The Second Variation for Minimal Submanifolds

ROBERT HERMANN*

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1. Introduction. Progress in multiple-integral calculus of variations problems has been retarded by the complexity of the calculations involved in the second variation formula. In the special case of geodesics of Riemannian manifolds experience has shown that there is great importance in identifying the geometric meaning of the various terms of the second variation formula. In multiple integral problems, the classical method does not seem to be very well suited to exhibiting this geometric information. In this paper, we will describe a formalism using differential forms and vector fields that seems much more efficient. As application, we derive the second variation formula for minimal submanifolds of Riemannian manifolds, which has been given in classical language by A. Duschek [2]. (I am indebted to M. Berger for pointing this out to me. When writing [4] I was unaware of this reference.)

As general reference for manifolds, vector fields and differential forms, refer to [5]. An announcement of some of the results has been given in [4].

The main motivation for this work is to extend the now-classical Morse theory for arc-length of curves [8, 9] to multiple integral variational problems. Now, this theory is successful for two independent reasons: a) the analytic existence theorems have been proved in sufficiently strong form, and b) the indices at the critical points can be analyzed in sufficiently useful form to apply to many geometric structures. For multiple integral problems, both a) and b) are much more difficult. There is little one can say about a) from the point of view of geometry: One must wait for the analysts to prepare the foundations. Meanwhile, there is much that can be done towards b), and perhaps provide the analysts some geometric motivation towards a). In Section 9 we will make explicit the connection with the second variation formula. Further work will be provided at a later time.

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2. Variational problems in canonical form. Let M be a manifold, and let $\phi : N \rightarrow M$ define N as a submanifold of M . Suppose that N is an oriented manifold with an oriented boundary ∂N . A *variational problem* involves assigning a number $L(\phi)$ to ϕ by integrating a function of ϕ and its derivatives over N . If $t \rightarrow \phi_t$ is a one-parameter family of submanifolds reducing to the given one at $t = 0$, the problem is to compute the *first* and *second variations*:

$$(2.1) \quad \left. \frac{d}{dt} L(\phi_t) \right|_{t=0},$$

$$(2.2) \quad \left. \frac{d^2}{dt^2} L(\phi_t) \right|_{t=0}.$$

Experience with the classical theory shows us that:

- a) There is a system of differential equations (the “Euler equations”) which, if satisfied by ϕ , enable us to write (2.1) as an integral over ∂N . Submanifolds satisfying these equations are called *extremals*.
- b) If ϕ is an extremal and if (2.1) is zero, then (2.2) can be written as the sum of integrals over N and ∂N , with the integrands differential operators involving the infinitesimal deformation vector field $p \rightarrow \partial \phi_t(p) |_{t=0}$ for $p \in N$.

Since we expect that (2.1) and (2.2) will only depend on the infinitesimal deformation vector field, it is reasonable to compute them using a special choice of deformation. In this paper, we shall consider deformations of the following type:

- (2.3) For each $p \in N$, the curve $t \rightarrow \phi_t(p)$ is an integral curve of a fixed vector field X on M , i.e. if $t \rightarrow \text{Exp}(tX)$ is the one-parameter group of diffeomorphisms of M generated by X , then $\phi_t = \text{Exp}(tX)\phi$.

Further, since we expect (2.1) and (2.2) to only depend on the *local* form of the integrand of (2.1) and (2.2), it is convenient to use local “moving frames” adapted to the one-parameter family of submanifolds, i.e., to suppose that M has a basis (ω_i) (Adopt the following range of indices and summation conventions: $1 \leq i, j, \dots \leq m = \dim M$, $1 \leq a, b, \dots \leq n = \dim N$, $n+1 \leq u, v, \dots \leq m$.) of differential one-forms with:

$$(2.4) \quad \phi_t^*(\omega_u) = 0.$$

Finally, we will suppose that the number $L(\phi_t)$ is given by integrating an n -differential form ω over N , *i.e.*

$$(2.5) \quad L(\phi_t) = \int_N \phi_t^*(\omega).$$

This method of dealing with variational problems has been suggested by E. Cartan in a very obscure paper [1]. Although we will not try to justify the point here, a general variational problem can be reduced to one of this type by "prolongation". An indication of this process for $n = 1$, *i.e.* for simple integral variational problems, will be found in [5]. See [7] for the classical work.

With these assumptions, (2.5) can easily be differentiated using the formulas of [5]: Let $X(\theta)$ denote the *Lie derivative* of the form θ by the vector field X , and let $X \lrcorner \theta$ denote the contraction of θ by X . $d\theta$ denotes exterior derivative, $\theta_1 \wedge \theta_2$ denotes exterior product. The basic calculation rules are:

$$\begin{aligned} X(\theta) &= X \lrcorner d\theta + d(X \lrcorner \theta), \\ X(\theta_1 \wedge \theta_2) &= X(\theta_1) \wedge \theta_2 + \theta_1 \wedge X(\theta_2), \\ X \lrcorner (\theta_1 \wedge \theta_2) &= (X \lrcorner \theta_1) \wedge \theta_2 + (-1)^p \theta_1 \wedge (X \lrcorner \theta_2), \end{aligned}$$

if θ_1 is a p -form.

$$\begin{aligned} dX(\theta) &= X(d\theta), \\ dd\theta &= 0, \\ Y(X \lrcorner \theta) &= [Y, X] \lrcorner \theta + X \lrcorner Y(\theta), \end{aligned}$$

where Y, X denotes the Jacobi bracket of vector fields $V(M)$ denotes the set of all vector fields on M . $F(M)$ denotes the ring of all C^∞ real-valued functions on M .

Now, return to (2.5):

$$\frac{d}{dt} \phi_t^*(\omega) = \phi_t^*(X(\omega)) = \phi_t^*(X \lrcorner d\omega) + d(X \lrcorner \omega).$$

Hence,

$$\begin{aligned} (2.6) \quad \frac{d}{dt} L(\phi_t) &= \int_N \phi_t^*(X \lrcorner d\omega) + d(X \lrcorner \omega) \\ &= \int_N \phi_t^*(X \lrcorner d\omega) + \int_{\partial N} \phi_t^*(X \lrcorner \omega). \end{aligned}$$

(Here, and in the following work, we use Stoke's formula

$$\int_N d\theta = \int_{\partial N} \theta.)$$

This is the first variation formula: From it we can read off the condition (Precisely, we are defining extremals as submanifolds satisfying (2.7).) that ϕ

be an extremal, namely:

$$(2.7) \quad \phi^*(X \rfloor d\omega) = 0 \quad \text{for all } X \in V(M).$$

A further condition that (2.6) be zero at $t = 0$ is that the vector field X be *transversal to the boundary* in the sense that:

$$(2.8) \quad \phi^*(X \rfloor \omega) = 0 \quad \text{when restricted to } N.$$

We can also easily compute (2.2):

$$\begin{aligned} \frac{d^2}{dt^2} L(\phi_t) &= \int_N \phi_i^*(X(X \rfloor d\omega)) + \int_{\partial N} \phi_i^*(X(X \rfloor \omega)) \\ &= \int_N \phi_i^*(X \rfloor d(X(\omega))) + \int_{\partial N} \phi_i^*(X \rfloor X(\omega)) \\ &= \int_N \phi_i^*(X \rfloor d(X \rfloor d\omega)) + \int_N \phi_i^*(X \rfloor d(X \rfloor \omega)). \end{aligned}$$

In particular,

$$(2.9) \quad \left. \frac{d^2}{dt^2} L(\phi_t) \right|_{t=0} = \int_N \phi^*(X \rfloor d(X \rfloor d\omega)) + \int_{\partial N} \phi^*(X \rfloor d(X \rfloor \omega)).$$

This exhibits the second variation formula in a form that is very convenient for calculation and that exhibits a pleasing symmetry between the integrands over N and ∂N . We shall now investigate some properties of these integrands.

3. General properties of the second variation form. Suppose that ϕ is an extremal, *i.e.* (2.7) is satisfied. Consider the form

$$(X, Y) \rightarrow X \rfloor d(Y \rfloor d\omega) \equiv SV(X, Y)$$

defined for two vector fields $X, Y \in V(M)$ which satisfy:

$$\phi^*(X(\omega_u)) = 0 = \phi^*(Y(\omega_u)).$$

(Such vector fields will be said to be *admissible*.)

Let us first investigate the symmetry under interchange of X and Y .

$$\begin{aligned} SV(X, Y) &= X \rfloor Y(d\omega) = -[Y, X] \rfloor d\omega + Y(X \rfloor d\omega) \\ &= -[Y, X] \rfloor d\omega + d(Y \rfloor X \rfloor d\omega) + Y \rfloor d(X \rfloor d\omega) \\ &= -[Y, X] \rfloor d\omega + d(Y \rfloor X \rfloor d\omega) + S(Y, X). \end{aligned}$$

Since $\phi^*([Y, X] \rfloor d\omega) = 0$, we have

$$(3.1) \quad \int_N \phi^*(SV(X, Y)) - SV(Y, X)) = \int_{\partial N} \phi^*(Y \rfloor X \rfloor d\omega).$$

Now, we have:

Theorem 3.1. $\int_N \phi^*(SV(X, Y)) = 0$ if either X or Y is tangent to N .

Proof. Suppose first that X is tangent to N . Then,

$$\begin{aligned} SV(X, Y) &= X \lrcorner d(Y \lrcorner d\omega) \\ &= X(Y \lrcorner d\omega) - d(X \lrcorner Y \lrcorner d\omega). \end{aligned}$$

Since ϕ is an extremal, $\phi^*(Y \lrcorner d\omega) = 0$. Since X is tangent to N ,

$$\phi^*(X(Y \lrcorner d\omega)) = X(\phi^*(Y \lrcorner d\omega)) = 0.$$

Then,

$$\int_N \phi^*(SV(X, Y)) = - \int_{\partial N} \phi^*(X \lrcorner Y \lrcorner d\omega),$$

which is zero since $\phi^*(Y \lrcorner d\omega) = 0$, and X is tangent to N .

Suppose now that Y is tangent to N . We see that the right hand side of (3.1) vanishes, using (2.7), hence we are reduced to the preceding case. Q.E.D.

Let

$$BV(X, Y) = X \lrcorner d(Y \lrcorner \omega).$$

Again, we investigate symmetry under interchange of X and Y .

$$\begin{aligned} BV(X, Y) &= X(Y \lrcorner \omega) - d(X \lrcorner Y \lrcorner \omega) \\ &= [X, Y] \lrcorner \omega + Y \lrcorner X(\omega) - d(X \lrcorner Y \lrcorner \omega) \\ &= [X, Y] \lrcorner \omega + Y \lrcorner d(X \lrcorner \omega) + Y \lrcorner X \lrcorner d\omega - d(X \lrcorner Y \lrcorner \omega) \\ &= [X, Y] \lrcorner \omega + BV(Y, X) + Y \lrcorner X \lrcorner d\omega - d(X \lrcorner Y \lrcorner \omega), \end{aligned}$$

hence

$$(3.2) \quad \int_{\partial N} \phi^*(BV(X, Y) - BV(Y, X)) = \int_{\partial N} \phi^*([X, Y] \lrcorner \omega) + \phi^*(Y \lrcorner X \lrcorner d\omega).$$

We now investigate whether the second variation formula $(X, Y) \rightarrow \int_N \phi^*(SV(X, Y))$ really only depends on the values of X and Y on the submanifold $\phi(N)$. To do this, it suffices to show that it is zero when fX or fY is substituted for X or Y , where f is a function on M that is zero on N .

$SV(fX, Y) = fX \lrcorner d(Y \lrcorner d\omega)$, which is zero when restricted to $\phi(N)$ if $\phi^*(f) = 0$.

$$\begin{aligned} SV(X, fY) &= X \lrcorner d(fY \lrcorner d\omega) = X \lrcorner (df \wedge Y \lrcorner d\omega) + X \lrcorner fd(Y \lrcorner d\omega) \\ &= X(f)(Y \lrcorner d\omega) - df \wedge (X \lrcorner Y \lrcorner d\omega) + fX \lrcorner d(Y \lrcorner d\omega), \end{aligned}$$

which again is zero when restricted to $\phi(N)$, providing ϕ is an extremal, *i.e.* satisfies (2.7).

A similar analysis is readily made for $BV(X, Y)$: When restricted to $\phi(\partial N)$

it only depends on the values of X and Y on the boundary providing Y is transversal to N at the boundary.

4. Computation of the second variation. Let us recapitulate our basic definitions: A real-valued function $\phi \rightarrow L(\phi)$ is defined on the space of sub-manifold mappings $N \rightarrow M$. Since experience with the classical theory shows us that the second variation formula has a local character, it is convenient to study those variational problems that can be locally formulated in the following form:

Suppose $\dim N = n$. Postulate the existence of 1-forms (ω_u) , $u = n + 1, \dots, m$, and an n -form ω such that for mappings $\phi : N \rightarrow M$ satisfying $\phi^*(\omega_u) = 0$, we have $\int_N \phi^*(\omega) = L(\phi)$. Suppose (ω_a) , $a = 1, \dots, n$, are additional 1-forms such that $\omega_1, \dots, \omega_n$ form a basis for 1-forms on M . Then, ω can be expressed in terms of these forms. Since the terms that involve $\omega_{m+1}, \dots, \omega_n$ will not contribute to $\int_N \phi^*(\omega)$, we can suppose that ω is a multiple of $\omega_1 \wedge \dots \wedge \omega_n$. In fact, we can normalize these forms so that $\omega = \omega_1 \wedge \dots \wedge \omega_n$.

This choice enables us to give the equations of the extremals more explicitly: Suppose $(\omega_{i,j})$ are 1-forms chosen so that

$$(4.1) \quad d\omega_i = \omega_{ij} \wedge \omega_j ; \quad \omega_{aa} = 0.$$

Then,

$$\begin{aligned} (4.2) \quad d\omega &= d\omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_n - \omega_1 \wedge d\omega_2 \wedge \dots \wedge \omega_n + \dots \\ &= \omega_{1i} \wedge \omega_i \wedge \omega_2 \wedge \dots \wedge \omega_n - \dots \\ &= \omega_{1u} \wedge \omega_u \wedge \omega_2 \wedge \dots \wedge \omega_n - \omega_1 \wedge \omega_{2u} \wedge \omega_u \wedge \omega_3 \wedge \dots \wedge \omega_n \\ &\quad + \dots \\ &= \omega_u \wedge \theta_u, \end{aligned}$$

with

$$(4.3) \quad \theta_u = -\omega_{1u} \wedge \omega_2 \wedge \dots \wedge \omega_n - \omega_1 \wedge \omega_{2u} \wedge \omega_3 \wedge \dots \wedge \omega_n - \dots.$$

Then,

$$\phi^*(X \lrcorner d\omega) = \phi^*(\omega_u(X)\theta_u).$$

Thus, ϕ is an extremal if and only if

$$(4.4) \quad \phi^*(\theta_u) = 0.$$

(Since our definition of extremal requires that $\phi^*(\omega_u(X)\theta_u) = 0$ for *all* vector fields X .) This exhibits the extremals as integral manifolds of the exterior differential system: $\theta_u = 0$. It would be interesting to investigate the applicability of Cartan's theory of integral manifolds of exterior differential systems, a task we hope to carry out in a later paper.

We can now carry out the computation of the second variation in a more

explicit form. Suppose that X is a vector field satisfying

$$0 = \phi^*(X(\omega_u)) = \phi^*(d(\omega_u(X)) - X \lrcorner d\omega_u).$$

Then,

$$\begin{aligned} \phi^*(X \lrcorner d(X \lrcorner d\omega)) &= \phi^*(X \lrcorner d(X(\omega))) \\ &= \phi^*(X \lrcorner X(d\omega)) \\ &= \phi^*(X \lrcorner X(\omega_u \wedge \theta_u)) \\ &= \phi^*(X \lrcorner (X(\omega_u) \wedge \theta_u + \omega_u \lrcorner X(\theta_u))) \\ &= \phi^*((X \lrcorner X(\omega_u)) \wedge \theta_u + \omega_u(X)X(\theta_u)) \\ &= \phi^*(\omega_u(X)X(\theta_u)) \quad \text{since } \phi^*(\theta_u) = 0. \end{aligned}$$

This is the basic formula. Let us recapitulate:

Theorem 4.1. *Suppose $\phi^*(\theta_u) = 0 = \phi^*(X(\omega_u)) = \phi^*(\omega_u) = \omega_{aa}$. Then,*

$$(4.5) \quad \int_N \phi^*(SV(X, X)) = \int_N \phi^*(\omega_u(X)X(\theta_u)).$$

5. The geometric meaning of the terms in the second variation formula.

Let us keep the notations of the previous section. In computing the form $\int_N \phi^*(SV(X, X))$ we have seen that we can ignore the component of X tangential to N . Hence, in computing it in terms of the moving frame introduced in the previous section, we may suppose that

$$\phi^*(\omega_a(X)) = 0.$$

Then,

$$\phi^*(X(\omega_a)) = \phi^*(X \lrcorner d\omega_a) = \phi^*(\omega_{ab}(X)\omega_b - \omega_u(X)\omega_{au}).$$

Lemma 5.1. *Let p be a point of N . Then, at most changing*

$$(5.1) \quad \omega_a \rightarrow \omega'_a = M_{ab}\omega_b,$$

where (M_{ab}) is an orthogonal matrix, we can arrange that

$$(5.2) \quad \phi^*(X(\omega_a))(p) = z_a \phi^*(\omega_a)(p) \quad (\text{no summation})$$

where $z_1, \dots, z_n$ are real numbers, and

$$(5.3) \quad \phi^*(d\omega_a)(p) = 0.$$

Proof. So far, we have not made a unique choice of forms (ω_i) . Let us, for the purposes of this lemma, choose them so that they are the connection forms of the Riemannian metric

$$ds^2 = \omega_i \cdot \omega_i,$$

i.e. so that $\omega_{ij} + \omega_{ji} = 0$. Suppose that $\omega_{ij} = \Gamma_{kij}\omega_k$, *i.e.* the (Γ_{kij}) are the usual Christoffel symbols. Then, the symmetric form $\phi^*(\omega_{au} \cdot \omega_a \omega_u(X))$ is independent of frame: It is just the second fundamental form of $\phi(N)$, denoted by $S_{X(p)}(,)$, at the point p , relative to the normal direction $X(p)$.

First, make a change of frame (5.1) so that (5.3) is satisfied. Next, we can make a change of frame (5.1) with constant (M_{ab}) : This leaves (5.3) unaltered, and, if the (M_{ab}) are suitably chosen, diagonalizes the second fundamental form $S_{X(p)}(,)$. But,

$$\phi^*(X(\omega_a))(p) = -\phi^*(\Gamma_{bau}\omega_b\omega_u(X))(p),$$

which is of form (5.2) if $\phi^*(\Gamma_{bau}\omega_u(X))(p)$, which is the matrix of the second fundamental form, is a diagonal matrix. Q.E.D.

Let p continue to be the same point used in Lemma 5.1, and suppose that (5.2) and (5.3) are satisfied. However, we do not want to necessarily suppose that the forms (ω_{ij}) are the connection forms of a Riemannian metric. (The metric was only used for technical convenience here: In the next section we will consider the case given by the metric in more detail.) Then,

$$z_a\phi^*(\omega_a)(p) = \phi^*(X \lrcorner d\omega_a)(p).$$

This gives the relation:

$$(5.4) \quad (\Gamma_{bau}\omega_u(X) + \Gamma_{uab}\omega_u(X))(p) = \begin{cases} 0 & \text{if } a \neq b \\ z_a & \text{if } a = b \end{cases}.$$

Thus,

$$-\phi^*(\theta_u) = \phi^*(\Gamma_{11u} + \cdots + \Gamma_{nnu})\omega),$$

hence the condition the ϕ be an extremal gives

$$(5.5) \quad (\Gamma_{11u} + \cdots + \Gamma_{nnu})(p) = 0.$$

Let us compute $X(\theta_u)(p)$: In these computations it will be convenient to leave out explicit mention that at every step the result is to be restricted to N and evaluated at p . Recall that

$$-\theta_u = \sum_a \omega_1 \wedge \cdots \wedge \omega_{au} \wedge \cdots \wedge \omega_n.$$

Thus,

$$(5.6) \quad -X(\theta_u) = \sum_a ((z_1 + \cdots + z_{a-1} + z_{a+1} + \cdots + z_n) \cdot \omega_1 \wedge \cdots \wedge \omega_{au} \wedge \cdots \wedge \omega_n + \omega_1 \wedge \cdots \wedge X(\omega_{au}) \wedge \cdots \wedge \omega_n).$$

This is as far as it seems profitable to carry the computations at this level of generality: In a later paper we plan to carry the computations to the end for more complicated geometric structures than Riemannian metrics. (For,

example, the case of the projective arc-length in a projective connection which is the case considered by Cartan [1], is an interesting candidate.)

6. The second variation formula for area. Suppose now that M is an oriented Riemannian manifold. Let $\phi : N \rightarrow M$ be an oriented submanifold. The *area* of $\phi(N)$ is obtained by considering the induced Riemannian metric on N , and constructing its volume element form. Locally, this amounts to the following construction: Choose an orthonormal basis (ω_i) of 1-forms of M which is positively oriented. (So that $\omega_1 \wedge \cdots \wedge \omega_m$ is the volume element of M .) Then, if $\phi^*(\omega_u) = 0$, i.e. if the last $m - n$ of the dual basis of vectors is orthogonal to $\phi(N)$, then the volume element of N is $\phi^*(\omega_1 \wedge \cdots \wedge \omega_n)$. If $t \rightarrow \phi_t$ is a deformation of ϕ , we see that the basis (ω_i) can be chosen so that $\phi_t^*(\omega_u) = 0$, so that the variational problems considered in the previous sections really does correspond to extremizing the area of $\phi(N)$. The formulas of Section 6, together with the standard formulas of Riemannian geometry in terms of moving frames, enable us to write the second variation completely in terms of invariantly defined geometric quantities. In fact, the $(\omega_{i,j})$ may be chosen as the connection forms associated with the Levi-Civita affine connection.

Obviously, a change of frame $\omega'_i = M_{ij}\omega_j$, with (M_{ij}) an orthogonal matrix and $M_{au} = 0 = M_{ua}$ does not affect the situation. In particular, the frame may be chosen so that, for a given point $p \in N$, the forms ω_{ab} and ω_{au} vanish at $\phi(p)$. By (5.4), we have then

$$\Gamma_{bau}\omega_u(X)(p) = \begin{cases} 0 & \text{if } a \neq b \\ z_a & \text{if } a = b \end{cases}.$$

In particular, $z_1 + \cdots + z_n = 0$ is the condition that $\phi^*(\theta_u) = 0$. We have already seen that $z_1, \dots, z_n$ are the eigenvalues of the second fundamental form of $\phi(N)$ in the direction $X(\phi(p))$, hence we have the following well-known result:

$\phi(N)$ is a *minimal submanifold* of M , i.e. an extremal for the area variational problem, if and only if the trace of the second fundamental form in each normal direction is zero.

Assuming that $\phi(N)$ is a minimal submanifold, we can proceed to the computation of $SV(X, X)$. By (5.5), we see that

$$X(\theta_u) = z_a\omega_1 \wedge \cdots \wedge \omega_{au} \wedge \cdots \wedge \omega_n - \omega_1 \wedge \cdots \wedge X(\omega_{au}) \wedge \cdots \wedge \omega_n,$$

(Continue the convention that these forms are considered to be restricted to $\phi(N)$ and evaluated at p .)

$$\omega_{au} = \Gamma_{bau}\omega_b.$$

Let $\Omega_{i,j} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj}$ be the *curvature forms*. The curvature tensor R_{ijkl} is given by

$$\Omega_{ij} = R_{ijk\ell} \omega_k \wedge \omega_\ell .$$

Then,

$$\begin{aligned} d\omega_{au} &= R_{au\iota j} \omega_\iota \wedge \omega_j \quad (\text{at } \phi(p)), \\ X(\omega_{au}) &= d(\omega_{au}(X)) + 2R_{auvb} \omega_v(X) \omega_b . \end{aligned}$$

Also,

$$0 = X(\omega_u) = d(\omega_u(X)) - \omega_{au}(X) \omega_a .$$

Put

$$\begin{aligned} A_u &= \omega_u(X), \\ dA_u &= A_{u,\iota} \omega_\iota , \\ dA_{u,\iota} &= A_{u,\iota j} \omega_j , \quad \text{etc.} \end{aligned}$$

Then,

$$\begin{aligned} \omega_{au}(X) &= A_{u,a} , \\ X(\omega_{au}) &= A_{u,ab} \omega_b + 2R_{auvb} A_v \omega_b . \end{aligned}$$

Finally,

$$(6.1) \quad SV(X, X) = (z_a z_a - A_u A_{u,aa} - 2R_{auva} A_u A_v) \omega .$$

The term $2 R_{auva} A_u A_v$ is readily recognized as $\|X(\phi(p))\|^2$ (the length of the normal vector $X(\phi(p))$ times

$$(-K(X_1(\phi(p)), X(\phi(p)) - \cdots - K(X_n(\phi(p)), X(\phi(p))),$$

where $K(,)$ is the sectional curvature of the plane spanned by the two tangent vectors in the argument. Let us denote this term by

$$\|X(\phi(p))\|^2 K(X(\phi(p))) .$$

Let us consider the term $A_u A_{u,aa}$. It is clear that it is, geometrically, to represent the inner product between the normal vector whose components are A_u , *i.e.* $X(\phi(p))$, and one whose components are $A_{u,aa}$. What is this latter vector? It is in fact the Laplace–Beltrami operator (evaluated at p) of the cross-section $p \rightarrow X(p)$ of the normal vector bundle to $\phi(N)$ as defined, for example, by J. Eels & J. Sampson [3]. We will denote it by ΔX . (In Section 7 we will go over its definition in more detail.) Then, the middle term in (6.1) can be written as:

$$-\langle X, \Delta X \rangle ,$$

where we denote the inner product on tangent vectors to M by $\langle, \rangle$. Let S^2 denote the sum of the squares of the eigenvalues of the second fundamental form $S_{X(p)}(,)$. Then, we have the main result of this paper:

$$(6.2) \quad \int_N \phi^*(SV(X, X)) = \int_N (S^2 - \langle X, \Delta X \rangle - \|X\|^2 K(X) ,$$

where the integration on the right hand side is understood to be taken over the volume element defined by the induced Riemannian metric on N .

Now let us look at the condition that the vector field X be transversal to $\phi(N)$ at the boundary. Recall that the general condition is

$$\phi^*(X \lrcorner \omega) = 0 \quad \text{on } N.$$

To compute it explicitly, with the choice of frame used before,

$$\omega = \omega_1 \wedge \omega_2 \cdots \wedge \omega_n,$$

$$X \lrcorner \omega = \omega_1(X)\omega_2 \wedge \cdots \wedge \omega_n - \omega_2(X)\omega_1 \wedge \omega_3 \wedge \cdots \wedge \omega_n + \cdots.$$

Thus the condition that X be transversal is:

$$(6.3) \quad \text{For } p \in \phi(\partial N), X(p) \text{ is perpendicular to } X_1(p), \text{ where } X_1 \text{ is the vector field on } N \text{ which is tangent to } N \text{ but perpendicular to } \partial N.$$

Let us compute the boundary variation:

$$\begin{aligned} BV(X, X) &= X \lrcorner d(X \lrcorner \omega) = X \lrcorner (X(\omega) - X \lrcorner d\omega) \\ &= X \lrcorner X(\omega). \end{aligned}$$

$$X(\omega) = X(\omega_1) \wedge \omega_2 \wedge \cdots \wedge \omega_n + \omega_1 \wedge X(\omega_2) \wedge \cdots \wedge \omega_n + \cdots,$$

$$\begin{aligned} X \lrcorner X(\omega) &= (X \lrcorner X(\omega_1)\omega_2 \wedge \cdots \wedge \omega_n \\ &\quad - (X \lrcorner X(\omega_2))\omega_1 \wedge \omega_3 \wedge \cdots \wedge \omega_n + \cdots). \end{aligned}$$

Suppose that we choose the frame so that $\omega_1 = 0$ when restricted to ∂N . Then, $\theta = \omega_2 \wedge \cdots \wedge \omega_n$ restricted to N is its volume element with respect to the induced Riemannian metric.

$$X \lrcorner X(\omega_a) = X(\omega_a(X)) = X(\langle X_a, X \rangle) = \langle \nabla_X X_a, X \rangle + \langle X_a, \nabla_X X \rangle$$

where $(X_1, \dots, X_m)$ is the basis on vector fields of M dual to the basis of forms, *i.e.* $\omega_i(X_j) = \langle X_i, X_j \rangle = \delta_{ij}$. Then, (6.3) takes the form:

$$\langle X, X_1 \rangle = 0.$$

$$(6.4) \quad \int_{\partial N} \phi^*(BV(X, X)) = \int_{\partial N} \phi^*(\langle \nabla_X X_1, X \rangle + \langle X_1, \nabla_X X \rangle \theta).$$

We will reduce this to a more convenient form in Section 9.

7. The Laplace-Beltrami operator for vector bundles. We will now explain in more detail how the differential operator $X \rightarrow \Delta X$ needed in formula (6.2) is defined. Our treatment will be independent of that by Eels & Sampson [3].

Let N be a manifold, and let $\pi : E \rightarrow N$ define a real vector bundle over N , each of whose fibres has a positive-definite inner product, which we shall denote by $\langle, \rangle$. Let $\Gamma(E)$ denote the space of cross-sections of the vector bundle, *i.e.* an

element ψ of $\Gamma(E)$ is a map $N \rightarrow E$ such that $\psi(p) \in \pi^{-1}(p)$ for $p \in N$. $\Gamma(E)$ is an $F(N)$ -module.

A (linear) *connection* for the vector bundle is defined by assigning, to each $X \in V(N)$, a linear mapping

$$\nabla_X : \Gamma(E) \rightarrow \Gamma(E)$$

("covariant differentiation") such that:

$$\left. \begin{aligned} \nabla_X(f\psi) &= X(f)\psi + f\nabla_X\psi \\ \nabla_{fX}\psi &= f\nabla_X\psi \end{aligned} \right\} \begin{array}{l} \text{for } X \in V(N), \quad \psi \in \Gamma(E), \\ f \in F(N). \end{array}$$

The connection *preserves the metric on the fibres of E* if:

$$X(\langle \psi_1, \psi_2 \rangle) = \langle \nabla_X \psi_1, \psi_2 \rangle + \langle \psi_1, \nabla_X \psi_2 \rangle$$

for $\psi_1, \psi_2 \in \Gamma(E)$, $X \in V(M)$. Suppose also that N has a Riemannian metric, determined by a positive definite inner product

$$(X, Y) \rightarrow \langle X, Y \rangle$$

on vector fields, and determining in the usual way the Levi-Civita *affine* connection $(X, Y) \rightarrow \nabla_X Y$. Using these two connections, we want to define the Laplace-Beltrami operator $\psi \rightarrow \Delta\psi$, as a linear map: $\Gamma(E) \rightarrow \Gamma(E)$. One method for doing this can be described as follows:

(7.1) Let p be a point of N . Choose a basis (X_a) for $V(N)$ such that $\langle X_a, X_b \rangle = \delta_{ab}$ ($1 \leq a, b, \dots \leq n = \dim N$) and $\nabla_{X_a} X_b(p) = 0$. Define $\Delta\psi(p)$ as $\nabla_{X_a} \nabla_{X_a} \psi(p)$. Let $\Delta\psi$ be the cross-section of $p \rightarrow \Delta\psi(p)$.

(It is readily verified that $\Delta\psi$ so defined is independent of the frame chosen.) Of course, it is desirable to have a global method for defining $\Delta\psi$. One such has been given by Eels & Sampson [3]. We will now briefly describe another one, although it is not strictly necessary for our immediate purposes.

The given Riemannian metric on N and inner product on the fibres of E defines a Riemannian metric on E . It can be described locally as follows:

Let (ω_a) be an orthonormal basis of 1-forms on N , and let (ψ_u) , $n+1 \leq u, v, \dots \leq m = \dim E$, be an orthonormal basis for cross-sections of E . Let y_u be the real-valued functions on E defined by:

$$y_u(v) = \langle \psi_u(p), v \rangle \quad \text{for } v \in \pi^{-1}(p), \quad p \in N.$$

Let ω_{uv} be the one-forms on N such that:

$$\omega_{uv}(X) = \langle \psi_u, \nabla_X \psi_v \rangle \quad \text{for } X \in V(N).$$

Then, the metric on E is defined by

$$ds^2 = \pi^*(\omega_a \cdot \omega_a) + [dy_u - \pi^*(\omega_{uv})y_v] \cdot [dy_u - \pi^*(\omega_{uv})y_v]$$

($\cdot$ denotes symmetric product of 1-forms). A cross-section $\psi \in \Gamma(E)$ defines

a real-valued function $f_\psi : v \rightarrow \langle \psi(p), v \rangle$ for $p \in N, v \in \pi^{-1}(v)$. (Such real-valued functions on E are characterized by being linear on the fibres of E .) One proves easily by explicit calculation that the result of applying the "ordinary" Laplace-Beltrami operator $\Delta = \text{div}(\text{grad})$ (e.g., see [5]) to f_ψ results in another function on E which is linear on the fibres, and that in fact

$$\Delta f_\psi = f_{\Delta\psi},$$

where $\psi \rightarrow \Delta\psi$ is the operator defined by (7.1). (Since calculations required to prove these facts are routine, we will not give them here.)

Now, suppose the objects ω_{uv} , ψ_u , X_a are as defined above. Suppose ψ is an arbitrary cross-section of π , of the form

$$\psi = A_u \psi_u.$$

Let p be a point of N at which we are supposing

$$\nabla_{X_a} X_b(p) = 0,$$

and

$$\omega_{uv}(p) = 0.$$

Then,

$$\begin{aligned} \nabla_{X_a} \psi &= X_a(A_u) \psi_u + A_u \nabla_{X_a} \psi_u \\ &= X_a(A_u) \psi_u + A_u \langle \nabla_{X_a} \psi_u, \psi_v \rangle \psi_v \\ &= X_a(A_u) \psi_u + A_u \omega_{uv}(X_a) \psi_v. \end{aligned}$$

Note that:

$$\begin{aligned} \omega_{uv} + \omega_{vu} &= 0. \\ \nabla_{X_a} \nabla_{X_a}(p) &= (X_a X_a(A_u) \psi_u + A_u X_a(\omega_{uv}(X_a)) \psi_v)(p). \end{aligned}$$

Hence,

$$(7.2) \quad \langle \Delta\psi, \psi \rangle(p) = A_u X_a X_a(A_u)(p).$$

Let us return to the situation considered in Section 6: Let N be a submanifold of a Riemannian manifold M , and let E be the vector bundle of the tangent vectors to M that are normal to N . It inherits from the Riemannian metric on M an inner product on its fibres. It also inherits a connection in the following way:

Let $X \in V(N)$, $\psi \in \Gamma(E)$. Extend ψ to a vector field Y on M . Define, for $p \in N$,

$$\nabla_X \psi(p) = \text{projection of } \nabla_X Y(p) \text{ on the normal vectors to } N \text{ at } p.$$

($\nabla_X Y$ is the covariant derivative relative to the Levi-Civita affine connection on M .) Using this definition, it is now clear that if N is a minimal submanifold of M , and if X is a vector field on M perpendicular to N that is used in the

calculation of the second variation formula (6.2), then $\langle \Delta X, X \rangle$ as defined here, *i.e.* by (7.2), is precisely as it should be to give the result indicated in (6.2), using (6.1). This completes our discussion of the geometry involved in (6.2) itself.

Next, we turn to expressing

$$\int_N \langle \Delta X, X \rangle \omega$$

as an integral over the boundary of N and an integral over N of a *positive* quantity. Since this invokes Green's formula, it is convenient to review the circle of ideas centering on Green's formula.

8. Green's formula for vector bundles. Let N be an oriented Riemannian manifold with oriented boundary ∂N . The Laplace–Beltrami operator for functions $f \in F(N)$ is defined as:

$$\Delta f = \operatorname{div} (\operatorname{grad} f),$$

where $\operatorname{grad} f$ is the vector field such that

$$X(f) = \langle \operatorname{grad} f, X \rangle \quad \text{for } X \in V(N),$$

and $\operatorname{div} (\operatorname{grad} f)$ is the function such that

$$\operatorname{grad} f(\omega) = \operatorname{div} (\operatorname{grad} f) \omega,$$

where ω is the n -form on N representing the volume element defined by the Riemannian metric. Thus, the formula

$$\operatorname{grad} f(\omega) = d(\operatorname{grad} f \lrcorner \omega),$$

gives

$$\int_N (\Delta f) \omega = \int_{\partial N} (\operatorname{grad} f \lrcorner \omega).$$

The term on the right hand side is just the integral over ∂N of the component of $\operatorname{grad} f$ normal to ∂N .

Now, let $\pi : E \rightarrow N$ be a vector bundle over N . As in Section 7, suppose it has a positive-definite inner product on its fibres and a connection which preserves the metric. If $\psi_1, \psi_2 \in \Gamma(E)$, our goal is to express $\int_N \langle \Delta \psi_1, \psi_2 \rangle$ (For convenience from now on we leave off explicit mention of the volume element on N and ∂N . It is fixed throughout the rest of the paper as the ones described above.) as an integral over the boundary, and an integral over N of a positive quantity and $\langle \psi_1, \Delta \psi_2 \rangle$.

Let p be a point of N . Choose an orthonormal basis for $V(N)$ such that:

$$\nabla_{X_a} X_b(p) = 0.$$

Unless mentioned otherwise, the following computation will be at p , using this choice of frame.

$$\begin{aligned}
 \Delta(\langle \psi_1, \psi_2 \rangle) &= X_a X_a (\langle \psi_1, \psi_2 \rangle) \\
 &= X_a (\langle \nabla_{X_a} \psi_1, \psi_2 \rangle + \langle \psi_1, \nabla_{X_a} \psi_2 \rangle) \\
 &= \langle \nabla_{X_a} \nabla_{X_a} \psi_1, \psi_2 \rangle + 2 \langle \nabla_{X_a} \psi_1, \nabla_{X_a} \psi_2 \rangle + \langle \psi_1, \nabla_{X_a} \nabla_{X_a} \psi_2 \rangle.
 \end{aligned}$$

Let $D\psi$ be the 1-form on N with values in the fibres of E defined by:

$$D\psi(X) = \nabla_X \psi, \quad \text{for } X \in V(N).$$

$D\psi$ can be regarded as a cross-section of the bundle over N resulting from taking the tensor product of E with the cotangent bundle to N . Since both bundles have a positive definite inner product, so does the tensor product bundle. Let $\langle D\psi_1, D\psi_2 \rangle$ be the inner product of these two cross-sections. Then, we have, independently of special choice of frame,

$$\Delta(\langle \psi_1, \psi_2 \rangle) = \langle \Delta \psi_1, \psi_2 \rangle + \langle \psi_1, \Delta \psi_2 \rangle + 2 \langle D\psi_1, D\psi_2 \rangle.$$

Hence

$$\begin{aligned}
 \int_N \langle \Delta \psi_1, \psi_2 \rangle + \langle \psi_1, \Delta \psi_2 \rangle &= \int_N \Delta(\langle \psi_1, \psi_2 \rangle) - 2 \langle D\psi_1, D\psi_2 \rangle \\
 &= \int_{\partial N} \text{grad}(\langle \psi_1, \psi_2 \rangle) \omega - 2 \int_N \langle D\psi_1, D\psi_2 \rangle.
 \end{aligned}$$

Suppose that we choose the bases (X_a) for $V(N)$ so that X_1 is perpendicular to ∂N and points inside N . Then, $\omega_2 \wedge \cdots \wedge \omega_n$ is the volume element of ∂N , hence:

$$(8.1) \quad \int_N \langle \Delta \psi_1, \psi_2 \rangle + \langle \psi_1, \Delta \psi_2 \rangle = -2 \int_N \langle D\psi_1, D\psi_2 \rangle + \int_N X_1(\langle \psi_1, \psi_2 \rangle).$$

(Thus, $X_1(\langle \psi_1, \psi_2 \rangle)$ is defined invariantly as the inward normal derivative to ∂N .) Further, note that:

$$(8.2) \quad \int_N \|D\psi\|^2 = 0 \quad \text{if and only if} \quad \nabla_X \psi = 0 \quad \text{for all } X \in V(N).$$

This immediately applies to the second variation formula (6.2), and enables us to write it as:

$$(8.3) \quad \int_N \varphi^*(SV(X, X)) = \int_N S^2 - \|X\|^2 K(X) + 2 \|DX\|^2 + \int_{\partial N} X_1(\langle X, X \rangle).$$

Now, we must develop Green's formula itself. The classical Green's formula expresses

$$\int_N f \Delta g - g \Delta f$$

as an integral over ∂N , for $f, g \in F(N)$. The generalization to vector bundles requires a calculation of

$$\int_N \langle \Delta \psi_1, \psi_2 \rangle - \langle \psi_1, \Delta \psi_2 \rangle.$$

Now

$$\begin{aligned}\langle \Delta \psi_1, \psi_2 \rangle &= \langle \nabla_{x_a} \nabla_{x_a} \psi_1, \psi_2 \rangle \\ &= X_a(\langle \nabla_{x_a} \psi_1, \psi_2 \rangle) - \langle \nabla_{x_a} \psi_1, \nabla_{x_a} \psi_2 \rangle, \\ \langle \Delta \psi_1, \psi_2 \rangle - \langle \psi_1, \Delta \psi_2 \rangle &= X_a(\langle \nabla_{x_a} \psi_1, \psi_2 \rangle - \langle \nabla_{x_a} \psi_2, \psi_1 \rangle).\end{aligned}$$

Given $\psi_1, \psi_2 \in \Gamma(E)$, define $Z(\psi_1, \psi_2)$ as the vector field on N such that

$$\langle Z(\psi_1, \psi_2), X \rangle = \langle \nabla_X \psi_1, \psi_2 \rangle.$$

Then,

$$Z(\psi_1, \psi_2) = \langle \nabla_{x_a} \psi_1, \psi_2 \rangle X_a.$$

Also then,

$$X_a(\langle \nabla_{x_a} \psi_1, \psi_2 \rangle) = \operatorname{div} Z(\psi_1, \psi_2).$$

Then,

$$\langle \Delta \psi_1, \psi_2 \rangle - \langle \psi_1, \Delta \psi_2 \rangle = \operatorname{div} (Z(\psi_1, \psi_2) - Z(\psi_2, \psi_1)).$$

Integrating, we have:

$$\begin{aligned}(8.4) \quad \int_N \langle \Delta \psi_1, \psi_2 \rangle - \langle \psi_1, \Delta \psi_2 \rangle &= \int_{\partial N} \langle Z(\psi_1, \psi_2), X_1 \rangle - \langle Z(\psi_2, \psi_1), X_1 \rangle \\ &= \int_{\partial N} \langle \nabla_{x_1} \psi_1, \psi_2 \rangle - \langle \nabla_{x_1} \psi_2, \psi_1 \rangle.\end{aligned}$$

This is Green's formula.

9. Towards a Morse theory for minimal submanifolds. As mentioned in the Introduction, there is much difficult work to be done before Morse theory for minimal submanifolds becomes a satisfactory theory. The results in this paper enable one to compute the Hessian form, as we show in this section. We hope to study it, motivated by one's geometric intuition. Of course, the method given in this paper enables one to calculate the Hessian for more general multiple integral variational problems. The surface area case is simply a good illustration for our methods.

Let M be a Riemannian manifold, and let N be an oriented manifold with oriented boundary ∂N , and let B be a submanifold of M . Let T be the space of submanifold maps $\varphi : N \rightarrow M$ such that:

$$\varphi(\partial N) \subset B.$$

As before, we consider the real-valued function

$$\varphi \rightarrow \mathbf{L}(\varphi) = \text{area of } \varphi(N).$$

If $t \rightarrow \varphi_t$ is a deformation of φ , *i.e.* a curve in T beginning at φ for $t = 0$, we consider:

$$(9.1) \quad \left. \frac{d^2}{dt^2} L(\varphi_t) \right|_{t=0},$$

if $(d/dt) L(\varphi_t)|_{t=0} = 0$ for all such deformations on T .

Suppose we consider deformations of the form $\varphi_t = \text{Exp}(tX)\varphi$, for $X \in V(M)$. There is then a symmetric bilinear form

$$(X, Y) \rightarrow h(X, Y),$$

such that $h(X, X)$ equals (9.1). h is called the *Hessian* of the minimal submanifold.

Now the condition that φ_t continue to be on B is that:

$$(9.2) \quad X \text{ is tangent to } B.$$

Hence, the condition that $\mathbf{L}$ have a critical point at φ is that:

- (9.3) a) φ be a minimal submanifold of M .
 b) $X_1(p)$ is normal to B for each $p \in \varphi(\partial N)$, where X_1 is the vector field on $\varphi(N)$ which on $\varphi(\partial N)$ points inward, normal to B .

Suppose now that these conditions are satisfied, and that $X \in V(M)$ satisfies (9.2). Then,

$$h(X, X) = \int_N [S^2 - \langle \Delta X, X \rangle - 2R(X, X)] + \int_{\partial N} \langle \nabla_X X_1, X \rangle + \langle X_1, \nabla_X X \rangle.$$

Suppose (X_i) is an orthonormal basis for vector fields such that $X_1, \dots, X_n$ are tangent to $\varphi(N)$. Then, from (6.1),

$$(9.4) \quad R(X, X) = \langle R(X_a, X)(X), X_a \rangle = -\langle R(X_a, X)(X_a), X \rangle,$$

$$(9.5) \quad S^2 = S_X(X_a, X_b)S_X(X_a, X_b) \equiv S^2(X, X),$$

where

$$S^2(X, Y) = S_X(X_a, X_b)S_Y(X_a, X_b).$$

Further,

$$\langle \nabla_X X, X_1 \rangle = S_{X_1}^B(X, X),$$

where $S_0^B(\cdot)$ is the second fundamental form of B .

$$\begin{aligned} \langle \nabla_X X_1, X \rangle &= \langle \nabla_{X_1} X, X \rangle + \langle [X, X_1], X \rangle \\ &= \langle \nabla_{X_1} X, X \rangle + \omega_u(X)\omega_u([X, X_1]) \\ &= \langle \nabla_{X_1} X, X \rangle + \omega_u(X)(\langle X(\omega_u)(X_1) - X(\omega_u(X_1)) \rangle). \end{aligned}$$

However, $\varphi^2(X(\omega_u)) = 0 - \omega_u(X_1)$. Then, for the purposes of integrating over ∂N , we have:

$$\langle \nabla_X X_1, X \rangle = \langle \nabla_{X_1} X, X \rangle.$$

Thus

$$h(X, X) = \int_N S^2(X, X) - \langle \Delta X, X \rangle - 2R(X, X) \\ + \int_{\partial N} S_{X_1}^B(X, X) + \langle \nabla_{X_1} X, X \rangle.$$

Put: $R(X, Y) = \langle R(X_a, X)(X_a), Y \rangle$. It is readily verified to be symmetric in X and Y . $S^2(X, Y)$ is obviously also symmetric in X and Y . So is $S_{X_1}^B(X, Y)$. Then,

$$(9.6) \quad h(X, Y) = \int_N S^2(X, Y) - \frac{1}{2}(\langle \Delta X, Y \rangle + \langle X, \Delta Y \rangle) - 2R(X, Y) \\ + \int_{\partial N} S_{X_1}^B(X, Y) + \frac{1}{2}(\langle \nabla_{X_1} X, Y \rangle + \langle \nabla_{X_1} Y, X \rangle)$$

equals, using Green's formula, (8.4),

$$\int_N S^2(X, Y) - \langle \Delta X, Y \rangle - 2R(X, Y) + \int_{\partial N} S_{X_1}^B(X, Y) + \langle \nabla_{X_1} X, Y \rangle.$$

This is the first formula, enabling us to carry forward "geometric" Morse theory. For example, the *index* of the extremal φ can be read off as the number of solutions of the eigenvalue equations.

$$(9.7) \quad \Delta X - 2R(X) + S^2(X) = \lambda X,$$

for *negative* λ , subject to the boundary condition:

$$(9.8) \quad S_{X_1(p)}^B(X(p), B_p) + \langle \nabla_{X_1} X(p), B_p \rangle = 0,$$

for all $p \in \varphi(\partial N)$, where $R(X)$ and $S^2(X)$ are the linear transformations such that:

$$R(X, Y) = \langle R(X), Y \rangle, \\ S^2(X, Y) = \langle S^2(X), Y \rangle.$$

φ is a *degenerate* extremal if and only if (9.7) and (9.8) have non-zero solutions with $\lambda = 0$.

Using (8.1), we can also express $h(X, X)$ as an integral involving only first derivatives of X . In fact, (8.1) gives:

$$\int_N \langle \Delta X, X \rangle = - \int_N \langle \nabla_{X_a} X, \nabla_{X_a} X \rangle + \int_{\partial N} \langle \nabla_{X_1} X, X \rangle,$$

whence:

$$(9.9) \quad h(X, X) = \int_N S^2(X, X) \\ + \langle \nabla_{X_a} X, \nabla_{X_a} X \rangle - 2R(X, X) + \int_{\partial N} S_{X_1}^B(X, X).$$

This immediately gives a simple interesting result in case the curvature of M is non-positive.

Theorem 9.1. *Suppose that the sectional curvature of M is non-positive. (In case $\dim N = \dim M - 1$, it would suffice to assume that the Ricci curvature of M is non-positive), and that either:*

- a) B is totally geodesic, or
- b) $B = \varphi(\partial N)$, i.e. the boundary conditions for X are:

$$X(p) = 0 \quad \text{for } p \in \varphi(N).$$

Then, $h(X, X) \geq 0$. $h(X, X) = 0$ if and only if $DX = 0$ (i.e. $\langle \nabla_{x_a} X, Y \rangle = 0$ for all vector fields Y normal to $\varphi(N)$) and $\varphi(N)$ is totally geodesic. (In case a), $DX = 0$ implies that $X = 0$ over $\varphi(N)$.)

In particular, this proves that if the curvature of M is ≤ 0 , there is no deformation $t \rightarrow \varphi_t$ of φ with the boundary fixed such that each submanifold is minimal. For geodesics, this is of course just the theorem that geodesics have no conjugate points. A more global version of Theorem 9.1 will be presented in [6].

The next step in the program shall be to generalize the Morse index theorem and the Rauch comparison theorem [8, 9]. S. Smale has given a version of the index theorem [10], using general theorems for elliptic operators but not the comparison theorem itself. Then, there remains much to be done, although this sort of problem should be much more tractable than the general existence theorems, since the analysis involved is linear.

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The University of California
 Santa Cruz
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