*Exact Solutions of Einstein-Maxwell Equations**

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1. Introduction. We determine all solutions of Einstein-Maxwell equations of Petrov class N when the propagation vector of the gravitational field is hypersurface orthogonal. This assumption is in fact needed only in the case where the characteristic Debever-vector of the Weyl tensor is not at the same time an eigenvector of the Ricci tensor. As an application we find all conformally flat space-times containing an electromagnetic field. We also give a new and simpler derivation of the vacuum, null-type metrics with hypersurface orthogonal rays. Concerning the problem of rotating solutions, we limit ourselves to exhibiting the general form of Minkowski line element built around a null, geodetic, shear-free congruence.

2. Integration method. There exists on our space-time manifold V_4 a preferred geodetic null congruence, namely the congruence generated by the Debever vector. The properties of this congruence are expressed in terms of the curvature form. It seems thus natural to try and find the simplest possible description of the curvature. The adjoint representation of $\mathfrak{L}$ (the arcwise connected part of the identity of the Lorentz group) being faithful we shall describe the curvature and the connection form as vectors of the Lie algebra $L(\mathfrak{L})$ of this group. $L(\mathfrak{L})$ is isomorphic to $L(SO_3(C))$; we can thus represent the curvature form as a vector of $L(SO_3(C))$. Furthermore the tangent space of self-dual 2-forms in a point x of V_4 is a realization of $L(SO_3(C))$. So what we really do is substituting for the usual bundle of orthonormal frames over V_4 , the bundle of self-dual 2-forms over V_4 . The formalism thus constructed has been described in detail elsewhere [1, 2, 3]. We shall simply write down here the most important formulae.

In each point x of V_4 we choose a "null" base $\{h_\alpha\}$ (Greek indices run from

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1 to 4, Latin indices from 1 to 3) in the tangent space Tx such that, if $\{\theta^a\}$ is the dual base, the metric of V_4 has the form

$$(2.1) \quad ds^2 = 2(\theta^1\theta^4 - \theta^2\theta^3).$$

$\{Z^i\}$ will denote the canonically associated base of the space of self dual 2-forms and γ^{ii} its metric tensor:

$$(2.2) \quad \gamma^{ii} = Z^i \cdot Z^i = 2\delta_1^i \delta_2^i - \frac{1}{2}\delta_3^i \delta_3^i.$$

The Riemannian connection defined on V_4 by the metric (2.1) is determined by a one-form ω with value in $L(\mathfrak{L})$ such that

$$(2.3) \quad d\theta = -\omega \wedge \theta,$$

and can also be determined by a one-form σ with value in $L(SO_3(C))$ such that

$$(2.4) \quad dZ = -\sigma \wedge Z.$$

The equations (2.3) are called "the metric equations" [4]. As usual the absolute differential of the connection form σ gives the curvature 2-form Σ

$$(2.5) \quad \Sigma = d\sigma + \sigma \wedge \sigma.$$

The components of the curvature form Σ in the base $\{Z^i\}$ are:

$$(2.6) \quad \Sigma_i = C_{ii}Z^i - \frac{R}{6}\gamma_{ii}Z^i + E_{i\bar{i}}\bar{Z}^i.$$

The symmetric trace free tensor C_{ii} corresponds to the self dual part of the Weyl tensor; R is the scalar curvature; $E_{i\bar{i}}$ is the image of the trace free Ricci tensor. From (2.5) one deduces Bianchi identities:

$$(2.7) \quad d\Sigma = \Sigma \wedge \sigma - \sigma \wedge \Sigma.$$

Let $\mathfrak{F}$ be an electromagnetic field defined on V_4 and solution of Maxwell equations in the absence of charges and current:

$$(2.8) \quad d(\mathfrak{F}_i Z^i) = 0.$$

$\mathfrak{F}_i$ are the components of the self dual part of this field. We assume that $\mathfrak{F}$ is the only source of the gravitational field so that Einstein equations are:

$$(2.9) \quad E_{i\bar{i}} = -\mathfrak{F}_i \bar{\mathfrak{F}}_{\bar{i}},$$

$$(2.10) \quad R = 0.$$

We further assume that the Weyl tensor C_{ii} of V_4 is of Petrov type N and is thus the square of a null bivector W

$$(2.11) \quad C_{ii} = \gamma W_i W_i.$$

As base vector Z^2 we choose W . Then:

$$(2.12) \quad C_{ii} = \gamma \delta_i^2 \delta_i^2.$$

There exists a group of transformations G_4 leaving (2.12) invariant. This consists of 2 independent null rotations around the characteristic Debever vector, one euclidian rotation q and a special Lorentz transformation p .

To integrate the field equations (2.8, 9, 10, 12) we start off by enlarging this system by adding the Bianchi identities (2.7) and the metric equations (2.3). In this new system we consider as independent variables the components of the curvature form (γ) of the electromagnetic field ($\mathfrak{F}_i$), of the connection form ($\sigma_{i\alpha}$) and of the base forms (h_α^β). Maxwell equations and Bianchi identities are divided into three groups. The first consists of linear relations between the $\sigma_{i\alpha}$. The second is a completely integrable system of Pfaffian forms; it gives the differential of 2 components of the electromagnetic field. The third group is composed of linear relations between derivatives of γ and derivatives of the third component of the electromagnetic field.

We use the field equations (2.8, 9, 10, 12) as integrability conditions for certain transformations of G_4 which simplify the connection form. It is also sometimes useful, although not necessary, to integrate the ones which give the dependence of certain rotation coefficients on the affine distance measured along the preferred congruence of geodesics.

The metric equations (2.3) are then easily integrated to obtain the form of the line element. The electromagnetic field has a given amplitude. The remaining Maxwell equations are then solved to obtain the phase.

We shall now present the metrics obtained by this method. Detailed calculations are given elsewhere [6]. Various cases have to be considered. The electromagnetic field may be singular or not. If not, one of its two characteristic directions may eventually coincide with the direction of the propagation of the gravitational field.

3. Solutions with Electromagnetic non-null field. No coincidence between the Characteristic Directions. With these assumptions

$$(3.1) \quad C_{ij}\mathfrak{F}^i \neq 0.$$

When the propagation vector of the gravitational field is hypersurface orthogonal, the line element is:

$$(3.2) \quad ds^2 = 2r \, dx \left(\frac{\partial^2 r}{\partial z \partial \bar{z}} dx + dy \right) - 2 \left(\frac{\partial r}{\partial \bar{z}} \sin y \, dx + \cos y \, dz \right) \left(\frac{\partial r}{\partial z} \sin y \, dx + \cos y \, d\bar{z} \right).$$

r is a function $r(x, z, \bar{z})$ satisfying the equation:

$$(3.3) \quad \frac{\partial}{\partial \bar{z}} \left(\frac{1}{r} \frac{\partial^2 r}{\partial z^2} \right) = 0.$$

The self dual part of the electromagnetic field is given by:

$$(3.4) \quad \mathfrak{F} = -\sqrt{2} \left[\cos y \, dx \wedge d\left(\frac{\partial r}{\partial z}\right) + dy \wedge \left(\frac{\partial r}{\partial z} \sin y \, dx + \cos y \, d\bar{z}\right) \right].$$

The complex amplitude of the gravitational field is given by:

$$(3.5) \quad \gamma = -\frac{2}{r} \left[tgy \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial^2 r}{\partial z^2} \right) + \frac{\partial r}{\partial \bar{z}} \cdot \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial^2 r}{\partial z^2} \right) \right],$$

and the amplitude of the electromagnetic field is

$$(3.6) \quad \beta \bar{\beta} = \frac{4}{r^2} \frac{\partial^2 r}{\partial z^2} \frac{\partial^2 r}{\partial \bar{z}^2}.$$

4. Solutions with Electromagnetic non-null field. The characteristic Debever vector is an eigenvector of the electromagnetic field. We have the condition:

$$(4.1) \quad C_{ij} \mathfrak{F}^i = 0.$$

The propagation vector of the gravitational field is necessarily recurrent. The general solution is:

$$(4.2) \quad ds^2 = 2 \, dx \left[\left(\frac{K}{2} y^2 + a \right) dx + \frac{1}{2K} \left(\frac{\partial \alpha}{\partial z} dz + \frac{\partial \bar{\alpha}}{\partial \bar{z}} d\bar{z} \right) + dy \right] - 2r^2 dz d\bar{z}.$$

K is the constant curvature of the surfaces $x = y = c^{st}$. a , α , r are functions of the three variables x , z , $\bar{z}$, and are related by the following set of equations:

$$(4.3) \quad \alpha + \bar{\alpha} = \frac{4}{r} \frac{\partial r}{\partial x},$$

$$(4.4) \quad \frac{\partial^2}{\partial z \partial \bar{z}} (\alpha - \bar{\alpha}) = -Kr^2 (\alpha - \bar{\alpha}),$$

$$(4.5) \quad \frac{\partial^2 a}{\partial z \partial \bar{z}} = \frac{1}{4K} \frac{\partial \alpha}{\partial z} \cdot \frac{\partial \bar{\alpha}}{\partial \bar{z}} - \frac{1}{4} r^2 \alpha \bar{\alpha}.$$

The function r is a solution of

$$(4.6) \quad \frac{1}{r^2} \frac{\partial^2}{\partial z \partial \bar{z}} (\log r) = -\frac{K}{2}.$$

The self-dual electromagnetic field is:

$$(4.7) \quad \mathfrak{F} = \sqrt{K} e^{i\varphi} \left[\left(\frac{3}{2K} \frac{\partial \alpha}{\partial z} dz + \frac{1}{2K} \frac{\partial \bar{\alpha}}{\partial \bar{z}} d\bar{z} + dy \right) \wedge dx - r^2 dz \wedge d\bar{z} \right],$$

where φ is a constant. Each sphere $x = y = c^{st}$ has the same curvature K ; its second fundamental form Ω depends in an essential way on the variable x :

$$(4.8) \quad \Omega = \frac{1}{4Kr^2 g} \left(\frac{\partial \alpha}{\partial z} dz + \frac{\partial \bar{\alpha}}{\partial \bar{z}} d\bar{z} \right) \left(\frac{\partial g}{\partial z} dz + \frac{\partial g}{\partial \bar{z}} d\bar{z} \right).$$

g is the determinant of the metric of the hypersurface $y = c^{st}$. The only non-vanishing component of the Weyl Tensor is:

$$(4.9) \quad \gamma = y \frac{\partial}{\partial z} \left(\frac{1}{r^2} \frac{\partial \alpha}{\partial z} \right) + \frac{1}{K} \frac{\partial}{\partial z} \left(\frac{1}{r^2} \frac{\partial^2 \alpha}{\partial x \partial z} \right) - 2 \frac{\partial}{\partial z} \left(\frac{1}{r^2} \frac{\partial a}{\partial z} \right) + \frac{1}{2K} \left(\frac{\partial \alpha}{\partial z} \right)^2.$$

A family of conformally flat spaces is obtained starting from the metric (4.2), assuming $\alpha = 0$ and $\gamma = 0$. One then finds that $a = a(x)$ and a transformation of the form:

$$\begin{aligned} y &\rightarrow \beta(x)y - \frac{1}{\beta} \frac{\partial \beta}{\partial x}, \\ \alpha &\rightarrow \int \beta(x) dx, \end{aligned}$$

gives the metric:

$$(4.10) \quad ds^2 = 2 dx \left(\frac{K}{2} y^2 dx + dy \right) - 2 \frac{dz d\bar{z}}{\left(1 + \frac{K}{2} z\bar{z} \right)^2}.$$

This is the most general conformally flat space containing a non-null electromagnetic field. This field is covariantly constant.

5. Solutions with Electromagnetic null field. The electromagnetic field is such that $\gamma^{ij}\mathfrak{F}_i\mathfrak{F}_j = 0$. One can prove rather easily that the characteristic null vector of the gravitational field must coincide with the characteristic null vector of the electromagnetic field. The metric has the form:

$$(5.1) \quad ds^2 = \left[1 + \frac{k}{2} (z + \bar{z}) \right] dx [b dx + dy] - [ky dx + dz][ky dx + d\bar{z}].$$

k is a constant, b is a function of $x, z, \bar{z}$ obeying the equation:

$$(5.2) \quad \frac{\partial^2}{\partial z \partial \bar{z}} \left[\ln \left\{ \left(1 + \frac{k}{2} (z + \bar{z}) \right) \frac{\partial^2 b}{\partial z \partial \bar{z}} \right\} \right] = 0.$$

The amplitude of the electromagnetic field is:

$$(5.3) \quad \beta\bar{\beta} = -\frac{4}{1 + \frac{k}{2} (z + \bar{z})} \frac{\partial^2 b}{\partial z \partial \bar{z}}$$

and the one of the gravitational field is:

$$(5.4) \quad \gamma = \frac{4}{1 + \frac{k}{2} (z + \bar{z})} \frac{\partial^2 b}{\partial z^2}.$$

These solutions represent “weakly plane” waves. Each wave front is a hyper-plane, but these wave fronts are not parallel to each other. When $k = 0$, we

obtain the well-known plane waves in the presence of an electromagnetic field. The characteristic vector of the gravitation field is then covariantly constant. We can obtain the most general conformally flat space-time containing an electromagnetic null field by a limiting process in (5.1). If γ vanishes b is bilinear in z and $\bar{z}$:

$$b = r^2(x)z\bar{z} + s(x)z + \bar{s}(x)\bar{z} + t(x).$$

The transformation:

$$\begin{aligned} z' &= z - p(x), \\ y' &= y - z \frac{\partial \bar{p}}{\partial x} - \bar{z} \frac{\partial p}{\partial x}, \end{aligned}$$

reduces the line element, when $k = 0$, to:

$$(5.5) \quad ds^2 = 2 dx [r^2(x)z\bar{z} dx + dy] - 2 dz d\bar{z}.$$

6. Derivation of vacuum expanding metrics with hypersurface orthogonal rays. We give here the detailed calculation as an illustration of the general method. The field equations (2.5) can be written as:

$$(6.1) \quad \Sigma_1 = 0 = d\sigma_1 - \frac{1}{2}\sigma_3 \wedge \sigma_1,$$

$$(6.2) \quad \Sigma_2 = \gamma Z^2 = d\sigma_2 + \frac{1}{2}\sigma_3 \wedge \sigma_2,$$

$$(6.3) \quad \Sigma_3 = 0 = d\sigma_3 + \sigma_1 \wedge \sigma_2.$$

The base $\{Z^i\}$ is defined up to an arbitrary transformation of G_4 . Equations (6.1) and (6.3) can be integrated immediately:

$$(6.4) \quad \sigma_1 = \alpha dz,$$

$$(6.5) \quad \sigma_2 = \frac{d\beta}{\alpha} + \lambda dz,$$

$$(6.6) \quad \sigma_3 = 2 \frac{d\alpha}{\alpha} + \beta dz.$$

α, β, λ are complex functions, z is a complex variable. If z is real, the expansion vanishes and we obtain the "weakly plane" waves mentioned previously. One can easily see that there always exists a transformation of G_4 such that the connection form σ has the reduced expression:

$$(6.7) \quad \sigma_1 = dz,$$

$$(6.8) \quad \sigma_2 = \lambda dz,$$

$$(6.9) \quad \sigma_3 = 0.$$

The field equation (6.2) becomes:

$$(6.10) \quad \gamma Z^2 = d\lambda \wedge dz.$$

From the metric equations (2.4) and the equations (6, 7, 8, 9), one deduces that:

$$(6.11) \quad \theta^1 = r \, dx,$$

$$(6.12) \quad \theta^2 = a \, dx + b \, dz,$$

$$(6.13) \quad \theta^4 = \rho \, dx + dy.$$

Differentiating (6.11, 12, 13) and comparing the obtained equations with (6.7, 8, 9, 10) one finds that:

$$b = -\frac{y}{2}, \quad \lambda = -\frac{4}{r} \frac{\partial^2 r}{\partial z^2}, \quad \rho = 4 \frac{\partial^2 r}{\partial z \partial \bar{z}}, \quad a = 2 \frac{\partial r}{\partial \bar{z}},$$

with $r = r(x, z, \bar{z})$. The Bianchi identities then show us that:

$$\frac{\partial}{\partial \bar{z}} \left(\frac{1}{r} \frac{\partial^2 r}{\partial z^2} \right) = 0.$$

The line element may thus be written as

$$(6.14) \quad ds^2 = 2r \, dx \left(\frac{\partial^2 r}{\partial z \partial \bar{z}} dx + dy \right) - 2 \left(\frac{\partial r}{\partial \bar{z}} dx - y \, dz \right) \left(\frac{\partial r}{\partial z} dx - y \, d\bar{z} \right).$$

The value of γ is:

$$(6.15) \quad \gamma = \frac{8}{ry} \frac{\partial}{\partial x} \left[\frac{1}{r} \frac{\partial^2 r}{\partial z^2} \right].$$

A transformation of the form $ry = \rho$ brings (6.14) to the usual Robinson-Trautman form.

7. Shear-free congruences of null lines in Minkowski space. One can show that the general form of the line element built around such a congruence is:

$$(7.1) \quad ds^2 = 2 \, dy \, [dx - \bar{\alpha} \, dz - \alpha \, d\bar{z}] - 2[d\alpha - y \, dz][d\bar{\alpha} - y \, d\bar{z}],$$

where α is a function of the three variables $x, z, \bar{z}$ and satisfies the relation

$$(7.2) \quad \mathfrak{F}(\alpha, x - \bar{z}\alpha, z) = 0.$$

If we choose $\alpha = (a + ib)z$ we can reduce the line element to

$$(7.3) \quad ds^2 = 2 \, dy \, [dx + ib(\bar{z} \, dz - z \, d\bar{z})] - 2(y^2 + b^2) \, dz \, d\bar{z}.$$

This is the so-called “twistors” metric given for the first time by I. Robinson.

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