Trace Bending and Spherical Modifications

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Let a manifold V be the trace of a sequence of spherical modifications from M_1 to M_2 . Then V is also the trace of a sequence from $M_1 \cup M_2$ to the empty set. We consider here the problem of finding a connection between these sequences. From geometric considerations, one would expect the second sequence to arise from the first by "bending" V at some manifold intermediate between M_1 and M_2 . A general result which verifies this expectation in certain cases is given in section three. Section one contains a generalization of the h -cobordism theorem which reduces the trace bending problem to a consideration of homology. In section two, the effect of a modification on the Morse number of a manifold is computed. All manifolds are assumed to be compact, C^∞ , and *simply connected*.

The results of this paper are part of the author's Ph.D. thesis, which was written at Indiana University under the direction of A. H. Wallace.

1. A generalized h -cobordism theorem. A smooth manifold triad $(V; M_1, M_2)$ will be called simply connected if V , M_1 , and M_2 are all simply connected (see [1]). A Morse function f on a triad $(V; M_1, M_2)$ will be called ordered if, for critical points p and q ,

$$\text{index } p < \text{index } q \text{ implies } f(p) < f(q).$$

A Morse function will be called minimal if the number of critical points of each index is minimal with respect to $H_*(V, M_1)$. That is, a minimal Morse function for the triad $(V; M_1, M_2)$ has

$$b_i(V, M_1) = \text{Rk } H_i(V, M_1) + \text{Tor } H_i(V, M_1) + \text{Tor } H_{i-1}(V, M_1)$$

critical points of index i , where $\text{Rk } G$ is the rank of the group G and $\text{Tor } G$ is the minimum possible number of generators of the torsion subgroup of G .

We have the following generalization of a theorem of Smale:

Theorem 1.1. *Let $T = (V; M_1, M_2)$ be a simply connected manifold triad with $\dim V \geq 6$. Then there is a minimal ordered Morse function for T .*

Remark. (See also [5].) Taking $M_1 = M_2$ empty, we have theorem 6.1 of [2]. If M_2 is empty, we have theorem 6.5 of the same paper. On the other hand,

if $H_*(V, M_1) = 0$, the result is the h -cobordism theorem. The above theorem obviously gives the Morse number of such a triad in terms of its homology.

The proof of the above theorem is a generalization of the proof given by Smale and uses the following generalization of Lemma 6.4 of [2].

Lemma 1.2. *Let $(V; M_1, M_2)$ be a simply connected smooth manifold triad with $\dim V \geq 6$. For $2m \leq \dim V$, there is a simply connected smooth manifold triad $(X_m; M_1, N_m)$ such that $N_m \cap M_2$ is empty and*

- a) $H_j(X_m, M_1) = 0, j > m, H_m(X_m, M_1)$ is torsion-free,
- b) $H_j(V, X_m) = 0, j \leq m,$
- c) *there is a minimal ordered Morse function for $(X_m; M_1, N_m)$.*

To apply the above theorem to sequences of modifications, we use the correspondence between Morse functions and modifications. (Given a Morse function for $(V; M_1, M_2)$, there is a sequence of modifications from M_1 to M_2 with V as trace, having one modification of type (r, s) for each critical point of index $r + 1$ and *vice versa*.) We have immediately the theorem of Wallace [3] on the rearrangement of modifications. A simple proof is also obtained for theorem 10.1 of [4]:

Theorem 1.3. *Let $(V; M_1, M_2)$ be a simply connected smooth manifold triad, $\dim V \geq 6$, and let V, M_1, M_2 , each contain a basic complex. Then there is a sequence of modifications with trace V which transforms M_1 into M_2 , and has no modifications of types other than*

$$\begin{aligned} (m-2, m), (m-1, m-1), (m, m-2), & \quad \text{if} \quad \dim M_1 = 2m-1, \\ (m-1, m), (m, m-1), & \quad \text{if} \quad \dim M_2 = 2m. \end{aligned}$$

Proof. The assumption that everything contains a basic complex gives, in the case $n = 2m$, that

$$H_i(V, M_1) = 0, \quad j \leq m-1, \quad j \geq m+2,$$

and

$$H_{m+1}(V, M_1) \text{ is torsion-free.}$$

The result for this case is then immediate from the theorem. The case $n = 2m - 1$ is similar.

Remark. In terms of modifications, Theorem 1.1 gives, in terms of homology, the minimum possible number of modifications in a sequence from M_1 to M_2 with trace V . Namely, it is the Morse number

$$\gamma(V, M_1) = \sum_r b_r(V, M).$$

2. The effect of a modification on the Morse number of a manifold. Let M_2 be obtained from M_1 by a single (r, s) modification, $1 \leq r \leq s$. Let

$$i_j^k : H_i(S^j) \rightarrow H_i(M_k), \quad j = r, s; \quad k = 1, 2,$$

denote the inclusion-induced map, where S^r , and S^s are the spheres involved in the modification. Let

$$t_i^k = \text{Tor } [H_i(M_k)/\text{im } i_i^k] - \text{Tor } H_i(M_k), \quad k = 1, 2; \quad j = r, s.$$

Theorem 2.1. *Let $r + 1 = s$. Then*

$$\begin{aligned} \text{Rk } H_i(M_2) &= \text{Rk } H_i(M_1) - \text{Rk } \ker i_s^2, & j = r, r + 2, \\ \text{Rk } H_{r+1}(M_2) &= \text{Rk } H_{r+1}(M_1) + 2 \text{Rk } \ker i_r^1, \\ \text{Tor } H_i(M_2) &= \text{Tor } H_i(M_1) + t_i^r, & j = r, r + 1, \end{aligned}$$

and all other corresponding ranks and torsions are equal.

Proof. Consider the effect of the (r, s) modification on the homology of M_1 and M_2 :

$$\begin{aligned} H_r(M_2) &\cong H_r(M_1)/\text{im } i_r^1, \\ H_{r+1}(M_2)/\text{im } i_s^2 &\cong H_{r+1}(M_1) \oplus \ker i_r^1, \\ H_{r+2}(M_2) \oplus \ker i_s^2 &\cong H_{r+2}(M_1). \end{aligned}$$

The results for the ranks are then immediate, since

$$\text{Rk } \ker i_s^2 = \text{Rk } \text{im } i_r^1,$$

by duality. For the torsions,

$$\begin{aligned} \text{Tor } H_{r+1}(M_2) &= \text{Tor } H_r(M_2) = \text{Tor } [H_r(M_1)/\text{im } i_r^1], \\ \text{Tor } H_r(M_1) &= \text{Tor } H_{r+1}(M_1) = \text{Tor } [H_{r+1}(M_2)/\text{im } i_s^2], \end{aligned}$$

so that

$$\begin{aligned} \text{Tor } H_r(M_2) &= \text{Tor } H_r(M_1) + t_r^1, \\ \text{Tor } H_{r+1}(M_2) &= \text{Tor } H_{r+1}(M_1) - t_s^2, \end{aligned}$$

and

$$t_r^1 = -t_s^2.$$

This completes the proof.

Corollary 2.2. *If $r + 1 = s$,*

$$\begin{aligned} b_i(M_2) &= b_i(M_1) - \text{Rk } \ker i_s^2 + t_r^1, & j = r, r + 2, \\ b_{r+1}(M_2) &= b_{r+1}(M_1) + 2 \text{Rk } \ker i_r^1 + 2t_r^1, \end{aligned}$$

and all other corresponding b_i 's are equal.

The following corollary gives the effect of a modification on the Morse number.

Corollary 2.3. *If $r + 1 = s$,*

$$\gamma(M_2) = \gamma(M_1) - 2 \text{Rk } \ker i_s^2 + 2 \text{Rk } \ker i_r^1 + 4t_r^1.$$

With a view toward considering sequences of more than one modification, let $\#(+, M_1, M_2)$ be the number of modifications between M_1 and M_2 which increase the Morse number, and $\#(-, M_1, M_2)$ the number which decrease it. (In the present situation, each is either zero or one.)

Corollary 2.4. *If $r + 1 = s$, and $\dim M_1 \geq 6$,*

$$b_j(M_2) = b_j(M_1) - \#(-, M_1, M_2), \quad j = r, r + 2,$$

$$b_{r+1}(M_2) = b_{r+1}(M_1) + 2\#(+, M_1, M_2),$$

and all others are equal.

Proof. By definition,

$$\text{Rk ker } i_r^1 = 0 \quad \text{implies} \quad t_r^1 \geq 0,$$

and

$$\text{Rk ker } i_r^1 = 1 \quad \text{implies} \quad t_r^1 \leq 0.$$

By duality,

$$\text{Rk ker } i_r^1 = 1 - \text{Rk ker } i_s^2.$$

From Corollary 2.3, then, a modification is increasing if and only if either

$$\text{Rk ker } i_r^1 = 1 \quad \text{and} \quad t_r^1 = 0,$$

or

$$\text{Rk ker } i_r^1 = 0 \quad \text{and} \quad t_r^1 = 1.$$

Also, a modification is decreasing if and only if either

$$\text{Rk ker } i_s^2 = 1 \quad \text{and} \quad t_r^1 = 0,$$

or

$$\text{Rk ker } i_s^2 = 0 \quad \text{and} \quad t_r^1 = -1.$$

Thus

$$\#(+, M_1, M_2) = \text{Rk ker } i_r^1 + t_r^1,$$

(1) and

$$\#(-, M_1, M_2) = \text{Rk ker } i_s^2 - t_r^1,$$

so the result follows from Corollary 2.2.

In a similar manner, we have the following results for the case $r + 1 < s$.

Theorem 2.5. *If $r + 1 < s$,*

$$\text{Rk } H_i(M_2) = \text{Rk } H_i(M_1) - \text{Rk ker } i_s^2, \quad j = r, s + 1,$$

$$\text{Rk } H_i(M_2) = \text{Rk } H_i(M_1) + \text{Rk ker } i_r^1, \quad j = r + 1, s,$$

$$\text{Tor } H_i(M_2) = \text{Tor } H_i(M_1) + t_r^1, \quad j = r, s.$$

Corollary 2.6. *If $r + 1 < s$,*

$$\begin{aligned} b_i(M_2) &= b_i(M_1) - \text{Rk ker } i_s^2 + t_r^1, & j &= r, s+1, \\ b_i(M_2) &= b_i(M_1) + \text{Rk ker } i_r^1 + t_r^1, & j &= r+1, s. \end{aligned}$$

Corollary 2.7. *If $r+1 < s$,*

$$\gamma(M_2) = \gamma(M_1) - 2 \text{Rk ker } i_s^2 + 2 \text{Rk ker } i_r^1 + 4t_r^1.$$

Remark. The conclusion here is the same as that for the case $r+1 = s$, in Corollary 2.3.

Corollary 2.8. *If $r+1 < s$ and $\dim M_1 \geq 6$,*

$$\begin{aligned} b_i(M_2) &= b_i(M_1) - \#(-, M_1, M_2), & j &= r, s+1, \\ b_i(M_2) &= b_i(M_1) + \#(+, M_1, M_2), & j &= r+1, s. \end{aligned}$$

For the case $r = s$, duality does not simplify the results, but we do have

Theorem 2.9. *For $r = s$,*

$$\begin{aligned} \text{Rk } H_i(M_2) &= \text{Rk } H_i(M_1) + \text{Rk ker } i_r^1 - \text{Rk ker } i_r^2, & j &= r, r+1, \\ \text{Tor } H_r(M_2) &= \text{Tor } H_r(M_1) + t_r^1 - t_r^2. \end{aligned}$$

(Here we have no *a priori* relation between t_r^1 and t_r^2 .)

Corollary 2.10. *For $r = s$,*

$$b_i(M_2) = b_i(M_1) + \text{Rk ker } i_r^1 - \text{Rk ker } i_r^2 + t_r^1 - t_r^2, \quad j = r, r+1.$$

Corollary 2.11. *If $r = s$,*

$$\gamma(M_2) = \gamma(M_1) + 2 \text{Rk ker } i_r^1 - 2 \text{Rk ker } i_r^2 + 2t_r^1 - 2t_r^2.$$

Remark. This is the same result as before on the gammas because we had then $t_r^1 = -t_r^2$.

Corollary 2.12. *If M_1 and M_2 cobound, then*

$$\gamma(M_1) \equiv \gamma(M_2), \pmod{2}.$$

Proof. A single modification changes $\gamma(M)$ by an even number.

3. Trace bending. In view of theorem 1.1, we may compare the homologies $H_*(T, M_1)$ and $H_*(T, M_1 \cup M_2)$ in order to find a connection between the minimal sequence from M_1 to M_2 , with trace T , and the minimal sequence from $M_1 \cup M_2$ to the empty set, also with trace T . We assume that T is the trace of a sequence of modifications from M_1 to M_2 , where the only type of modification occurring is (r, s) , $1 \leq r \leq s$.

Theorem 3.1. *Let $r+1 = s$ and $\dim M_1 \geq 6$. Then*

$$\begin{aligned} b_{r+2}(T, M_1 \cup M_2) &= b_{r+1}(M_1) + \#(+, M_1, M_2), \\ b_{r+3}(T, M_1 \cup M_2) &= b_{r+2}(M_1) - \#(-, M_1, M_2), \end{aligned}$$

and all other corresponding b_i 's are equal.

Proof. We have

$$H_{r+3}(T, M_1 \cup M_2) \oplus \ker i_* \cong H_{r+2}(M_1),$$

$$H_{r+2}(T, M_1 \cup M_2)/\text{im } i_* \cong H_{r+1}(M_1),$$

$$H_{r+2}(T, M_1 \cup M_2) \oplus \ker i'_* \cong H_{r+1}(M_2),$$

$$H_{r+1}(T, M_1 \cup M_2)/\text{im } i'_* \cong H_r(M_2),$$

where all other corresponding groups are isomorphic and

$$i_* : H_{r+2}(T, M_2) \rightarrow H_{r+2}(T, M_1 \cup M_2),$$

$$i'_* : H_{r+1}(T, M_1) \rightarrow H_{r+1}(T, M_1 \cup M_2).$$

Thus

$$H_{r+2}(M_2) \oplus \ker i_* \cong H_{r+2}(M_1),$$

so

$$\text{Rk } H_{r+2}(M_2) = \text{Rk } H_{r+2}(M_1) - \text{Rk } \ker i_*.$$

But theorem 2.1 implies that

$$\text{Rk } H_{r+2}(M_2) = \text{Rk } H_{r+2}(M_2) - (\# \ker i_*^2 = Z),$$

so that

$$\text{Rk } \ker i_* = (\# \ker i_r^1 = 0).$$

Thus, still using 2.1,

$$\text{Rk } H_{r+3}(T, M_1 \cup M_2) = \text{Rk } H_{r+2}(M_1) - (\# \ker i_*^2 = Z),$$

$$\text{Rk } H_{r+2}(T, M_1 \cup M_2) = \text{Rk } H_{r+1}(M_1) + (\# \ker i_r^1 = Z),$$

$$\text{Tor } H_{r+2}(T, M_1 \cup M_2) = \text{Tor } H_{r+1}(M_1) + (\text{sum of } t_r^1\text{'s}).$$

Thus the result follows from (1).

Corollary 3.2. *Let $r + 1 = s$, $\dim M_1 \geq 6$ and M be any intermediate manifold between M_1 and M_2 . Then*

$$b_{r+1}(T, M_1 \cup M_2) = b_r(M) + \#(-, M_1, M),$$

$$b_{r+2}(T, M_1 \cup M_2) = b_{r+1}(M) + \#(+, M, M_2) - \#(+, M_1, M),$$

$$b_{r+3}(T, M_1 \cup M_2) = b_{r+2}(M) - \#(-, M, M_2).$$

Proof. Use corollary 2.4 with $M_2 = M$.

Remark. If the k modifications leading to M all decrease the Morse number and the remaining j all increase it, then

$$b_{r+1}(T, M_1 \cup M_2) = b_r(M) + k,$$

$$b_{r+2}(T, M_1 \cup M_2) = b_{r+1}(M) + j,$$

$$b_{r+3}(T, M_1 \cup M_2) = b_{r+2}(M),$$

so that one has, in the minimal sequence from $M_1 \cup M_2$ to the empty set, $k(r, s)$ modifications, $j(s, r)$ modifications, and one $(p, n - p - 1)$ for each $(p - 1, n - p - 1)$ in the minimal sequence from the empty set to itself through M as trace. That is, in order to get the minimal sequence from $M_1 \cup M_2$ to the empty set through T , one may "bend T at M ."

Similarly, we have the following

Theorem 3.3. *If $r + 1 < s$,*

$$b_{r+1}(T, M_1 \cup M_2) = b_r(M) + \#(-, M_1, M),$$

$$b_{r+2}(T, M_1 \cup M_2) = b_{r+1}(M) - \#(+, M_1, M),$$

$$b_{s+1}(T, M_1 \cup M_2) = b_s(M) + \#(+, M, M_2),$$

$$b_{s+2}(T, M_1 \cup M_2) = b_{s+1}(M) - \#(-, M, M_2).$$

Remark. Again, 3.3 and 3.2 give the same result for the Morse number. Namely, if $r + 1 \leq s$ and $\dim M_1 \geq 6$,

$$\begin{aligned} \gamma(T, M_1 \cup M_2) = \gamma(M) + \#(-, M_1, M) - \#(+, M_1, M) \\ + \#(+, M, M_2) - \#(-, M, M_2). \end{aligned}$$

Also, 3.3 gives the expected result on bending T .

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