The Null Bundle of an Einstein-Riemann Space II: Normed Null Bases

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1. Introduction. This paper is a direct continuation of the results established in reference [1]. We shall refer to [1] by the Roman numeral I and cite explicit references to theorems and definitions from I by hyphenation with I. The notation is the same as that established in I and will be used without further comment. As an aid to the reader, we note here that the symbol E is used throughout to denote a given 4-dimensional, hyperbolic-normal metric space with metric tensor g_{AB} .

The purpose of this paper is to develop the theory of normed null bases and to apply the results to the problem of solving the systems of partial differential equations developed in I. This is accomplished by means of a fundamental existence theorem that is proved in Sec. 5. This theorem provides a fairly simple computational basis: all calculations can be reduced to considerations of anholonomic (scalar) fields defined over a Minkowski space. We apply these results at the end of the paper to obtain several new and interesting exact solutions of the Einstein vacuum field equations. One of the solutions is of particular interest since it represents a non-radiating gravitational field of two parallel rods in gravitational equilibrium with their mutual rotation about the common center.

2. Representation. Consider a system of four, covariant, linearly independent, null vector fields v_A^a ($a = 1, 2, 3, 4$) on E such that v_A^1 and v_A^2 are complex conjugates while v_A^3 and v_A^4 are real. Here and throughout this paper, capital Roman letters are used for tensor indices, lower case Roman letters are used for labeling indices and for anholonomic indices, while lower case Greek letters are used exclusively for labeling indices. All indices are assumed to obey the summation convention over the range 1 through 4. The system v_a^A dual to v_A^a is defined by the requirements

$$(2.1) \quad v_a^A v_A^b = \delta_a^b \quad \text{or} \quad v_a^A v_B^a = \delta_B^A.$$

Definition 2.1. A set of null vector fields v_A^a , as defined above, is said to

form a *normed null basis* on E if and only if

$$(2.2) \quad g^{AB} v_A^1 v_B^2 = g^{AB} v_A^3 v_B^4 = 1$$

and the (real) bivectors

$$[1, 2]_{AB} = i v_{[A}^1 v_{B]}^2 \quad \text{and} \quad [3, 4]_{AB} = v_{[A}^3 v_{B]}^4$$

are totally perpendicular.

In order to obtain a characterization of all normed null bases on a space E , we first require necessary and sufficient conditions that an arbitrary collection of four complex-valued vector fields forms a normed null basis.

Theorem 2.1. *Let the quantities g_{ab} and g^{ab} be defined by the relations*

$$(2.3) \quad g_{ab} = g_{AB} v_a^A v_b^B, \quad g^{ab} = g^{AB} v_A^a v_B^b,$$

and set

$$(2.4) \quad I_a^b = \delta_a^1 \delta_2^b + \delta_a^2 \delta_1^b + \delta_a^3 \delta_3^b + \delta_a^4 \delta_4^b.$$

A collection of four complex-valued vector fields v_A^a forms a normed null basis on E if and only if

$$(2.5) \quad g_{ab} = \delta_a^1 \delta_b^2 + \delta_a^2 \delta_b^1 + \delta_a^3 \delta_b^4 + \delta_a^4 \delta_b^3 = g^{ab}$$

and

$$(2.6) \quad \bar{v}_A^a = I_b^a v_A^b,$$

where the superior bar denotes complex conjugation.

Proof. When (2.4) is substituted into (2.6), it is easily seen that (2.6) states the requirement that v_a^1 and v_a^2 are complex conjugate fields while v_A^3 and v_A^4 are real. The condition $g_{ab} = g^{ab}$ follows directly from (2.1), (2.3), and the fact that g_{AB} and g^{AB} are reciprocal tensor fields. The zeros on the main diagonal of $((g^{ab}))$ show that the v 's are all null vectors, while the 1's simply state that the v 's satisfy the conditions (2.2). The remaining zeros state the conditions for total perpendicularity of the bivectors $[1, 2]_{AB}$ and $[3, 4]_{AB}$. When (2.5) is substituted into the second of (2.3) and we take determinants of both sides of the resulting system of equations, we have

$$1 = \det(g^{AB}) (\det(v_A^a))^2,$$

and hence the v 's are linearly independent. Thus all conditions of Definition 2.1 are met and the theorem is established.

The following theorem provides us with a characterization of *all* normed null bases on E . This result is of central importance in this series of papers, since it provides a basis for a rather general existence theorem.

Theorem 2.2. *Let v_A^a be a normed null basis on E . Then the set of all normed null bases on E is given by*

$$(2.7) \quad V_A^\alpha = C_a^\alpha v_A^b, \quad V_A^A = C_a^A v_A^b, \quad C_b^\alpha C_\beta^b = \delta_\beta^\alpha,$$

where $((C_a^\alpha))$ ranges through all collections of 16 complex-valued functions on E that satisfy the conditions

$$(2.8) \quad C_a^\alpha C_b^\beta g^{ab} = g^{\alpha\beta}, \quad g^{\alpha\beta} = \delta_a^\alpha \delta_b^\beta g^{ab},$$

$$(2.9) \quad \bar{C}_a^\alpha I_f^a = I_\beta^\alpha C_f^\beta, \quad I_\beta^\alpha = \delta_a^\alpha \delta_\beta^b I_b^a.$$

Proof. We have

$$g^{AB} V_A^\alpha V_B^\beta = g^{AB} v_A^a v_B^b C_a^\alpha C_b^\beta = g^{ab} C_a^\alpha C_b^\beta = g^{\alpha\beta}$$

and

$$\bar{V}_A^\alpha - I_\beta^\alpha V_A^\beta = \bar{C}_a^\alpha \bar{v}_A^a - I_\beta^\alpha C_a^\beta v_A^a = (\bar{C}_a^\alpha I_f^a - I_\beta^\alpha C_f^\beta) v_A^f.$$

By Theorem 2.1, V_A^α forms a normed null basis on E if and only if (2.3) and (2.4) hold for $g^{\alpha\beta}$ given by the second of (2.8) and (2.5). When these conditions are combined with the above calculations and the linear independence of the v 's is used, we obtain (2.8) and (2.9). It then follows that the totality of all systems of four complex-valued vector fields defined by (2.7), when the C 's range through all collections of complex-valued functions satisfying (2.8) and (2.9), will form normed null bases on E . Conversely, if we have any normed null basis V_A^α , then we may define a matrix of scalars by the relations

$$C_a^\alpha = V_A^\alpha v_A^A,$$

where v_A^A is a fixed normed null basis on E . It is then easily shown by use of Theorem 2.1 that the C 's satisfy the conditions (2.8) and (2.9). This demonstration involves, of course, the invertability of the matrix $((C_a^\alpha))$, which follows from taking determinants of the above relations defining the C 's and noting that $\det(V_A^\alpha) \neq 0$ and $\det(v_b^B) \neq 0$ throughout E .

The following result will be needed later.

Theorem 2.3. *The system (2.8) is equivalent to the system*

$$(2.10) \quad C_a^\alpha C_b^\beta g_{\alpha\beta} = g_{ab}, \quad g_{\alpha\beta} = g^{\alpha\beta}.$$

Proof. We have

$$g_{AB} V_A^\alpha V_B^\beta = g_{AB} v_A^a v_B^b C_a^\alpha C_b^\beta = g_{ab} C_a^\alpha C_b^\beta.$$

Since this must be equal to $g_{\alpha\beta}$ if V_A^α is to form a dual normed null basis on E , we obtain (2.10). From the proof of (2.8), however, the conditions (2.8) and (2.10) give identical requirements on the C 's since $g_{ab} = g^{ab}$ implies and is implied by $g_{\alpha\beta} = g^{\alpha\beta}$, as is easily seen from the second of (2.8) and (2.5).

3. The form of the matrix C. Theorem 2.2 gave a representation of all normed null bases in terms of a given normed null basis and a collection of functions C_a^α , where the C 's are arbitrary to within satisfaction of the conditions

(2.8) and (2.9). In order to obtain a general set of C 's that satisfy these conditions, it is convenient to introduce matrix formulations.

Let $\mathbf{E}$ denote the 4×4 identity matrix, and set

$$(3.1) \quad \mathbf{G} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{T} = 2^{-1/2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad i^2 = -1.$$

It is then easily seen that

$$(3.2) \quad \mathbf{G} = \mathbf{G}' = \mathbf{G}^{-1} = \bar{\mathbf{G}}, \quad \mathbf{I} = \mathbf{I}' = \mathbf{I}^{-1} = \bar{\mathbf{I}}, \quad \mathbf{T}^{-1} = \bar{\mathbf{T}}',$$

where t denotes transposition, -1 denotes inversion, and the bar denotes complex conjugation. Further, we have

$$(3.3) \quad \mathbf{H} = \mathbf{TGT}', \quad \mathbf{E} = \mathbf{TIT}' = \bar{\mathbf{T}}\bar{\mathbf{I}}\bar{\mathbf{T}}' = \bar{\mathbf{T}}\mathbf{I}\mathbf{T}^{-1},$$

and

$$(3.4) \quad \mathbf{GI} = \mathbf{IG}, \quad \mathbf{T}'\mathbf{T} = \mathbf{I}.$$

We note at this point that $\mathbf{H}$ is the real matrix representation of the Minkowski metric in Cartesian coordinates with unit light speed, while $\mathbf{G}$ is the matrix equivalent of (g^{ab}) , as is easily seen by a comparison of (3.1) and (2.5). Thus, since $\mathbf{T}$ is nonsingular, it follows from the first of (3.3) that $\mathbf{H}$ and $\mathbf{G}$ are equivalent.

Let us define the matrix $\mathbf{C}$ by $\mathbf{C} = ((C_{\alpha}^a))$, where a is the column index and α is the row index. A direct inspection shows that the system of equations (2.8) and (2.9) is equivalent to the matrix system

$$(3.5) \quad \mathbf{CGC}' = \mathbf{G}, \quad \bar{\mathbf{C}}\mathbf{I} = \mathbf{IC}.$$

Theorem 3.1. *Let $\mathfrak{B}$ be the set of all real-valued, 4×4 matrices such that*

$$(3.6) \quad \mathbf{BHB}' = \mathbf{H},$$

and let $\mathfrak{C}$ be the set of all complex-valued, 4×4 matrices that satisfy the conditions (3.5). Then

$$(3.7) \quad \mathbf{B} = \mathbf{TCT}^{-1}$$

is an element of $\mathfrak{B}$ for all $\mathbf{C} \in \mathfrak{C}$, and (3.7) establishes an isomorphism between $\mathfrak{B}$ and $\mathfrak{C}$.

Proof. Since $\mathbf{T}$ is nonsingular, (3.7) is equivalent to the system

$$(3.8) \quad \mathbf{C} = \mathbf{T}^{-1}\mathbf{B}\mathbf{T}.$$

It is easily seen from the first of (3.5) that $\mathbf{C}$ is nonsingular, and hence $\mathbf{B}$ is nonsingular by (3.7). Since the second of (3.5) is equivalent to $\bar{\mathbf{C}}\mathbf{C}^{-1} = \mathbf{I}$, (3.3) and (3.8) combine to give

$$\mathbf{0} = \bar{\mathbf{C}}\mathbf{C}^{-1} - \mathbf{I} = \bar{\mathbf{T}}^{-1}\bar{\mathbf{B}}\bar{\mathbf{T}}\mathbf{T}^{-1}\mathbf{B}^{-1}\mathbf{T} - \mathbf{I} = \bar{\mathbf{T}}^{-1}\bar{\mathbf{B}}\mathbf{B}^{-1}\mathbf{T} - \mathbf{I}.$$

When (3.3) is again used, it is seen that $\mathbf{B}$ must be a real-valued matrix. Similarly, the first of (3.5) and (3.3) give

$$\mathbf{0} = \mathbf{C}\mathbf{G}\mathbf{C}' - \mathbf{G} = \mathbf{T}^{-1}\mathbf{B}\mathbf{T}\mathbf{G}\mathbf{T}'\mathbf{B}'\mathbf{T}^{-1'} - \mathbf{G} = \mathbf{T}^{-1}\mathbf{B}\mathbf{H}\mathbf{B}'\mathbf{T}^{-1'} - \mathbf{G}.$$

When the first of (3.3) is used again, we obtain $\mathbf{B}\mathbf{H}\mathbf{B}' - \mathbf{H} = \mathbf{0}$, and hence $\mathbf{B}$ satisfies (3.6). Thus, since $\mathbf{B}$ has been shown to be real, $\mathbf{B}$ is an element of $\mathfrak{B}$ for all $\mathbf{C}$ contained in $\mathfrak{C}$. Since $\mathbf{T}$ is nonsingular, the mapping defined by (3.7) is one-to-one and onto. Further, since (3.7) takes $\mathbf{C} = \mathbf{E}$ into $\mathbf{B} = \mathbf{E}$, the mapping is an isomorphism.

It clearly follows from (3.5) and (3.6) that the sets of matrices $\mathfrak{C}$ and $\mathfrak{B}$ form groups, which we denote by $\mathfrak{C}$ and $\mathfrak{B}$.

Corollary 3.1. *The groups $\mathfrak{B}$ and $\mathfrak{C}$ are isomorphic groups of 4×4 matrices under the mappings (3.7).*

It is easily seen from the third of (3.1) and (3.6) that the Jacobian matrix of any element of the Lorentz group is an element of $\mathfrak{B}$. Now, the components of the matrices in the groups $\mathfrak{B}$ and $\mathfrak{C}$ can be arbitrary functions of position in the space E since there is no requirement that these matrices arise from coordinate transformations in E . However, if P is a generic point of E , then the group $\mathfrak{B}$ evaluated at the point P is isomorphic to the group of Lorentz Jacobian matrices. This follows from the completeness of the Lorentz Jacobian matrices with respect to solutions of (3.6) that are independent of position in the space E [2].

Theorem 3.2. *Let $\mathfrak{L}$ denote the group of Lorentz Jacobian matrices and let $\mathfrak{M}(\mathfrak{L})$ denote a mapping of the parameters of $\mathfrak{L}$ (mapping of the group space coordinates) into the space of real-valued functions defined on E ; then $\mathfrak{B} = \cup \mathfrak{M}(\mathfrak{L})$, where the union is over all maps $\mathfrak{M}$.*

We now combine Theorems 3.1 and 3.2 to obtain the following characterization of the $\mathfrak{C}$'s.

Theorem 3.3. *The group of matrices $\mathfrak{C}$ is isomorphic (under (3.7)) to the group of matrices obtained by all mappings of the parameters of the group of Lorentz Jacobian matrices into the space of real valued functions defined over E .*

With the above theorems, we can obtain an explicit representation of the matrices $\mathbf{C}$ which, although not necessarily the simplest, is adequate for our

present purposes. Let (x, y, z, t) denote the usual coordinates of Minkowski space. If we rotate the x, y axes through an angle α , perform a Lorentz transformation of the z, t plane, and then reflect the new z axis if $\epsilon_1 = -1$ and the new t axis if $\epsilon_2 = -1$, the last of equations (3.1) and (3.8) yield the following matrix for $\mathbf{C}$:

$$(3.9) \quad 2\mathbf{N}(\alpha, \beta; \epsilon_1, \epsilon_2) = \begin{bmatrix} 2e^{i\alpha} & 0 & 0 & 0 \\ 0 & 2e^{-i\alpha} & 0 & 0 \\ 0 & 0 & (\epsilon_1 + \epsilon_2)e^\beta & (\epsilon_1 - \epsilon_2)e^{-\beta} \\ 0 & 0 & (\epsilon_1 - \epsilon_2)e^\beta & (\epsilon_1 + \epsilon_2)e^{-\beta} \end{bmatrix},$$

with $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$. For $\epsilon_1\epsilon_2 = 1$, the matrix $\mathbf{N}$ has been obtained by Hlavatý [3] and has been extensively used by him in his analysis of the Einstein unified field theory and in the construction of general 4-component spin space over a differentiable manifold. If we rotate the (y, z) plane through an angle ϕ_2 followed by a rotation of the resulting (x, z) plane through an angle ϕ_1 , the $\mathbf{C}$ -matrix corresponding to this is

$$(3.10) \quad 2\mathbf{W}(\phi_1, \phi_2) = \begin{bmatrix} C_1 + C_2 - iS_1S_2 & C_1 - C_2 + iS_1S_2 & S_1C_2 - iS_2 & S_1C_2 - iS_2 \\ C_1 - C_2 - iS_1S_2 & C_1 + C_2 + iS_1S_2 & S_1C_2 + iS_2 & S_1C_2 - iS_2 \\ -S_1 - iC_1S_2 & -S_1 + iC_1S_2 & C_1C_2 + 1 & C_1C_2 - 1 \\ -S_1 - iC_1S_2 & -S_1 + iC_1S_2 & C_1C_2 - 1 & C_1C_2 + 1 \end{bmatrix},$$

where $C_1 = \cos \phi_1$, $S_1 = \sin \phi_1$, $C_2 = \cos \phi_2$, $S_2 = \sin \phi_2$. It then follows from the known representations of the Lorentz group in terms of the little Lorentz group, the orthogonal space rotation group, and the discrete reflection group [4], that the above theorems give

$$(3.11) \quad \mathbf{C} = \mathbf{W}(\phi_3, \phi_4)\mathbf{N}(\alpha, \beta; \epsilon_1, \epsilon_2)\mathbf{W}(\phi_1, \phi_2)$$

as a representation of elements of $\mathcal{C}$. Here, $\phi_1, \phi_2, \phi_3, \phi_4, \alpha$, and β are arbitrary real-valued functions defined over E while ϵ_1 and ϵ_2 are real-valued functions defined on E with the discrete range $+1$ and -1 .

Another useful form of the $\mathbf{C}$ -matrix can be obtained directly from the work of Bazanski [5], when allowance is made for the differences in the choice of signs between his work and that reported here. The form is given by

$$(3.12) \quad (C_b^\alpha) = \begin{bmatrix} e^{i\phi} & 0 & -\bar{\gamma}e^{i\phi} & 0 \\ 0 & e^{-i\phi} & -\gamma e^{-i\phi} & 0 \\ 0 & 0 & e^\psi & 0 \\ \gamma e^{-\psi} & \bar{\gamma}e^{-\psi} & -\gamma\bar{\gamma}e^{-\psi} & e^{-\psi} \end{bmatrix},$$

where ϕ and ψ are real valued functions on E and γ is a complex-valued function on E . At each point of E , this matrix represents the proper orthochronous Lorentz transformations that result under the requirement that the directions of v_A^3 and V_A^3 coincide. The full six parameter group at a point in E is then obviously obtainable upon specifying the relative orientation of V_A^3 with respect to v_A^3 and making use of the group properties of the C 's.

Another useful representation for the component of the group of $\mathbf{C}$ -matrices that is continuously connected to the identity element is given by

$$(3.13) \quad \mathbf{C} = \exp \begin{pmatrix} -ia & 0 & d & e \\ 0 & ia & \bar{d} & \bar{e} \\ -\bar{e} & -e & -f & 0 \\ -\bar{d} & -d & 0 & f \end{pmatrix},$$

where a and f are real-valued functions and d and e are complex-valued functions on E . Note that (3.12) is just a subgroup of the group given by (3.13). The general method whereby (3.13) is obtained will be presented in a forthcoming paper under the title *Exponential Representations and Canonical Forms of Matrix Groups*, and stems from the fact that $\mathbf{C} = \exp(\mathbf{G}^{-1}\mathbf{Q})$, $\mathbf{Q}' = -\mathbf{Q}$ satisfies $\mathbf{C}\mathbf{G}\mathbf{C}' = \mathbf{G}$ and $\mathbf{G}$ commutes with $\mathbf{I}$.

4. The anholonomic frame generated by a normed null basis. A normed null basis defines an anholonomic frame on E , since, in general, we will not have $\nabla_{[A}V_{B]}^\alpha = 0$. With any such frame, we may convert any geometric object valued function into its corresponding anholonomic representation by strangling every living index by inner multiplication with V_A^α or V_α^A , i.e.,

$$T_\alpha = T_A V_\alpha^A, \quad T_A = T_\alpha V_\alpha^A.$$

Similarly, anholonomic differentiation may be defined in E by the relation

$$(4.1) \quad \nabla_A T_B = V_A^\alpha V_B^\beta \nabla_\alpha T_\beta,$$

from which it follows (see [6], pp. 169f) that

$$(4.2) \quad \nabla_\alpha T_\beta = V_\alpha^A \nabla_A T_\beta - Z_{\alpha\beta}^\gamma T_\gamma,$$

where the anholonomic connection coefficients $Z_{\alpha\beta}^\gamma$ are defined by

$$(4.3) \quad Z_{\alpha\beta}^\gamma = \gamma_{AB}^C V_C^\gamma V_\alpha^A V_\beta^B - V_\alpha^A V_\beta^B \partial_A V_B^\gamma = -V_\alpha^A V_\beta^B \nabla_A V_C^\gamma.$$

(Recall that γ_{AB}^C are the components of affine connection on E .) A straightforward calculation then gives the following form for the anholonomic representation of the curvature tensor (see [6], p. 172),

$$(4.4) \quad K_{\alpha\beta\gamma}^\mu = 2\partial_{[\alpha} Z_{\beta]\gamma}^\mu + 2Z_{[\alpha|\nu]}^\mu Z_{\beta]\gamma}^\nu + 2Z_{[\alpha\beta]}^\nu Z_{\nu\gamma}^\mu.$$

In view of the results established in Theorem 2.2, the anholonomic frame

generated by any normed null basis on E can be written in terms of the anholonomic frame generated by a given normed null basis v_A^a and the quantities C_a^α . We may thus view the v 's as a canonical (anholonomic) frame, in which case the C 's become the anholonomic components of the V 's with respect to the v 's, *i.e.*,

$$C_a^\alpha = V_A^\alpha v_A^A,$$

so that C_a^α is the α -th anholonomic component of the α -th V_A with respect to the dual basis v_a^A . This viewpoint is the key to our further considerations since it allows us to catch all considerations in terms of the known space E with the known normed null basis v_A^a , in which we seek the quantities C_a^α which are *scalar fields* on E under coordinate transformations.

The next theorem, which is a natural continuation of Theorem 3.2, will be of repeated use in what follows.

Theorem 4.1. *Let v_A^a be a given normed null basis on E and let z_{ab}^c be the anholonomic connection formed from this basis. Let V_A^α be any other normed null basis on E (formed from v_A^a by use of (2.7)) and let $Z_{\alpha\beta}^\gamma$ be the anholonomic connection formed from V_A^α . We have*

$$(4.5) \quad Z_{\alpha\beta}^\gamma = C_a^\alpha C_\beta^b (C_{ab}^\gamma - \partial_a C_b^\gamma), \quad \partial_a \equiv v_A^A \partial_A.$$

Proof. When (2.7) is substituted into (4.3), we obtain

$$(4.6) \quad Z_{\alpha\beta}^\gamma = -C_a^\alpha C_\beta^b \nabla_A (C_{ab}^\gamma v_b^A) v_b^B.$$

If we expand the covariant derivative of the product $C_{ab}^\gamma v_b^A$ and remember that $\nabla_A C_{ab}^\gamma = \partial_A C_{ab}^\gamma$ since the C 's transform as scalar functions under coordinate transformations in E , the use of (2.1), (4.2), and (4.3) gives (4.5).

In referring considerations of anholonomic quantities to the anholonomic frame generated by the given normed null basis v_A^a , we invariably run across quantities with both anholonomic indices and Greek labeling indices. If we remember that all anholonomic indices are denoted by lower case Roman indices, we may use the process of anholonomic covariant differentiation based on the basis v_A^a , which we shall denote by ∇_a . When (4.5) is combined with the definition of anholonomic covariant differentiation with respect to the basis v_A^a , the following result is obtained.

Theorem 4.2. *Let v_A^a be a given normed null basis on E , let V_A^α be any other normed null basis on E and let $Z_{\alpha\beta}^\gamma$ be the anholonomic connection formed from V_A^α . We have*

$$(4.7) \quad Z_{\alpha\beta}^\gamma = -C_a^\alpha C_\beta^b \nabla_a C_b^\gamma.$$

Starting with an anholonomic connection $Z_{\alpha\beta}^\gamma$, the objects of anholonomy $\Omega_{\alpha\beta}^\gamma$ are defined by (see [6], p. 169)

$$(4.8) \quad \Omega_{\alpha\beta}^\gamma = -Z_{[\alpha\beta]}^\gamma.$$

A straightforward combination of Theorem 4.2 and (4.8) leads to the following result.

Theorem 4.3. *Let v_A^α be a given normed null basis on E and let ω_{ab}^α be the corresponding objects of anholonomy. If V_A^α is any other normed null basis on E , its objects of anholonomy are given by*

$$(4.9) \quad \Omega_{\alpha\beta}^\gamma = C_a^\alpha C_\beta^b (C_c^\gamma \omega_{ab}^c + \partial_{[a} C_{b]}^\gamma) = C_a^\alpha C_\beta^b \nabla_{[a} C_{b]}^\gamma.$$

Theorem 4.4. *Let v_A^α be a given normed null basis on E . The normed null basis V_A^α , obtained from v_A^α by (2.7), is holonomic if and only if the C 's are such that*

$$(4.10) \quad \nabla_{[a} C_{b]}^\gamma = 0.$$

If these conditions are satisfied, then

$$(4.11) \quad C_a^\alpha = v_a^A \partial_A y^\alpha,$$

where the y 's are four, complex-valued, functionally independent functions of the coordinates of E such that

$$(4.12) \quad \tilde{y}^\alpha = I_\beta^\alpha y^\beta, \quad g^{AB} \partial_A y^\alpha \partial_B y^\beta = g^{\alpha\beta},$$

and

$$(4.13) \quad y^\alpha = \int C_a^\alpha v_a^A dx^A.$$

Further, if the v 's are such that they are also holonomic, then the C 's are constant valued functions on E .

Proof. As is well known, $\Omega_{\alpha\beta}^\gamma = 0$ are both necessary and sufficient conditions that the V 's form a holonomic basis, in which event there exist functions y^α such that

$$(4.14) \quad V_A^\alpha = \partial_A y^\alpha.$$

Since the V 's form a normed null basis, the y 's must consequently satisfy the requirement obtained by combining (4.14) and Theorem 2.1. This leads to the conditions (4.12). Since the C 's form a nonsingular array, (4.9) shows that we can have $\Omega_{\alpha\beta}^\gamma = 0$ if and only if (4.10) holds. A general solution of the system (4.10) is easily seen to be given by (4.11). Further, (4.11) and (2.7) combine to give precisely (4.14). It is also easily seen that (4.10) constitutes the first set of integrability conditions for the equations $\partial_A y^\alpha = C_a^\alpha v_a^A$, and hence the functions y^α exist and are given by (4.13) to within an additive constant. If the v 's are holonomic, then there exist functionally independent functions ϕ^a on E such that $v_a^A = \partial x^A / \partial \phi^a$. When this is substituted into (4.11), we then have $C_a^\alpha = \partial y^\alpha / \partial \phi^a$, and hence the C 's are the components of a Jacobian matrix. By Theorem 3.1, we have $\mathbf{C} = \mathbf{T}^{-1} \mathbf{B} \mathbf{T}$, where $\mathbf{T}$ is a constant-valued

matrix, and hence $\mathbf{B}$ must be a Jacobian matrix. Since $\mathbf{B}$ must satisfy (3.6), the requirement that it be a Jacobian matrix demands that $\mathbf{B}$ be a constant-valued matrix (see [2], pp. 403–410). It thus follows that the C 's are constant valued functions on E in this case.

Corollary 4.1. *Let v_A^a be a holonomic normed null basis on E , then E is flat and the set of all holonomic normed null bases on E is obtained by*

$$(4.15) \quad V_A^\alpha = C^\alpha v_A^a,$$

where the C 's range through matrix representations of the Lorentz group with respect to the coordinates

$$(4.16) \quad \phi^a = \int v_A^a dx^A,$$

i.e., the C 's are given by

$$(4.17) \quad \mathbf{C} = \mathbf{T}^{-1}\mathbf{B}\mathbf{T}$$

where $\mathbf{B}$ is any Jacobian matrix of the Lorentz group with respect to the coordinates (x, y, z, t) .

Proof. If v_A^a is holonomic, then there exist functions ϕ^a defined by (4.16) such that $v_A^a dx^A = d\phi^a$ and conversely. Accordingly, the line element on E is given by

$$ds^2 = g_{AB} dx^A dx^B = g_{ab} v_A^a v_B^b dx^A dx^B = g_{ab} d\phi^a d\phi^b.$$

Hence, since $((g_{ab}))$ is a constant valued matrix on E , the space E is flat. The remaining conclusions of the theorem follow directly from Theorem 3.3 and the fact that the C 's must be constants if the v 's and the V 's are holonomic.

Corollary 4.2. *The system of equations (4.10) are completely integrable if and only if E is flat, and hence a holonomic normed null basis exists on E if and only if E is flat.*

Proof. By Theorem 4.3, there exists a holonomic normed null basis on E if and only if (4.10) is completely integrable. On the other hand, it is evident from the proof of Corollary 4.1 that a holonomic normed null basis exists on E if and only if E is flat. The result then follows.

The following results will be needed later.

Theorem 4.5. *We have*

$$(4.18) \quad \nabla_A V_B^\gamma = -Z_{\alpha\beta}^\gamma V_A^\alpha V_B^\beta,$$

$$(4.19) \quad \nabla_a V_B^\gamma = -C_a^\alpha Z_{\alpha\beta}^\gamma V_B^\beta,$$

$$(4.20) \quad \nabla_\bullet g^{ab} = \nabla_\bullet g_{ab} = \nabla_\bullet g^{a\beta} = \nabla_\bullet g_{\alpha\beta} = 0,$$

$$(4.21) \quad Z_{\alpha\beta}^{(\gamma} g^{\lambda)\beta} = z_{bc}^{(a} g^{e)b} = 0,$$

$$(4.22) \quad V_{\beta}^B \nabla_B V_C^{\gamma} = -Z_{\beta\alpha}^{\gamma} V_C^{\alpha},$$

$$(4.23) \quad \begin{aligned} g_{AB} &= 2V_{(A}^1 V_{B)}^2 + 2V_{(A}^3 V_{B)}^4 \\ &= 2v_{(A}^1 v_{B)}^2 + 2v_{(A}^3 v_{B)}^4. \end{aligned}$$

Proof. Equation (4.18) follows immediately from (4.3); (4.19) then follows from the definition of the anholonomic covariant derivative with respect to the v 's and (4.18). The system of relations (4.20) just expresses Ricci's lemma, and (4.21) is a direct consequence of (4.18) and the relations $0 = \nabla_c g^{\alpha\beta} = \nabla_c (g^{AB} V_A^{\alpha} V_B^{\beta})$. The system (4.22) follows directly from (4.18), while (4.23) is just another way of writing (2.3) and (2.5).

5. Existence theorems. We are now in a position to prove an existence theorem of fundamental importance.

Theorem 5.1. *Let v_A^{α} be a given normed null basis on E . There exists a normed null basis V_A^{α} on E that satisfies the conditions*

$$(5.1) \quad F(V_A^{\alpha}, Z_{\alpha\beta}^{\gamma}, \dots) = 0$$

if and only if there exists a matrix $((C_{\alpha}^a))$ of functions defined on E that satisfies (2.8) and (2.9) and is such that

$$(5.2) \quad F(C_{\alpha}^a v_A^{\alpha}, -C_{\alpha}^a C_{\beta}^b \nabla_a C_b^{\gamma}, \dots) = 0.$$

Proof. By Theorem 2.2, any normed null basis on E can be represented in terms of the given normed null basis and the C 's. Similarly, Theorem 4.2 shows that the anholonomic connection based on the V 's can be represented in terms of the anholonomic connection based on the v 's together with the C 's and their first derivatives according to (4.7). It thus follows that there exists a system of V 's that satisfies (5.1) if and only if there exists a system of C 's that satisfies (5.2).

In order to point out the usefulness of this theorem, let us assume for the time being that the space E is the flat Minkowski space referred to Cartesian coordinates (x, y, z, t) where the light speed is taken to be unity. A covariant constant, and hence holonomic, normed null basis is given on E by the following prescription:

$$(5.3) \quad \begin{aligned} v_A^1 &= 2^{-1/2}(1, i, 0, 0), & v_A^2 &= 2^{-1/2}(1, -i, 0, 0), \\ v_A^3 &= 2^{-1/2}(0, 0, 1, 1), & v_A^4 &= 2^{-1/2}(0, 0, 1, -1). \end{aligned}$$

Since the v 's are holonomic, there exist functions y^a such that $v_A^{\alpha} = \partial_A y^{\alpha}$, i.e., $y^{\alpha} = \int v_A^{\alpha} dx^A$. We therefore have

$$(5.4) \quad \begin{aligned} y^1 &= 2^{-1/2}(x + iy), & y^2 &= 2^{-1/2}(x - iy) \\ y^3 &= 2^{-1/2}(z + t), & y^4 &= 2^{-1/2}(z - t). \end{aligned}$$

Further, since ∂_a is defined by $\partial_a = v_a^A \partial_A = v_a^A (\partial_A y^b) \partial_b$, where we have used the notation $\partial_b = \partial/\partial y^b$, we see that

$$(5.5) \quad \partial_a = \partial_a$$

since $\partial_A y^b = v_A^b$. Hence, anholonomic partial differentiation becomes ordinary partial differentiation with respect to the y 's.

Lemma 5.1. *Let E be a Minkowski space with unit light speed and let the normed null basis v_A^a be given by (5.3). We then have*

$$(5.6) \quad z_{ab}^c = 0, \quad K_{kii}^h = 0, \quad \nabla_a = \partial_a.$$

If V_A^α is any other normed null basis on E , then

$$(5.7) \quad Z_{\alpha\beta}^\gamma = -C_\alpha^a C_\beta^b \partial_a C_b^\gamma.$$

Proof. Under the given hypotheses, the first of (5.6) follows directly from the defining equations (4.3). The second and third equations of (5.6) then follow from the first of (5.6), the definitions of anholonomic covariant differentiation and the anholonomic curvature tensor, and (5.5). When (5.6) is used, (5.7) follows from Theorem 4.2.

Starting with the space E , we may define a tensor field on E by the relations

$$(5.8) \quad h_{AB} = g_{AB} + k_{AB}, \quad k_{[AB]} = 0.$$

If we require the k 's to be such that $\det(h_{AB}) \neq 0$ and signature $(h_{AB}) = \text{signature}(g_{AB})$, then h_{AB} may serve as the metric tensor for a new space E^* under the Einstein theory. If we introduce the basis v_A^a on E and another basis V_A^α , and note that

$$g_{AB} = g_{ab} v_A^a v_B^b = g_{\alpha\beta} V_A^\alpha V_B^\beta,$$

we may write

$$(5.9) \quad h_{AB} = (g_{ab} + k_{\alpha\beta} C_a^\alpha C_b^\beta) v_A^a v_B^b$$

in view of (5.15). Hence, noting that $v_A^a dx^A = dy^a$ for the above choice of the v 's, we have

$$(5.10) \quad ds^2 = h_{AB} dx^A dx^B = (g_{ab} + k_{\alpha\beta} C_a^\alpha C_b^\beta) dy^a dy^b$$

as the line element on E^* .

The importance of this subterfuge is that it allows us to translate all relations that hold in the space E^* with metric tensor h_{AB} to relations that hold with respect to tensor fields defined on the space E . This is indeed an economy for the space E is the simplest possible space that may be entertained in the theory of relativity.

Let us go a little further and actually obtain the relevant equations as they appear under the Einstein theory. Since h_{AB} is nonsingular, the affine connection Γ_{BC}^A of the space E^* can be written in the form

$$\Gamma_{BC}^A = \gamma_{BC}^A + T_{BC}^A ,$$

where γ_{BC}^A is the affine connection on E and T_{BC}^A is a tensor valued function of the V 's, their covariant derivatives in E , the g 's, the scalar valued functions $k_{\alpha\beta}$ on E , and the derivatives of these scalars. Since $\gamma_{BC}^A = 0$ in the present instance, we then have

$$(5.11) \quad \Gamma_{BC}^A = T_{BC}^A ,$$

for the coordinates y^a on E . If R_{AB} denotes the Ricci tensor of E^* and K_{AB} denotes the Ricci tensor of E , then, since $K_{AB} = 0$ in the present case, we have (see [6], p. 141)

$$(5.12) \quad R_{AB} = \nabla_C T_{AB}^C - \nabla_A T_{CB}^C + T_{CE}^C T_{AB}^E - T_{AE}^C T_{CB}^E .$$

Accordingly, by the last of (5.6), R_{AB} is determined in terms of the V 's and their first and second *partial derivatives*, the g 's, and the scalar functions $k_{\alpha\beta}$ and their first and second partial derivatives. Now, from the definition of $Z_{\alpha\beta}^\gamma$ and Lemma 5.1, it is easily seen that

$$(5.13) \quad \nabla_B V_C^\gamma = \partial_b C^\gamma \nu_B^b \nu_C^c , \quad \nabla_E \nabla_B V_C^\gamma = \partial_a \partial_b (C^\gamma) \nu_B^b \nu_C^c \nu_E^e ,$$

while $V_A^\alpha = C_a^\alpha \nu_A^a$. It thus follows that we can write

$$(5.14) \quad R_{AB} = F_{mn}(C_a^\alpha , \partial_b C_a^\alpha , \partial_c \partial_b C_a^\alpha , k_{\alpha\beta} , \partial_b k_{\alpha\beta} , \partial_c \partial_b k_{\alpha\beta} , g_{ab}) \nu_A^m \nu_B^n .$$

The following result is now a specific realization of Theorem 5.1.

Theorem 5.2. *Let E be a Minkowski space referred to cartesian coordinates (x, y, z, t) with unit light speed and let E^* be a space with metric tensor*

$$(5.15) \quad h_{AB} = g_{AB} + k_{\alpha\beta} V_A^\alpha V_B^\beta ,$$

where the V 's form a normed null basis on E . Then a solution of the system of equations

$$(5.16) \quad R_{AB} = W_{AB}$$

exists if and only if there exists a matrix of scalar functions $((C_a^\alpha))$ on E that satisfies (2.8) and (2.9) and a collection of scalar functions $k_{\alpha\beta} = k_{\beta\alpha}$ on E such that

$$(5.17) \quad F_{mn} = W_{AB} \nu_m^A \nu_n^B ,$$

where the F 's are the system of second order partial differential operators given by (5.14).

By this theorem, the problem of solving the Einstein field equations is reduced to the problem of solving systems of second order partial differential equations in the Minkowski space E referred to the coordinates (y^1, y^2, y^3, y^4) , and all of the unknowns are *scalar* functions defined on E . We shall make extensive use of this simplification in the remaining papers of this series. In the particular

case where $k_{\alpha\beta} = \delta_\alpha^4 \delta_\beta^4$, (5.15) reduces to

$$h_{AB} = g_{AB} + V_A^4 V_B^4,$$

and hence we recover the problem considered in I since V_A^4 is a null vector field on E .

6. Normed null bases and the Ricci class. In the previous paper I, we defined a space E_i to be an element of the null bundle of a given hyperbolic normal metric space E if and only if the metric tensor h_{AB} of E_i could be written in the form

$$(6.1) \quad h_{AB} = g_{AB} + \ell_A \ell_B,$$

where g_{AB} is the metric tensor of E and ℓ_A is a null vector field on E . Such a space was said to be an element of the Ricci class of E if and only if E_i had the same Ricci tensor as E . As shown in Theorem I-5.7, necessary and sufficient conditions that E_i be an element of the Ricci class of E are

$$(6.2) \quad L_A - \rho \ell_A = 0,$$

$$(6.3) \quad (\ell_B \nabla^B + U_{.B}^B) \Omega_{CE} + (\ell_C \nabla^B + U_{.C}^B) \Omega_{BE} + (U_{E.}^B + D) \mathcal{L}_{BC} = 0,$$

where the various symbols are defined as follows:

$$(6.4) \quad U_{AB} = \nabla_A \ell_B, \quad \Omega_{AB} = 2U_{[AB]}, \quad \mathcal{L}_{AB} = 2U_{(AB)}, \quad L_A = \ell^B U_{BA},$$

and D is the invariant differential operator $\ell^A \nabla_A$.

Agreement 6.1. The space E is a Minkowski space with unit light speed and is referred to Cartesian coordinates (x, y, z, t) .

Remark. Under this agreement, all of the results of the previous section may be applied.

Theorem 6.1. Under Agreement 6.1, every solution of the system (6.2) and (6.3) gives a null vector field ℓ_A such that the space E_i with metric tensor h_{AB} given by (6.1) is a space in which the vacuum Einstein field equations hold.

Proof. By Agreement 6.1, the space E is flat, and hence its Ricci tensor K_{AB} vanishes identically. Since the system (6.2) and (6.3) are both necessary and sufficient conditions for the Ricci tensor R_{AB} of E_i to be equal to the Ricci tensor of E , we have $R_{AB} = 0$.

Agreement 6.2. Throughout the remainder of this Section we take

$$(6.5) \quad \ell_A = V_A^4$$

at all points of E except those for which $\ell_A = 0$.

This identification is obviously consistent with the conditions defining ℓ_A since V_A^4 is a real-valued null vector field in E . It is also consistent with the

conditions defining V_A^4 since $V_A^4 \neq 0$ for some value of A at every point of E .

The following Lemma is an immediate consequence of Agreements 6.1 and 6.2, equations (6.4), and the results obtained in the previous sections.

Lemma 6.1. *We have*

$$(6.6) \quad \ell_A = C_a^4 v_A^a, \quad \ell^A = g^{ab} C_b^4 v_a^A,$$

$$(6.7) \quad U_{AB} = \partial_a C_b^4 v_A^a v_B^b, \quad U_{.B}^E = g^{er} \partial_r C_b^4 v_B^b v_{.e}^E,$$

$$(6.8) \quad \mathcal{L}_{AB} = 2 \partial_{(a} C_{b)}^4 v_A^a v_B^b, \quad \Omega_{AB} = 2 \partial_{[a} C_{b]}^4 v_A^a v_B^b,$$

$$(6.9) \quad L_A = g^{bf} C_f^4 \partial_b C_a^4 v_A^a.$$

$$(6.10) \quad \nabla_A = v_A^a \partial_a, \quad \nabla^E = v_{.e}^E g^{er} \partial_r, \quad D = g^{ab} C_b^4 \partial_a.$$

Theorem 6.2. *The system of equations (6.2) and (6.3) is equivalent to the system*

$$(6.11) \quad (g^{bf} C_f^4 \partial_b - \rho)(C_a^4) = 0.$$

$$(6.12) \quad g^{er} \{ (C_b^4 \partial_r + \partial_r C_b^4) \partial_{[c} C_{e]}^4 + (C_c^4 \partial_r + \partial_r C_c^4) \partial_{[b} C_{e]}^4 + (C_e^4 \partial_r + \partial_r C_e^4) \partial_{[b} C_{c]}^4 \} = 0.$$

Proof. In order to establish the Theorem, we need only substitute the results given by Lemma 6.1 into the system (6.2) and (6.3) and note the linear independence of the v 's.

The general solution of the system (6.11) and (6.12) obviously involves a full representation of the C 's. We shall return to this general problem in a future communication after certain additional results are established. For the remainder of this section, we shall confine our attention to the case in which the C 's are given by (3.9) with $\epsilon_1 \epsilon_2 = 1$. In this case, we have

$$(6.13) \quad C_a^4 = \delta_a^4 \lambda,$$

where $\lambda = \pm e^{-u}$.

Lemma 6.2. *Under the substitution (6.13), the system (6.11) is equivalent to the system*

$$(6.14) \quad \lambda(\partial_3 \lambda - \rho) = 0,$$

and hence we have either $\lambda = 0$ or

$$(6.15) \quad \partial_3 \lambda = \rho.$$

Proof. When (6.13) is substituted into (6.11), we obtain

$$\delta_a^4 (g^{b4} \lambda \partial_b - \rho) \lambda = 0.$$

From (2.5), we see that $g^{b4} = \delta_3^b$, and hence (6.14) is obtained.

Lemma 6.3. *Under the substitution (6.13), the system (6.12) is equivalent*

to the system

$$(6.15) \quad \delta_b^4 \partial_3 \partial_b (\lambda^2) + \delta_c^4 \partial_3 \partial_b (\lambda^2) - \delta_c^4 \delta_b^4 g^{*r} \partial_r \partial_c (\lambda^2) = 0.$$

Proof. In the interests of brevity, we shall work out only the last term in (6.12); the other two terms follow identical lines. Under (6.13), we have

$$\begin{aligned} 2g^{*r}(C_a^4 \partial_r + \partial_r C_a^4) \partial_{(b} C_{c)}^4 &= (\lambda g^{4r} \partial_r + g^{4r} \partial_r \lambda)(\delta_c^4 \partial_b \lambda + \delta_b^4 \partial_c \lambda) \\ &= (\lambda \partial_3 + \partial_3 \lambda)(\delta_c^4 \partial_b \lambda + \delta_b^4 \partial_c \lambda) \\ &= \frac{1}{2} \delta_c^4 \partial_3 \partial_b (\lambda^2) + \frac{1}{2} \delta_b^4 \partial_3 \partial_c (\lambda^2). \end{aligned}$$

When similar results are combined from the first two terms of (6.12) and the common numerical factor is eliminated, we obtain (6.15).

Lemma 6.4. *The system (6.15) is equivalent to the system*

$$(6.16) \quad 0 = \partial_1 \partial_2 (\lambda^2) = \partial_3 \partial_1 (\lambda^2) = \partial_3 \partial_2 (\lambda^2) = \partial_3 \partial_3 (\lambda^2).$$

Proof. We first note that by (2.5) we have

$$g^{*r} \partial_r \partial_c (\lambda^2) = 2(\partial_1 \partial_2 + \partial_3 \partial_4)(\lambda^2).$$

The first of (6.16) is thus obtained when we set $b = c = 4$ in (6.15). The second, third, and fourth equations in (6.16) follow directly from (6.15) when we set $c = 4$ and $b = 1, 2, 3$. Since the system (6.15) is symmetric in c and b and vanishes identically unless either b or c is equal to 4, the result is established.

Theorem 6.3. *A general solution of the system (6.11) and (6.12) under the substitution*

$$(6.17) \quad C_a^4 = \lambda \delta_a^4$$

is given by

$$(6.18) \quad \lambda^2 = A(y^4)\alpha(y^1) + B(y^4)\beta(y^2) + U(y^4)y^3 + W(y^4),$$

$$(6.19) \quad 2\lambda\rho = U(y^4).$$

where the functions A, B, U, W, α , and β are of class C^1 in their respective arguments.

Proof. Under the substitution (6.17), Lemmas 6.3 and 6.4 show that the system (6.12) is equivalent to the system (6.16). The first of this system gives

$$\lambda^2 = A(y^3, y^4)\alpha(y^1) + B(y^3, y^4)\beta(y^2) + F(y^3, y^4).$$

When this is substituted into the remaining equations of the system (6.16) and account is taken of the various cases that can arise, all cases are seen to lead to (6.18) or to special cases of this solution that are obtained by allowing either A or B or both to vanish. The second half of the solution, namely ρ , is obtained from (6.18) when use is made of Lemma 6.2. The continuity conditions are obvious.

In order that $C_a^4 = \lambda \delta_a^4$ be obtained from a C -matrix, we must have

$\lambda = \pm \exp(-u)$ for some real-valued function u . It thus follows that λ^2 must be real. Now, by (5.4), the variables y^1 and y^2 are complex with $y^2 = \bar{y}^1$. Hence, applying the condition $S = \bar{S}$, $S = \lambda^2$ to (6.18) gives

$$A(y^4)\alpha(y^1) + B(y^4)\beta(\bar{y}^1) = A(y^4)\bar{\alpha}(y^1) + B(y^4)\bar{\beta}(\bar{y}^1).$$

Rewriting this equation to obtain

$$A(y^4)\alpha(y^1) - B(y^4)\bar{\beta}(\bar{y}^1) = A(y^4)\bar{\alpha}(y^1) - B(y^4)\beta(\bar{y}^1),$$

and noting that the right-hand side is the complex conjugate of the left-hand side, the well-known theorem in complex variable theory tells us that $S = \bar{S}$ if and only if

$$B(y^4)\beta(\bar{y}^1) = A(y^4)\bar{\alpha}(y^1) + T(y^4).$$

When this is substituted into (6.18) and we combine Theorems 6.3 and 6.1 and with Lemma 6.1, we have the following results.

Theorem 6.4. *Let E be a Minkowski space with metric tensor g_{AB} referred to the system of Cartesian coordinates (x, y, z, t) and let the quantities V_A^a and y^a be defined by (5.3) and (5.4). Let the quantity λ^2 be defined by the relation*

$$(6.20) \quad \lambda^2 = A(y^4) \operatorname{Re} [\alpha(y^1)] + U(y^4)y^3 + W(y^4),$$

where A , U , and W are arbitrary C^1 functions of their indicated arguments, α is a function of the complex variable y^1 , and $\operatorname{Re} [\]$ denotes "the real part of". Then any region of the space E , with metric tensor

$$(6.21) \quad h_{AB} = g_{AB} + \lambda^2 v_A^4 v_B^4$$

satisfies the Einstein vacuum field equations provided $\lambda^2 > 0$.

The restriction in the above theorem to regions in E , (in E) where $\lambda^2 > 0$ arises from the fact that at points on the locus defined by $\lambda^2 = 0$, we may not identify ℓ_A with V_A^4 . We shall now eliminate this condition. It was shown in the previous paper that if ℓ_A defined an element of the Ricci class of E , then any (real) constant multiple of ℓ_A would also define an element of the Ricci class of E (see Theorem I-5.8). Now, it is evident from the proof of this theorem that the same result holds when we use a purely imaginary multiplicative constant. When the constant is $-i$, this simply has the effect of changing the form $\ell_A \ell_B$ into $-\ell_A \ell_B$ and hence the resulting metric tensor

$$h_{AB} = g_{AB} - \ell_A \ell_B$$

is again real. Let us denote the regions of E where the function

$$\ell = A(y^4) \operatorname{Re} [\alpha(y^1)] + U(y^4)y^3 + F(y^4)$$

is negative by N and the regions where it is positive by P . Theorem 6.4 gives us a solution everywhere in the open set P . It is also evident that (6.20) gives

a solution everywhere in the open set N when we replace λ^2 by ℓ , since this is equivalent to replacing ℓ_A by $i\ell_A$ in (6.1). Thus, since ℓ is a C^1 function on E and agrees with λ^2 on the set P , we have the following result.

Theorem 6.5. *Let E be a Minkowski space with metric tensor g_{ab} referred to the system of Cartesian coordinates (x, y, z, t) and let the quantities v_A^a and y^a be defined by (5.3) and (5.4). Let the function ℓ be defined on E by the relation*

$$(6.21) \quad \ell = A(y^4) \operatorname{Re} [\alpha(y^1)] + U(y^4)y^3 + W(y^4),$$

where A , U , and W are arbitrary C^1 functions of their indicated arguments and α is an arbitrary C^2 function of the complex variable y^1 . Then any space E with metric tensor

$$(6.22) \quad h_{AB} = g_{AB} + \ell v_A^4 v_B^4$$

satisfies the Einstein vacuum field equations

The question naturally arises as to whether these solutions are really distinct from E in the sense that they do not have vanishing curvature tensors.

Theorem 6.6. *Necessary and sufficient conditions that a space E_t with metric tensor given by (6.22) be a flat space is that the function ℓ be a constant function on E .*

Proof. It is evident that we need only prove that the curvature tensor of E_t does not vanish over some finite region. Let us therefore consider the region P in which the function ℓ is positive. When previous results are used, we easily see that

$$(6.23) \quad \begin{aligned} h_{AB} &= (g_{ab} + C_a^4 C_b^4) v_A^a v_B^b = (g_{\alpha\beta} C_\alpha^a C_\beta^b + C_a^4 C_b^4) v_A^a v_B^b \\ &= (g_{\alpha\beta} + \delta_\alpha^4 \delta_\beta^4) C_\alpha^a C_\beta^b v_A^a v_B^b = (g_{\alpha\beta} + \delta_\alpha^4 \delta_\beta^4) V_A^\alpha V_B^\beta. \end{aligned}$$

As is well-known, necessary and sufficient conditions that a region P of E_t be flat is that there exist coordinate systems on P such that the metric tensor of E_t is a constant valued tensor field on P . Multiplying (6.23) by $dx^A dx^B$, we have

$$ds_t^2 = (g_{\alpha\beta} + \delta_\alpha^4 \delta_\beta^4) (V_A^\alpha dx^A) (V_B^\beta dx^B)$$

on P , and hence such a coordinate system exists if and only if $V_A^\alpha dx^A = d\phi^\alpha$ constitutes a globally integrable system on P . If these conditions are satisfied, then the V 's are holonomic on P , and Theorem 4.4 shows that the C 's must be constant-valued functions on P . We thus obtain $C_a^4 = \lambda \delta_a^4 = \text{constant}$ and the result is established.

7. Cylindrical vacuum solutions. Consider those spaces in which ℓ is given by

$$(7.1) \quad \ell = 2A \operatorname{Re} [\alpha(2^{1/2} y^1)],$$

where A is taken as a constant. If we set $ds_t^2 = h_{AB} dx^A dx^B$, use (5.3) and (5.4),

and introduce coordinates r and θ by the substitution $2^{\frac{1}{2}}y^1 = r \exp(i\theta)$, we obtain

$$(7.2) \quad ds_t^2 = dr^2 + r^2 d\theta^2 + (1 + A \operatorname{Re} [\alpha(re^{i\theta})]) dz^2 \\ + (A \operatorname{Re} [\alpha(re^{i\theta})] - 1) dt^2 - 2A \operatorname{Re} [\alpha(re^{i\theta})] dz dt,$$

where $r^2 = x^2 + y^2$. Since the metric coefficients are independent of t , such a space-time will be stationary. It is not necessarily static, however, owing to the occurrence of the mixed term $-2A \operatorname{Re} [\alpha(re^{i\theta})] dz dt$. It is also evident that such space-times are cylindrical since the metric coefficients are independent of the z -coordinate.

Theorem 7.1. *All solutions of the Einstein vacuum field equations obtained under Theorem 6.5 with ℓ given by (7.1) are stationary cylindrical space-times.*

We shall now obtain some particular solutions. Let us set $A = 2mc^{-2}$, $\alpha(re^{i\theta}) = \ln(re^{i\theta})$, we then have

$$(7.4) \quad \ell = 4mc^{-2} \ln(r)$$

and (7.2) gives

$$(7.5) \quad ds_t^2 = dr^2 + r^2 d\theta^2 + \left(1 + \frac{2m}{c^2} \ln(r)\right) dz^2 \\ + \left(\frac{2m}{c^2} \ln(r) - 1\right) dt^2 - \frac{4m}{c^2} \ln(r) dz dt.$$

It is easily seen that (7.5), or for that matter (7.2), is of signature $(+++)$ throughout E . Further, in the region

$$\exp(-c^2/2m) \ll r \ll \exp(c^2/2m),$$

the solution (7.5) is easily identified with the gravitational field of a rod of uniform mass $M = m/G$ per unit length that is coincident with the z -axis provided $c =$ speed of light. From this and the occurrence of the mixed term $dz dt$, it is easily seen that (7.5) represents the gravitational field of a uniform rod coincident with the z -axis throughout the domain of the variables. We note here the cylindrical analog of the second Schwarzschild singularity, namely $r = \exp(-c^2/2m)$, and in addition, a new surface of interest given by $r = \exp(c^2/2m)$.

As a further example, consider the case in which

$$A = mc^{-2}, \quad \alpha(u) = \ln(u^2 - k^2), \quad u = 2^{1/2}y^1,$$

for k a real constant. This gives

$$(7.6) \quad \ell = 2mc^{-2} \ln(r^4 - 2r^2k^2 \cos(2\theta) + k^4).$$

When this is substituted into (7.2), the coefficient of dt^2 is

$$mc^{-2} \ln(r^4 - 2r^2k^2 \cos(2\theta) + k^4) - 1,$$

and in the limit as k goes to zero, it goes over into

$$4mc^{-2} \ln(r) - 1.$$

It is evident from (7.6) that the coefficients depending on ℓ will have singularities at $x = \pm k$, $y = 0$. It thus follows, by the same reasoning as used in the case of (7.4), that the solution given by (7.2), with ℓ given by (7.6), represents the gravitational field of two rods of uniform mass $M = m/G$ per unit length that are parallel to the z -axis and are situated at ($y = 0$, $x = \pm k$). Since the momentum-energy tensor vanishes, except at the singularities, the coordinate system in use must be one for which the rods are instantaneously at rest. The significant point here is that the problem of two rods that rotate about their common center in such a fashion that they are in gravitational equilibrium admits a solution which *does not exhibit gravitational waves in the comoving coordinate system*. This brings to the fore some serious questions concerning current thinking on the generation of gravitational waves. In this context, it is of interest to mention that by taking $m/G = \delta(z - t)c^{-2}E$, which is permissible since we may have $A = A(y^4)$ by Theorem 6.5, the solution obtained with (7.6) represents the gravitational field of two photons of energy E that move along the lines $y = 0$, $x = \pm k$ in the corotating coordinate system.

The solutions reported here are given only as an indication of the use of the existence theorems established in Sec. 5. We shall return to the problem of constructing general solutions of the Einstein field equations in succeeding papers in this series. This problem will be significantly simplified by a complete presentation of the scheme that was outlined in Sec. 5, and a detailed exposition of the equations of motion and the equations of geodesics in the augmented spaces E^* . This will also simplify the interpretation of solutions, since we will be able to investigate the resulting fields by examining the intrinsic (anholonomic) paths of test particles and light rays in such spaces.

REFERENCES

- [1] EDELEN, D. G. B., The Null Bundle of an Einstein-Riemann Space I: General Theory, RM-4870-PR, *The RAND Corporation*, J. Math. Mech., **16** (1966) 351-363.
- [2] FOCK, V., *The Theory of Space, Time, and Gravitation*, Second Revised Edition, The McMillan Co., New York, 1964.
- [3] HLAVATÝ, V., *Geometry of Einstein's Unified Field Theory*, P. Nordhoff Ltd., Gronigen, Holland, 1957.
- [4] RZEWUSKI, J., *Field Theory, Part I, Classical Theory*, Panstwowe Wydawnictwo Naukowe, Warsaw, 1958.
- [5] BAZANSKI, S. L., Decomposition of the Lorentz Transformation Matrix into Skew-Symmetric Tensors, *J. Math. Phys.*, **6** (1965) 1201-1203.
- [6] SCHOUTEN, J. A., *Ricci-Calculus*, Second Edition, Springer-Verlag, Berlin, 1954.

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