \beta |z_2 - z_1|^\mu + |\lambda| |F[z_2, w_n(z_2), \lambda] - F[z_1, w_n(z_1), \lambda]| + |\lambda| |\gamma_0(z_2, \lambda) - \gamma_0(z_1, \lambda)|,$$

where

$$(3.8) \quad \gamma_0(z, \lambda) = \frac{1}{2\pi i} \int_{\Gamma} \frac{N[t, z, w_n(t), \lambda]}{t - z} dt.$$

Setting $N[t, z, w_n(t), \lambda] = \Delta_n^*[t, z, \lambda]$, the hypotheses of lemma 2 are satisfied for $|\lambda| \leq \lambda_0$ since (i) by the induction hypothesis $w_n(t) \in \Omega$ for t on Γ and thus from (H.6)

$$(3.9) \quad |N[t, z, w_n(t)]| < \bar{N}_0,$$

where $\bar{N}_0$ is independent of n ; (ii) by (H.5)

$$(3.10) \quad |N[t_2, z_2, w_n(t_2), \lambda] - N[t_1, z_1, w_n(t_1), \lambda]| < \eta_1' |z_2 - z_1|^{\mu_1} + \eta_2' |t_2 - t_1|^\mu + \eta_3' |w_n(t_2) - w_n(t_1)| < \eta_1' |z_2 - z_1|^{\mu_1} + (\eta_2' + K\eta_3') |t_2 - t_1|^\mu,$$

where the last inequality follows by the induction hypothesis.

Thus, if we let, $g = 1$, and

$$(3.11) \quad K_n = \bar{N}_0, \quad a_2 = \eta_1' / \bar{N}_0, \quad a_1 = (\eta_2' + K\eta_3') / \bar{N}_0,$$

we see the hypotheses of lemma 2 are satisfied and thus

$$(3.12) \quad |\gamma_0(z, \lambda)| < \bar{N}_0 A,$$

$$(3.13) \quad |\gamma_0(z_2, \lambda) - \gamma_0(z_1, \lambda)| < \bar{N}_0 A^* |z_2 - z_1|^\mu,$$

where A and A^* are given by (2.6.1) and (2.6.2) respectively, with a_1, a_2 given by (3.11) and A^* is independent of n and λ .

From (3.7), (3.13), (H.3), and the induction hypothesis, we conclude that

$$|w_{n+1}(z_2) - w_{n+1}(z_1)| < W |z_2 - z_1|^\mu,$$

where

$$W = \beta + |\lambda| \{ \eta_2^{*'} + \eta_3^{*'} K + \bar{N}_0 A^* \}.$$

By hypothesis $K > \beta$ thus by choosing $|\lambda| \leq \lambda_1$ where

$$\lambda_1 = \frac{K - \beta}{\eta_2^{*'} + \eta_3^{*'} K + A^* \bar{N}_0} > 0,$$

we get:

$$|w_{n+1}(z_2) - w_{n+1}(z_1)| < K |z_2 - z_1|^\mu$$

which is (3.5).

Next we show that $w_{n+1}(z) \in \Omega$ for z on Γ . From (H.4) (H.7) and the induction hypothesis:

$$(3.14) \quad |F[z, w_n(z), \lambda]| < \bar{F}_0, \quad \text{for } z \text{ on } \Gamma, |\lambda| \leq \lambda_0,$$

$$(3.14.A) \quad |C(w_n)| \leq C_0.$$

Combining (3.12), (3.14), and (3.14.A) we can conclude

$$(3.15) \quad |w_{n+1}(z) - B(z)| < |\lambda| [\bar{F}_0 + \bar{N}_0 A + C_0].$$

Thus by choosing $|\lambda| \leq \lambda_2$ where $\lambda_2 = \epsilon / (\bar{F}_0 + \bar{N}_0 A + C_0)$ we see that $|w_{n+1}(z) - B(z)| < \epsilon$. But since $B(z) \in F_\epsilon$ for z on Γ and $\rho[F_\epsilon, \bar{\Omega}^c] \geq \epsilon$, we conclude that $w_{n+1}(z) \in \Omega$ for z on Γ . Thus the lemma is proved.

Lemma 4. *Let the conditions of lemma 1, as well as (H.3), (H.4), (H.5) and (H.8) hold, then for*

$$\begin{aligned} \tilde{\Delta}_{n+1}(z) \equiv & \lambda [C(w_n) - C(w_{n-1})] + \lambda \{ F[z, w_n(z), \lambda] - F[z, w_{n-1}(z), \lambda] \} \\ & + \frac{\lambda}{2\pi i} \int_{\Gamma} \frac{\Delta_n^*[t, z, \lambda]}{t - z} dt, \end{aligned}$$

$$(4.1) \quad |\tilde{\Delta}_{n+1}(z)| < |\lambda| b K_n,$$

$$(4.2) \quad |\tilde{\Delta}_{n+1}(z_2) - \tilde{\Delta}_{n+1}(z_1)| < |\lambda| \bar{b} K_n |z_2 - z_1|^\mu,$$

where $b = A + \eta_3^{*'} + C_1$ and $\bar{b} = 4\bar{F}\bar{g} + 4\eta_2^* + 12\eta_3^* K + A^*$ are numerical constants independent of n and λ . A and A^* are given by (2.6.1) and (2.6.2) respectively with $g = \eta_3'$ and with α_1 and α_2 given by (1.5.a) and (1.5.b) respectively.

Proof. From (H.3) and (1.2.a)

$$(4.3) \quad |F[z, w_n(z), \lambda] - F[z, w_{n-1}(z), \lambda]| < \eta_3^{*'} |w_n(z) - w_{n-1}(z)| < \eta_3^{*'} K_n.$$

From (H.8) and (1.2.a)

$$(4.3.a) \quad |C(w_n) - C(w_{n-1})| \leq C_1 ||w_n - w_{n-1}|| < C_1 K_n,$$

for z on Γ , $|\lambda| \leq \lambda_0$. Moreover from (H.5)

$$(4.4) \quad |\Delta_n^*(t, z, \lambda)| \equiv |N(t, z, w_n(t), \lambda) - N(t, z, w_{n-1}(t), \lambda)| \\ < \eta'_3 |w_n(t) - w_{n-1}(t)| < \eta'_3 K_n.$$

Using (4.4) and (1.5) we conclude from lemma 2

$$(4.5) \quad \frac{1}{2\pi} \left| \int_{\Gamma} \frac{\Delta_n^*[t, z, \lambda]}{t - z} dt \right| < K_n A,$$

where A is given by (2.6.1), with $g = \eta'_3$, a_1 by (1.5.a), and a_2 by (1.5.b). From (4.3), (4.3.a), and (4.5) we conclude (4.1).

In order to prove (4.2) let

$$\delta_n^*(t, \lambda) \equiv F[t, w_n(t), \lambda] - F[t, w_{n-1}(t), \lambda].$$

Then from lemma 1 we conclude

$$(4.6) \quad |\delta_n^*(t_2, \lambda) - \delta_n^*(t_1, \lambda)| < K_n a_1^* |t_2 - t_1|^\mu,$$

where

$$(4.7) \quad a_1^* = 4\bar{F}\bar{g} + 4\eta_3^* + 12\eta_3^*K.$$

Thus

$$(4.8) \quad |\tilde{\Delta}_{n+1}(z_2) - \tilde{\Delta}_{n+1}(z_1)| \\ \leq |\lambda| |\delta_n^*(z_2, \lambda) - \delta_n^*(z_1, \lambda)| + |\lambda| |\gamma(z_2, \lambda) - \gamma(z_1, \lambda)|,$$

where $\gamma(z, \lambda)$ is defined in lemma 2. With the aid of (4.6) and (2.6), (4.8) becomes

$$|\tilde{\Delta}_{n+1}(z_2) - \tilde{\Delta}_{n+1}(z_1)| < |\lambda| K_n [a_1^* + A^*] |z_2 - z_1|^\mu,$$

where a_1^* and A^* are given by (4.7) and (2.6.2) respectively with a_1 and a_2 given by (1.5.a) and (1.5.b) respectively.

Proof of the theorem by the method of successive approximations. Let

$$w_0(z) = B(z) \varepsilon F, \quad \text{for } z \text{ on } \Gamma, \quad |w_0(z_2) + w_n(z_1)| < K |z_2 - z_1|^\mu$$

for z_1 and z_2 on Γ , $K > \beta$.

(I) We shall show first that the sequence of successive substitutions

$$(I.1) \quad w_{n+1}(z) = B(z) + \lambda C(w_n) + \lambda F[z, w_n(z), \lambda] + \frac{\lambda}{2\pi i} \int_{\Gamma} \frac{N[t, z, w_n(t), \lambda]}{t - z} dt$$

converges uniformly.

We must show first that $w_1(z) \varepsilon \Omega$. Note that since $w_0(z) \varepsilon \Omega$ for z on Γ then by (H.4) and (H.7),

$$(I.2) \quad |C(w_0) + F[z, w_0(z), \lambda]| < C_0 + \bar{F}_0, \quad \text{for } z \text{ on } \Gamma, \quad |\lambda| \leq \lambda_0.$$

Also note that by (H.6),

$$|N[t, z, w_0(t), \lambda]| < \bar{N}_0, \quad \text{for } z \text{ and } t \text{ on } \Gamma, \quad |\lambda| \leq \lambda_0,$$

and by (H.5) and the Hölder assumption for $w_0(z)$,

$$\begin{aligned} |N[t_2, z_2, w_0(t_2), \lambda] - N[t_1, z_1, w_0(t_1), \lambda]| \\ < \eta'_2 |t_2 - t_1|^\mu + \eta'_1 |z_2 - z_1|^{\mu_1} + \eta'_3 K |t_2 - t_1|^\mu, \end{aligned}$$

for z_1, z_2, t_1, t_2 on Γ , $|\lambda| \leq \lambda_0$. Thus lemma 2 yields for

$$\tilde{\gamma}(z, \lambda) = \frac{1}{2\pi i} \int_{\Gamma} \frac{N[t, z, w_0(t), \lambda]}{t - z} dt,$$

$$(I.3) \quad |\tilde{\gamma}(z, \lambda)| < \bar{N}_0 A(\tilde{a}_1),$$

$$(I.4) \quad |\tilde{\gamma}(z_2, \lambda) - \tilde{\gamma}(z_1, \lambda)| < \bar{N}_0 A^*(\tilde{a}_1, \tilde{a}_2) |z_2 - z_1|^\mu,$$

where A and A^* are the constants (2.6.1), (2.6.2) with $g = 1$ and

$$\begin{aligned} \tilde{a}_1 &= (\eta'_2 + \eta'_3 K) \bar{N}_0^{-1}, \\ \tilde{a}_2 &= \eta'_1 \bar{N}_0^{-1}. \end{aligned}$$

Combining (I.1) and (I.3) we get that

$$\begin{aligned} |w_1(z) - B(z)| &< |\lambda| [C_0 + \bar{F}_0 + \bar{N}_0 A(\tilde{a}_1)] \\ &\leq \lambda_2 [C_0 + \bar{F}_0 + \bar{N}_0 A(\tilde{a}_1)] < \epsilon, \quad \text{for } |\lambda| \leq \lambda_2, \end{aligned}$$

by the definition of λ_2 given in the statement of lemma 3. Thus since $B(z) \in F_*$, $w_1(z) \in \Omega$ for z on Γ .

Next we show $|w_1(z_2) - w_1(z_1)| < K |z_2 - z_1|^\mu$ for z_1 and z_2 on Γ . Using (H.2), (H.3) and (I.4) we get:

$$\begin{aligned} |w_1(z_2) - w_1(z_1)| &< \beta |z_2 - z_1|^\mu + |\lambda| \eta_2^{*'} |z_2 - z_1|^\mu \\ &\quad + |\lambda| K \eta_3^{*'} |z_2 - z_1|^\mu + |\lambda| \bar{N}_0 A^*(\tilde{a}_1, \tilde{a}_2) |z_2 - z_1|^\mu. \end{aligned}$$

From lemma 3 this becomes

$$|w_1(z_2) - w_1(z_1)| < K |z_2 - z_1|^\mu, \quad \text{for } |\lambda| \leq \lambda_1.$$

Thus lemma 3 applies for $|\lambda| \leq \min \{\lambda_0, \lambda_1, \lambda_2\}$ where

$$\begin{aligned} \lambda_1 &= \frac{K - \beta}{\eta_2^{*'} + \eta_3^{*'} K + A^*(\tilde{a}_1, \tilde{a}_2) \bar{N}_0}, \\ \lambda_2 &= \frac{\epsilon}{\bar{F}_0 + \bar{N}_0 A(1, \tilde{a}_1) + C_0}, \end{aligned}$$

and we get $|w_k(t_2) - w_k(t_1)| < K |t_2 - t_1|^\mu$, $k = 0, 1, 2, \dots$ as well as the fact that $w_k(t) \in \Omega$. Consider

$$\begin{aligned} (I.5) \quad w_0(z) &+ (w_1(z) - w_0(z)) + (w_2(z) - w_1(z)) + (w(z)_3 - w(z)_2) + \dots \\ &+ (w_n(z) - w_{n-1}(z)) + (w_{n+1}(z) - w_n(z)) + \dots \end{aligned}$$

By lemma 4

$$||\tilde{\Delta}_{n+1}(z)|| = |w_{n+1}(z) - w_n(z)| < |\lambda| \tilde{b} K_n ,$$

where $\tilde{b} = b + \bar{b}$ (b and $\bar{b}$ of lemma 4, with, $\bar{g} = 1$) and K_n is such that:

$$||w_n(z) - w_{n-1}(z)|| \leq K_n ,$$

with

$$||w_n(z) - w_{n-1}(z)|| = \max_{z \text{ on } \Gamma} |w_n(z) - w_{n-1}(z)| + \max_{\substack{t_1, t_2 \in \Gamma \\ t_1 \neq t_2}} \frac{|\Delta_n(t_2) - \Delta_n(t_1)|}{|t_2 - t_1|^\mu}.$$

Then we may take $K_n = ||w_n(z) - w_{n-1}(z)||$. Thus

$$(I.6) \quad K_{n+1} = ||\tilde{\Delta}_{n+1}(z)|| < K_n |\lambda| \tilde{b} = ||\tilde{\Delta}_n(z)|| |\lambda| \tilde{b}.$$

Now for $|\lambda| < 1/\tilde{b}$ we have that there is a q such that $|\lambda| \tilde{b} \leq q < 1$, then

$$(I.7) \quad ||\tilde{\Delta}_{n+1}(z)|| < q ||\tilde{\Delta}_n(z)|| < q^2 ||\tilde{\Delta}_{n-1}(z)|| < \dots < q^n ||\tilde{\Delta}_1(z)||.$$

Then (I.5) is dominated by:

$$\begin{aligned} |w_0(z)| + ||\tilde{\Delta}_1(z)|| + q ||\tilde{\Delta}_1(z)|| + q^2 ||\tilde{\Delta}_1(z)|| + \dots \\ = |w_0(z)| + ||\tilde{\Delta}_1(z)|| \frac{1}{1-q}. \end{aligned}$$

Thus (I.5) converges uniformly and absolutely for z on Γ to something, call it $w(z)$ so that

$$w_n(z) \xrightarrow{unif} w(z).$$

(II) We next show that $w(z)$ is a solution of the integral equation (E).

From (3.5) we see that $|w(z_2) - w(z_1)| \leq K |z_2 - z_1|^\mu$. Moreover

$$\begin{aligned} |w(z) - w_0(z)| &= |(w_1(z) - w_0(z)) + (w_2(z) - w_1(z)) + \dots| \\ &\leq |\lambda| \tilde{b} K_0 + |\lambda|^2 \tilde{b}^2 K_0 + \dots \\ &= |\lambda| \tilde{b} \frac{K_0}{1 - |\lambda| \tilde{b}}, \end{aligned}$$

where $||w_1(z) - w_0(z)|| = K_0 < \epsilon$ provided $|\lambda| \leq \min \{ \lambda_0, \lambda_1, \lambda_2/2, \epsilon \lambda_1 / \lambda (k - \beta) \}$. Thus

$$|w(z) - w_0(z)| < |\lambda| \tilde{b} \frac{\epsilon}{1 - |\lambda| \tilde{b}}$$

and since $|\lambda| < 1/2\tilde{b}$, $|w(z) - w_0(z)| < \epsilon$. Therefore $w(z) \in \Omega$ for $z \in \Gamma$ and thus the integral is meaningful.

Consider

$$(I.8) \quad I_{n+1}(z) = \lambda[C(w) - C(w_n)] + \lambda\{F[z, w(z), \lambda] - F[z, w_n(z), \lambda]\} \\ + \frac{\lambda}{2\pi i} \int_{\Gamma} dt \frac{N[t, z, w(t), \lambda] - N[t, z, w_n(t), \lambda]}{t - z}.$$

Then by (H.3)

$$(I.9) \quad |I_{n+1}(z)| \leq |\lambda| C_1 \|w - w_n\|_c \\ + |\lambda| \eta_3^{*'} |w(z) - w_n(z)| + |\lambda| (2\pi)^{-1} \left| \int_{\Gamma} dt \frac{\bar{\Delta}_n^*[t, z, \lambda]}{t - z} \right|,$$

where

$$\bar{\Delta}_n^*[t, z, \lambda] = N[t, z, w(t), \lambda] - N[t, z, w_n(t), \lambda].$$

Next note that for z on Γ

$$|w(z) - w_n(z)| \leq |(w_{n+1}(z) - w_n(z))| + |(w_{n+2}(z) - w_{n+1}(z))| + \dots \\ \leq K_{n+1} + K_{n+2} + \dots \\ \leq |\lambda| \bar{\delta} K_n \{1 + |\lambda| \bar{\delta} + (|\lambda| \bar{\delta})^2 + (|\lambda| \bar{\delta})^3 + \dots\} \\ < |\lambda| \bar{\delta} K_n \frac{1}{1 - |\lambda| \bar{\delta}} < q K_n \frac{1}{1 - q}, \quad \text{where } 1 > q \geq |\lambda| \bar{\delta}.$$

Thus we see for $K_{n_1} = q K_n / (1 - q)$:

$$(I.10) \quad |w(z) - w_n(z)| < K_{n_1}, \\ |w(z) - w_{n+1}(z)| < |\lambda| \bar{\delta} K_{n_1}.$$

Next we shall derive a condition for the Hölder constants for

$$\bar{\Delta}_n(z) \equiv w(z) - w_n(z).$$

Note that

$$\bar{\Delta}_n(z_2) - \bar{\Delta}_n(z_1) = (w_{n+1}(z_2) - w_n(z_2)) + (w_{n+2}(z_2) - w_{n+1}(z_2)) + \dots \\ - (w_{n+1}(z_1) - w_n(z_1)) - (w_{n+2}(z_1) - w_{n+1}(z_1)) + \dots \\ = \bar{\Delta}_{n+1}(z_2) + \bar{\Delta}_{n+2}(z_2) + \dots \\ - (\bar{\Delta}_{n+1}(z_1) + \bar{\Delta}_{n+2}(z_1) + \dots) \\ = (\bar{\Delta}_{n+1}(z_2) - \bar{\Delta}_{n+1}(z_1)) + (\bar{\Delta}_{n+2}(z_2) - \bar{\Delta}_{n+2}(z_1)) + \dots.$$

Thus by lemma 4 and (I.6)

$$(I.11) \quad |\bar{\Delta}_n(z_2) - \bar{\Delta}_n(z_1)| \leq |\lambda| \bar{\delta} |z_2 - z_1|^\mu (K_n + K_{n+1} + K_{n+2} + \dots) \\ < |\lambda| \bar{\delta} |z_2 - z_1|^\mu K_n \frac{1}{1 - |\lambda| \bar{\delta}}.$$

Let

$$K_{n_2} \equiv \frac{|\lambda| \bar{b}}{1 - q} K_n ,$$

where q is as above and let $\tilde{q} = \bar{b}/(1 - q)$. Then

$$(I.12) \quad \tilde{K}_n = \max \{K_{n_1} , K_{n_2}\} = |\lambda| \tilde{q} K_n .$$

We have by (I.10) and (I.11) that

$$(I.13) \quad |\bar{\Delta}_n(z)| < \tilde{K}_n \quad \text{and} \quad |\bar{\Delta}_n(z_2) - \bar{\Delta}_n(z_1)| < \tilde{K}_n |z_2 - z_1|^\mu ,$$

for z_1 and z_2 on Γ . Thus we can use lemma 1 to conclude

$$(I.14) \quad |\bar{\Delta}_n^*[t_2, z_2, \lambda] - \bar{\Delta}_n^*[t_1, z_1, \lambda]| < \tilde{K}_n [a_2 |z_2 - z_1|^{\mu_1} + a_1 |t_2 - t_1|^\mu] ,$$

where a_1 and a_2 are given by (1.5.a) and (1.5.b) respectively with $\bar{g} = 1$ since

$$|w_n(z_2) - w_n(z_1)| < K |z_2 - z_1|^\mu \quad \text{and} \quad |w(z_2) - w(z_1)| < K |z_2 - z_1|^\mu .$$

Then from lemma 2 we conclude that

$$(I.15) \quad |\gamma^*(z, \lambda)| < \tilde{K}_n A ,$$

$$(I.16) \quad |\gamma^*(z_2, \lambda) - \gamma^*(z_1, \lambda)| < \tilde{K}_n A^* |z_2 - z_1|^\mu , \quad \text{for } z_1 \text{ and } z_2 \text{ on } \Gamma ,$$

where

$$\gamma^*(z, \lambda) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\bar{\Delta}_n^*[t, z, \lambda]}{t - z} dt$$

and A and A^* are given by (2.6.1) and (2.6.2) respectively with $g = \lambda_0 \bar{q} \eta'_3$.

Thus from (I.9), (I.10), (I.12), and (I.15) we see

$$(I.17) \quad |I_{n+1}(z)| \leq |\lambda| C_1 K_{n_1} + |\lambda| \eta_3^* K_{n_1} \\ + |\lambda| \tilde{K}_n A \leq |\lambda| \tilde{K}_n \{\eta_3^{*'} + A + C_1\} .$$

We wish to find next a constant $K_{n+1,2}$ for which

$$(I.18) \quad |I_{n+1}(z_2) - I_{n+1}(z_1)| \leq |\lambda| K_{n+1,2} |z_2 - z_1|^\mu .$$

To this end let

$$\bar{\delta}_n^*(t, \lambda) = F[t, w(t), \lambda] - F[t, w_n(t), \lambda] .$$

Then since (I.13) holds, $|w(t_2) - w(t_1)| < K |t_2 - t_1|^\mu$ and $|w_n(t_2) - w_n(t_1)| < K |t_2 - t_1|^\mu$. We get from Lemma 1

$$(I.19) \quad |\delta_n^*(z_2, \lambda) - \delta_n^*(z_1, \lambda)| < \tilde{K}_n a_1^* |z_2 - z_1|^\mu ,$$

where $a_1^* = 4\bar{F} + 4\eta_2^* + 12\eta_3^* K$. Combining (I.19) and (I.16) we see that the $K_{n+1,2}$ of (I.18) can be taken to be

$$K_{n+1,2} = \tilde{K}_n [a_1^* + A^*(a_1, a_2)] .$$

Next let

$$(I.20) \quad \bar{q} = \max \{ \eta_3^{*'} + A(1, a_1) + C_1, a_1^* + A^*(a_1, a_2) \}.$$

Thus we have shown

$$(I.21) \quad \left. \begin{aligned} |\bar{\Delta}_n(z)| &< \bar{K}_n \\ |\bar{\Delta}_n(z_2) - \bar{\Delta}_n(z_1)| &< \bar{K}_n |z_2 - z_1|^\mu \end{aligned} \right\} \Rightarrow \begin{aligned} |I_{n+1}(z)| &< \bar{q} \bar{K}_n |\lambda| \\ |I_{n+1}(z_2) - I_{n+1}(z_1)| &< \bar{q} \bar{K}_n |\lambda| |z_2 - z_1|^\mu. \end{aligned}$$

Thus if we let

$$(I.22) \quad w^*(z) = B(z) + \lambda C(w) + \lambda F[z, w(z), \lambda] + \frac{\lambda}{2\pi i} \int_{\Gamma} dt \frac{N[t, z, w(t), \lambda]}{t - z},$$

then (I.8) becomes:

$$(I.23) \quad I_{n+1}(z) \equiv w^*(z) - w_{n+1}(z).$$

By (I.21) and (I.12) we have

$$|I_{n+1}(z)| < |\lambda| \bar{q} \bar{K}_n \equiv |\lambda| \bar{q} |\lambda| \bar{q} K_n.$$

But by (I.6)

$$K_n < K_{n-1} |\lambda| \bar{b}.$$

Thus we get:

$$(I.24) \quad |I_{n+1}(z)| < |\lambda|^2 \bar{q} \bar{q} K_n < |\lambda|^2 \bar{q} \bar{q} (|\lambda| \bar{b}) K_{n-1} < \cdots |\lambda|^2 \bar{q} \bar{q} (|\lambda| \bar{b})^{n-1} K_1$$

and

$$(I.25) \quad |I_{n+1}(z_2) - I_{n+1}(z_1)| < |\lambda| \bar{q} \bar{K}_n |z_2 - z_1|^\mu < \cdots |\lambda|^2 \bar{q} \bar{q} (|\lambda| \bar{b})^{n-1} K_1 |z_2 - z_1|^\mu.$$

Hence by (I.24) and (I.25) we see that for $|\lambda| < \bar{b}^{-1}$, $w_n(z) \rightarrow w^*(z)$ (unif.) and thus for $|\lambda| < \bar{b}^{-1}$, $w^*(z) \equiv w(z)$. But by (I.20), $\bar{q} = \max \{ \eta_3^{*'} + A + C_1, a_1^* + A^* \}$, i.e., $\bar{q} = \max \{ b, \bar{b} \}$ and thus $w(z)$ is a solution of the integral equation (E).

(III) Uniqueness. We shall next show that among the functions that satisfy a Hölder condition with exponent μ and constant K there is at most one solution to the integral equation (E) for those $|\lambda|$ considered.

Let $v(z)$ be a function defined and single valued on Γ for which

$$|v(z_2) - v(z_1)| < K |z_2 - z_1|^\mu, \quad \text{for } z_1 \text{ and } z_2 \text{ on } \Gamma$$

and $v(z) \not\equiv w(z)$, where $w(z)$ is the solution

$$w(z) = \lim_{n \rightarrow \infty} w_n(z).$$

Let

$$d_n(z) \equiv v(z) - w_{n-1}(z), \quad n = 0, 1, 2, \cdots.$$

For suitable C_n^* and t , t_1 and t_2 on Γ

$$|d_n(t)| \leq C_n^* ,$$

and

$$|d_n(t_2) - d_n(t_1)| \leq C_n^* |t_2 - t_1|^\mu .$$

Thus the conditions of lemma 1 are satisfied with $\bar{g} = 1$ and we may utilize lemma 4 to conclude, if we set:

$$\delta_{n+1}^*(t, z, \lambda) \equiv N[t, z, v(t), \lambda] - N[t, z, w_{n-1}(t), \lambda],$$

and

$$\begin{aligned} \tilde{\delta}_{n+1}(z) &\equiv \lambda \{C(v) - C(w_{n-1})\} \\ &+ \lambda \{F[z, v(z), \lambda] - F[z, w_{n-1}(z), \lambda]\} + \frac{\lambda}{2\pi i} \int_{\Gamma} \frac{\delta_n^*[t, z, \lambda]}{t - z} dt, \end{aligned}$$

that

$$\begin{aligned} |\tilde{\delta}_{n+1}(z)| &< |\lambda| b C_n^* , \\ |\tilde{\delta}_{n+1}(z_2) - \tilde{\delta}_{n+1}(z_1)| &< |\lambda| \bar{b} C_n^* |z_2 - z_1|^\mu , \end{aligned}$$

where b and $\bar{b}$ are given in lemma 4. Thus if we let $b_0 = \max \{b, \bar{b}\}$ we can conclude that

$$C_{n+1}^* \equiv |\lambda| b_0 C_n^* \leq (|\lambda| b_0)^2 C_{n-1}^* \leq \cdots \leq (|\lambda| b_0)^n C_1^* .$$

But by assumption $|\lambda| < \min \{b^{-1}, \bar{b}^{-1}\}$ and thus $|\lambda| b_0 < 1$, from which it follows that

$$\lim_{n \rightarrow \infty} |\tilde{\delta}_n(z)| = 0 .$$

But by the definition of $\tilde{\delta}_n(z)$ we get

$$\lim_{n \rightarrow \infty} w_n(z) = v(z) .$$

Thus $w(z) = v(z)$ which proves the uniqueness. This completes the proof of the theorem.

Remark. An existence proof without uniqueness can be gotten by appealing to the Schauder Fixed Point Theorem as well as to lemma 1 and lemma 2. The Banach space B to be considered, is the set of all complex-valued functions defined on Γ that satisfy an $H(\mu)$ condition on Γ with norm $\varphi \in B$,

$$\|\varphi\| = \max_{t \text{ on } \Gamma} |\varphi(t)| + \max_{\substack{t_1, t_2 \text{ on } \Gamma \\ t_1 \neq t_2}} \frac{|\varphi(t_2) - \varphi(t_1)|}{|t_2 - t_1|^\mu} .$$

Due to length we shall eliminate this proof.

Application. The Deformation of an Elastic Circular Cylinder in a Two-Dimensional Ideal Fluid Flow.

In the mathematical formulation for the determination of the equilibrium position of a deformed elastic circular cylinder due to the irrotational two-dimensional flow of an incompressible, inviscid fluid we are led to a non-linear, singular integral equation of the form considered above. This problem is an example of a class of elasticity problems in which the boundary condition for the elastic body changes as the body deforms.

We consider the original circle in the z plane given by $z = e^{i\theta}$ and assume that on the boundary the pressure is given by Bernoulli's theorem; *viz*:

$$(A.1) \quad p[w(z)]_{z=e^{i\theta}} = \Pi + \frac{\rho U^2}{2} \left\{ 1 - \left| \frac{e^{2i\theta} - 1}{w'(z)} \right|^2 \right\}_{z=e^{i\theta}},$$

where

$$(A.1.a) \quad \begin{aligned} \Pi &= \text{pressure at infinity,} \\ \rho &= \text{density of fluid,} \\ U &= \text{fluid velocity at infinity,} \\ w(z) &= \text{conformal map of exterior of undeformed region onto the} \\ &\quad \text{exterior of the deformed region and } w(z) \text{ has a continuous} \\ &\quad \text{derivative.} \end{aligned}$$

We also assume the elastic circular cylinder to be isotropic homogeneous and to obey the usual Hooke's law assumption; the pressure given by (A.1) is the only external force acting on the boundary and there are no body forces acting on the cylinder.

Since the flow is circulation free, the total force on the cylinder is zero, and the cylinder is in static equilibrium.

The biharmonic equation $\Delta^2 u = 0$ governs the state of stress and strain within the cylinder. Following Muskhelishvili [13], the general solution is of the form

$$u = \text{Re} \{ \bar{z}\varphi(z) + \Psi(z) \},$$

where $\varphi(z)$ and $\Psi(z)$ are functions regular in the circle. If we assume $\varphi'(z)$ and $\Psi(z)$ are continuous on the closed disc, the boundary conditions are

$$(A.2) \quad \varphi(\sigma) + \overline{\sigma\varphi'(\sigma)} + \overline{\Psi(\sigma)} = f(\sigma),$$

$$(A.3) \quad \kappa\varphi(\sigma) - \overline{\sigma\varphi'(\sigma)} - \overline{\Psi(\sigma)} = 2\mu g(\sigma),$$

where $\sigma = e^{i\theta}$,

$$(A.4) \quad f'(\sigma) = N - iT,$$

N and T are the external stresses acting on this circle in the direction of the outward normal and tangent to the left of the normal respectively, $\kappa = (\lambda + 3\mu)/(\lambda + \mu) > 1$, where λ and μ are the Lamé constants, and $g(\theta)$

is the complex displacement of the boundary. These equations are coupled since $g(\sigma) = w(\sigma) - \sigma$ and $X_n + iY_n$ is given by (A.1).

Our problem is to determine $g(\sigma)$ from (A.2) and (A.3) and to show from the $g(\sigma)$ so determined, that (A.1) is satisfied. $g(\sigma)$ will give the distortion due to the hydrodynamic pressure. We shall show $g(\sigma) \in H(1 + \mu)$ and $f(\sigma) \in H(1 + \mu)$ (i.e., f and g have first derivatives in $H(\mu)$), thus (A.1.a) will be satisfied.

First we shall derive an integral equation that is a necessary condition for (A.2) and (A.3) to be satisfied assuming $g'(\sigma)$, $\varphi'(\sigma)$ and $f'(\sigma) \in H(\mu)$ on Γ . We first solve (A.2), assuming f given, exactly as Muskhelishvili does [13]. To this end integrate (A.2) and get

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\sigma)}{\sigma - z} d\sigma + \frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{\sigma \varphi'(\sigma)}}{\sigma - z} d\sigma + \frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{\Psi(\sigma)}}{\sigma - z} d\sigma = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\sigma)}{\sigma - z} d\sigma.$$

Since we are assuming $\varphi'(z)$ and $\Psi(z)$ are regular in $|z| < 1$ and continuous in $|z| \leq 1$ we obtain for $|z| < 1$

$$(A.5) \quad \varphi(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{\overline{\sigma \varphi'(\sigma)}}{\sigma - z} d\sigma + \overline{\Psi(0)} = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\sigma)}{\sigma - z} d\sigma.$$

If

$$\varphi(z) = a_0 + a_1 z + a_2 z^2 + \dots,$$

then (A.5) becomes:

$$(A.6) \quad \varphi(z) + 2\bar{a}_2 + \bar{a}_1 z + \overline{\Psi(0)} = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\sigma)}{\sigma - z} d\sigma,$$

with the assumption that $\varphi(0) = 0$, and since a constant does not affect the stress distribution, let us drop $2\bar{a}_2 + \overline{\Psi(0)}$. Then we get

$$(A.7) \quad \varphi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\sigma)}{\sigma - z} d\sigma - \frac{z}{4\pi i} \int_{\Gamma} \frac{f(\sigma)}{\sigma^2} d\sigma.$$

In order to get $\Psi(z)$ we integrate the conjugate of (A.2) and remember that $f(\sigma)$ is real

$$(A.8) \quad \Psi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\sigma)}{\sigma - z} d\sigma - \frac{\varphi'(z)}{z} + \frac{a_1}{z}.$$

Note that $\Psi(z)$ is regular in $|z| < 1$ since $a_1 = \varphi'(0)$.

Next we wish to introduce the expressions for $\varphi(z)$ and $\Psi(z)$ given in (A.2) and (A.8) into (A.3). For this purpose we first differentiate (A.7) and then integrate the right hand side by parts to get

$$(A.9) \quad \varphi'(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(\sigma)}{\sigma - z} d\sigma - \frac{1}{4\pi i} \int_{\Gamma} \frac{f'(\sigma)}{\sigma} d\sigma,$$

since $f(\sigma)$ is periodic and $f'(\sigma) \in H(\mu)$.

Making use of the Plemelj formulas, for (A.7) and (A.8) when we let $z \rightarrow t$ on Γ , we get, remembering the meaning of a_1

$$(A.10) \quad \varphi(t) = \frac{1}{2}f(t) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\sigma)}{\sigma - t} d\sigma - a_1 t,$$

$$(A.11) \quad \Psi(t) = \frac{1}{2}f(t) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\sigma)}{\sigma - t} d\sigma - l\varphi'(t) + la_1.$$

From these we see that the boundary condition (A.2) is satisfied and thus $\varphi(z)$ and $\Psi(z)$ given by (A.7) and (A.8) give rise to the solution. We want in addition that (A.8) be satisfied where $f(\sigma)$ and $g(\sigma)$ are related by

$$(A.12) \quad f'(\sigma) = -\Pi - \frac{\rho U^2}{2} \left\{ 1 - \left| \frac{\sigma^2 - 1}{1 + g'(\sigma)} \right|^2 \right\},$$

where we have made use of (A.1) and (A.4) noting that $T = 0$, $N = -p[w(z)]$, and $g(\sigma) = w(\sigma) - \sigma$. (Note that though $g(\sigma)$ is given by the expression $g(\sigma) = w(\sigma) - \sigma$ for σ on the boundary, where $w(z)$ and z are regular functions in $|z| < 1$, this does not mean that $g(z) = w(z) - z$ and thus it does not mean that $g(z)$ is a regular function with boundary values $g(\sigma)$.)

Upon adding (A.2) and (A.3) we get

$$(A.13) \quad (1 + \kappa)\varphi(\sigma) = f(\sigma) + 2\mu g(\sigma).$$

Since $\varphi'(\sigma)$, $f'(\sigma)$, $g'(\sigma)$ are assumed to be in $H(\mu)$ we can differentiate and obtain:

$$(A.14) \quad (1 + \kappa)\varphi'(\sigma) = f'(\sigma) + 2\mu g'(\sigma).$$

In (A.9) let $z \rightarrow t$ on Γ and use the Plemelj formula,

$$(A.15) \quad \varphi'(t) = f'(t) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(\sigma)}{\sigma - t} d\sigma - \frac{1}{4\pi i} \int_{\Gamma} \frac{f'(\sigma)}{\sigma} d\sigma.$$

Introducing (A.15) into (A.14) yields

$$(A.16) \quad g'(t) = \frac{\kappa}{2\mu} f'(t) - \frac{1 + \kappa}{8\mu\pi i} \int_{\Gamma} \frac{f'(\sigma)}{\sigma} d\sigma + \frac{(1 + \kappa)}{4\mu\pi i} \int_{\Gamma} \frac{f'(\sigma)}{\sigma - t} d\sigma.$$

(A.16) together with (A.12) gives after collecting terms

$$(A.17) \quad g'(t) = B + U^2 C(g') + U^2 F[t, g'(t)] + \frac{U^2}{2\pi i} \int_{\Gamma} \frac{N[\sigma, g'(\sigma)]}{\sigma - t} d\sigma,$$

where

$$B = -\frac{\kappa}{2\mu} \Pi,$$

$$C(g') = -\frac{(1 + \kappa)\rho}{8\mu} \frac{1}{2\pi i} \int_{\Gamma} \frac{\left| \frac{\sigma^2 - 1}{1 + g'(\sigma)} \right|^2}{\sigma} d\sigma,$$

$$F[\sigma, g'(\sigma)] = \frac{\rho\kappa}{4\mu} \left| \frac{\sigma^2 - 1}{1 + g'(\sigma)} \right|^2 - \frac{\kappa\rho}{4\mu},$$

$$N[\sigma, g'(\sigma)] = \frac{(1 + \kappa)}{\mu} \frac{\rho}{2} \left| \frac{\sigma^2 - 1}{1 + g'(\sigma)} \right|^2.$$

It is easy to see that solving (A.17) will give a solution to our problem. To apply the theorem we must assume

$$B = -\frac{\kappa}{2\mu} \Pi \neq -1.$$

If $0 > B > -1$, let $\Omega = \Omega_{\delta+}$, where

$$\Omega_{\delta+} = \{(x, y): (x - B)^2 + y^2 \leq (1 + B - \delta)^2, \quad 0 < \delta < (1 + B)\},$$

and if $B < -1$, let $\Omega = \Omega_{\delta-}$, where

$$\Omega_{\delta-} = \{(x, y): (x - B)^2 + y^2 \leq (1 + B + \delta)^2, \quad 0 < \delta < |1 + B|\}.$$

Then for F_* we can take $F_* = \{B\}$. Then (H.1) becomes

$$\rho(F_*, \bar{\Omega}_{\delta+}^*) = 1 + B - \delta > 0 \quad \text{and} \quad \rho(F_*, \bar{\Omega}_{\delta-}^*) = |1 + B + \delta| > 0.$$

H.2 is clearly satisfied. Since $|1 + w| \geq \delta > 0$ for $w \in \Omega$ and thus since $\rho > 0$, $\mu > 0$, $\kappa > 1$,

$$|F[\sigma, g'(\sigma)]| \leq \pm \frac{\rho\kappa}{4\mu} \left\{ \left(\frac{2}{\delta} \right)^2 - 1 \right\},$$

where the $+$ sign is taken if $\delta < 2$ and the $-$ sign is taken if $\delta > 2$. Note $+$ sign is always taken if $-1 < B < 0$ since then $0 < \delta < 1 + B < 1$. Also since F has continuous partials with respect to $\text{Re } w(\sigma)$ and $\text{Im } w(\sigma)$ provided $w \in \Omega$, we see that (H.4) is satisfied. Likewise (H.6) holds. If we let $w = u + iv$, then

$$\left| \frac{\sigma^2 - 1}{1 + w} \right|^2 = \frac{2 - (\sigma^2 + \sigma^{-2})}{(1 + u)^2 + v^2},$$

which clearly has continuous second partial derivatives provided $w \in \Omega$. Thus (H.3) is satisfied. Similarly (H.5) is satisfied. Note that since $\kappa > 1$, $\rho > 0$, $\mu > 0$,

$$|C(w)| = \frac{(1 + \kappa)}{8\mu} \rho \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sigma^2 - 1}{1 + w(\sigma)} \right|^2 d\theta, \quad \text{where } \sigma = e^{i\theta},$$

and for $w \in \Omega$

$$|C(w)| \leq \frac{1 + \kappa}{8\mu} \rho \left(\frac{2}{\delta} \right)^2 = \frac{1 + \kappa}{2\delta^2\mu} \rho,$$

and thus (H.7) is satisfied. Finally note that

$$\begin{aligned} |C(w_2) - C(w_1)| &\leq \frac{1 + \kappa}{8\mu} \rho \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sigma^2 - 1}{1 + w_2(\sigma)} \right|^2 d\theta \\ &\leq \frac{1 + \kappa}{8\mu} \rho |\sigma_0^2 - 1| \left| \frac{1}{|1 + w_2(\sigma_0)|^2} - \frac{1}{|1 + w_1(\sigma_0)|^2} \right|, \end{aligned}$$

where $\sigma_0 = e^{i\theta_0}$, $0 < \theta_0 < 2\pi$, by the mean value theorem. But $|1/(1 + w)|$ has continuous partial derivatives with respect to u and v , $w = u + iv \in \Omega$ and

thus for some α_0

$$\left| \left| \frac{1}{1 + w_2(\sigma_0)} \right|^2 - \left| \frac{1}{1 + w_1(\sigma_0)} \right|^2 \right| \leq \alpha_0 |w_2(\sigma_0) - w_1(\sigma_0)| \leq \alpha_0 \|w_2 - w_1\|_c,$$

from which (H.8) follows. Thus the theorem applies, the integral equation (A.17) has a unique Hölder continuous solution and thus the hydroelastic problem is solved.

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