On the Zeros of Cross-Product Bessel Functions

JOHANN MARTINEK,
HENRY P. THIELMAN &
EUGENE C. HUEBSCHMAN

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ABSTRACT

To obtain a useful solution for physical interpretation and eventual numerical evaluation of the wave equation with cylindrical layered geometry, the zeros of certain cross-products of Bessel functions and their derivatives must be determined. In this paper an examination of a set of cross-products related to mixed boundary value problems as analytical functions of the index is carried out. Also the locations of the zeros of these product expressions are compared with the zeros studied by Cochran [1] for different boundary conditions.

1. Introduction. In boundary value problems pertaining to cylindrical annuli one encounters the following combinations of functions

$$(1) \quad f_{\nu}(r, kz) = O_{\nu} O_{kz} [J_{\nu}(r) Y'_{\nu}(kz) - J'_{\nu}(kz) Y_{\nu}(r)]$$

with $O_r = d/d(r)$ or 1, and $O_{kz} = d/d(kz)$ or 1, where J_{ν} and Y_{ν} are Bessel functions. These expressions arise regularly in a variety of physical problems like in transmission line theory, acoustics, electric and magnetic theory.

One is usually interested in either the nature of the r, kz -zeros of the expressions $f_{\nu}(r, kz)$ for fixed ν or that of their ν -zeros with r and kz given. We are interested in the latter and will derive information about these zeros without the use of numerical computation.

There are then two combinations of operators related to different boundary value problems in two dimensions. For the case of the outer (or inner) boundary condition being Dirichlet's and the inner (or outer) boundary condition being Neumann's we find the following results. When z and r are real and $k > 0$ the appropriate expression

$$J_{\nu}(r) Y'_{\nu}(kz) - J'_{\nu}(kz) Y_{\nu}(r)$$

has a countable number of ν -zeros of which only at most a finite number are real, the remainder being purely imaginary. All of those zeros are shown to be simple except for that occurring at the origin ($\nu = 0$), which can appear only as a zero of the second order. Asymptotic expressions are derived for the large imaginary ν -zeros. Finally the distribution of the imaginary ν -zeros shows an interlacing with the imaginary ν -zeros for the boundary value problems dealt with by Cochran, 1964. This later phenomena appears to be the essential difference in the nature of the three sets of imaginary zeros under consideration.

2. Absence of complex ν -zeros, simplicity of others. The integral formula (Watson, p. 139)

$$(1) \quad \int_1^s \left\{ (k^2 - l^2)\rho - \frac{\mu^2 - \nu^2}{\rho} \right\} \cdot C_\mu(k\rho) \cdot \bar{C}_\nu(l\rho) d\rho \\ = z \left\{ C_\mu(kz) \frac{\partial \bar{C}_\nu(lz)}{\partial z} - \bar{C}_\nu(lz) \frac{\partial C_\mu(kz)}{\partial z} \right\},$$

with μ and ν both in general complex and where $C_\mu(kz)$ and $C_\nu(lz)$ are cylinder functions, which can be any of the Bessel functions of arbitrary order, is one of the few equations in the literature (except for the Bessel equation itself) which relate cylinder functions explicitly to the order of that function. Using in equation (1) the following pairs of functions from left to right

$$(2) \quad \begin{aligned} C_\mu(kz) &= J_\mu(kz), & \bar{C}_\nu(lz) &= J_\nu(lz), \\ &= Y_\mu(kz), & &= J_\nu(lz), \\ &= J_\mu(kz), & &= Y_\nu(lz), \\ \text{and} & & & \\ &= Y_\mu(kz), & &= Y_\nu(lz), \end{aligned}$$

in succession we obtain the equations

$$\begin{aligned} 1) \quad & \int_1^s \left\{ (k^2 - l^2)\rho - \frac{\mu^2 - \nu^2}{\rho} \right\} J_\mu(k\rho) J_\nu(l\rho) d\rho \\ &= \left\{ \rho \left[J_\mu(k\rho) \frac{\partial J_\nu(l\rho)}{\partial \rho} - \frac{\partial J_\mu(k\rho)}{\partial \rho} J_\nu(l\rho) \right] \right\}_{\rho=s} \\ (3) \quad 2) \quad & - \int_1^s \left\{ (k^2 - l^2)\rho - \frac{\mu^2 - \nu^2}{\rho} \right\} Y_\mu(k\rho) J_\nu(l\rho) d\rho \\ &= - \left\{ \rho \left[Y_\mu(k\rho) \frac{\partial J_\nu(l\rho)}{\partial \rho} - \frac{\partial Y_\mu(k\rho)}{\partial \rho} J_\nu(l\rho) \right] \right\}_{\rho=s} \\ 3) \quad & - \int_1^s \left\{ (k^2 - l^2)\rho - \frac{\mu^2 - \nu^2}{\rho} \right\} J_\mu(k\rho) Y_\nu(l\rho) d\rho \\ &= - \left\{ \rho \left[J_\mu(k\rho) \frac{\partial Y_\nu(l\rho)}{\partial \rho} - Y_\nu(l\rho) \frac{\partial J_\mu(k\rho)}{\partial \rho} \right] \right\}_{\rho=s} \end{aligned}$$

and

$$4) \quad \int_1^z \left\{ (k^2 - l^2)\rho - \frac{\mu^2 - \nu^2}{\rho} \right\} Y_\mu(k\rho) Y_\nu(l\rho) d\rho \\ = \left\{ \rho \left[Y_\mu(k\rho) \frac{\partial Y_\nu(l\rho)}{\partial \rho} - Y_\nu(l\rho) \frac{\partial Y_\mu(k\rho)}{\partial \rho} \right] \right\}_{\rho=z}.$$

Multiplying 1) by $Y_\mu(r) \cdot Y_\nu(R)$, 2) by $J_\mu(r) \cdot Y_\nu(R)$, 3) by $Y_\mu(r) \cdot J_\nu(R)$ and 4) by $J_\mu(r) \cdot J_\nu(R)$ and adding all four equations (3) we find, after rearranging terms, the equation

$$(4) \quad \int_1^z \left\{ (k^2 - l^2)\rho - \frac{\mu^2 - \nu^2}{\rho} \right\} [Y_\mu(r) J_\mu(k\rho) - J_\mu(r) Y_\mu(k\rho)] \\ \cdot [Y_\nu(R) J_\nu(l\rho) - J_\nu(R) Y_\nu(l\rho)] d\rho \\ = (lz) \cdot [Y_\mu(r) J_\mu(kz) - J_\mu(r) Y_\mu(kz)] \cdot [Y_\nu(R) J'_\nu(lz) - J_\nu(R) Y'_\nu(lz)] \\ - (kz) \cdot [Y_\nu(R) J_\nu(lz) - J_\nu(R) Y_\nu(lz)] \cdot [Y_\mu(r) J'_\mu(kz) - J_\mu(r) Y'_\mu(kz)].$$

After applying the operator $d/dr \cdot d/dR$ to both sides of equation (4) we yield

$$(5) \quad \int_1^z \left\{ (k^2 - l^2)\rho - \frac{\mu^2 - \nu^2}{\rho} \right\} [Y'_\mu(r) J_\mu(k\rho) - J'_\mu(r) Y_\mu(k\rho)] \\ \cdot [Y'_\nu(R) J_\nu(l\rho) - J'_\nu(R) Y_\nu(l\rho)] d\rho \\ = (lz) \cdot [Y'_\mu(r) J_\mu(kz) - J'_\mu(r) Y_\mu(kz)] \cdot [Y'_\nu(R) J'_\nu(lz) - J'_\nu(R) Y'_\nu(lz)] \\ - (kz) \cdot [Y'_\nu(R) J_\nu(lz) - J'_\nu(R) Y_\nu(lz)] \cdot [Y'_\mu(r) J'_\mu(kz) - J'_\mu(r) Y'_\mu(kz)].$$

On the righthand sides of equations (4) and (5) we immediately recognize the expression

$$(6) \quad Y_\mu(r) J_\mu(kz) - J_\mu(r) Y_\mu(kz)$$

as the denominator which appears in the solution for a cylindrical annulus with Dirichlet's conditions on both boundaries. Furthermore we have the expressions

$$(7a,b) \quad \pm Y_\mu(r) J'_\mu(kz) \mp J_\mu(r) Y'_\mu(kz),$$

which are the denominators appearing in the solution for the same annulus with either Dirichlet's condition on the inner and Neumann's condition on the outer boundary of the annulus or *vice versa*. Finally there is the expression

$$(8) \quad Y'_\mu(r) J'_\mu(kz) - J'_\mu(r) Y'_\mu(kz),$$

which is the denominator which occurs in the solution for the annulus with Neumann's conditions on both boundaries.

We are now in a position to study the zeros of the expressions (6), (7a, b) and (8) by means of either equation (4) or (5). Depending on the type of boundary value problem to be considered one of the expressions (6), (7a, b)

or (8) must be put to zero in (4) or (5). Let us equate *e.g.*, the expression (7b) to zero in equation (5)

$$(9) \quad J_\mu(r) Y'_\mu(kz) - J'_\mu(kz) Y_\mu(r) = 0.$$

Assuming also in equation (5) $\mu = \nu^*$, the complex conjugate of ν , $r = R$ and $k = 1$, then the righthand side of equation (5) becomes zero. The lefthand side of (5) contains now the multiplying factor

$$(10) \quad |J_\nu(l\rho) Y'_\nu(R) - J'_\nu(R) Y_\nu(l\rho)|^2.$$

Hence the integral in the lefthand side of (5) is positive definite and a contradiction results. The other factor is, if we write for ν in general complex

$$(11) \quad \nu = A + iB, \quad \nu^* = A - iB,$$

the term $4iAB/\rho$, which has to be zero in order to comply with the righthand side of equation (5). Hence either A or B must be zero for lz and R real. This implies however the absence of complex ν -zeros. Hence the ν -zeros for lz and R real are either real or purely imaginary.

If we would have chosen any of the other expressions *i.e.*, (6), (7a) or (8), then the reasoning would be identical and we would have obtained the same result (Cochran 1964). Hence we can conclude that none of the changes of the boundary conditions considered earlier, would remove the ν -zeros from the imaginary or real axis of the complex ν -plane.

3. The distribution of the ν -zeros. It remains to be seen now if either the distribution of the ν -zeros on these axes is different as compared to the case where both boundaries of the annulus have either Dirichlet's or Neumann's conditions imposed. For the answer, we will apply Stirling's results for the Γ -functions to the familiar series expansion of J_ν . We have for imaginary index ($\mu = \mu_0 + i\mu_1$)

$$(12) \quad J_{\pm i\mu_1}(x) = (2\pi\mu_1) \cdot \exp \left[\frac{\mu_1\pi}{2} \pm i \left(\mu_1 + \mu_1 \log \left(\frac{x}{2\mu_1} \right) - \frac{\pi}{4} \right) \right] [1 + O(\mu_1^{-1})].$$

This result was used by Cochran 1964 to derive the distribution of the infinity of imaginary ν -zeros ($i\mu_1$) for real x . They are given asymptotically in our case for equation (9), with r replaced by x and kz replaced by kx , by the equation

$$(13) \quad J_\mu(x) Y'_\mu(kx) - J'_\mu(kx) Y_\mu(x) = J_{i\mu_1}(x) J'_{-i\mu_1}(kx) - J'_{i\mu_1}(kx) J_{-i\mu_1}(x) \\ = - \left(\frac{i}{\pi kx} \right) \cdot e^{\mu_1\pi} [\cos(\mu_1 \log k) + O(\mu_1^{-1})].$$

From equation (13) we find for $\nu = i\mu_1$

$$(14) \quad \mu_1 = \frac{(s + \frac{1}{2})\pi}{\log k} + O(s^{-1})$$

where s is a large positive integer. The asymptotic distribution of imaginary

ν -zeros, equation (14), turned out to be exactly the same as the ones obtained by Cochran 1964 for both of his sets of boundary value problems, namely

$$(15) \quad \mu_1 = \frac{s\pi}{\log k} + O(s^{-1}).$$

The only difference which exists here is essentially a shift of $+\pi/2$ or $-\pi/2$ along the imaginary ν -axis which corresponds to an interlacing of zeros.

As regards to the ν -zeros on the real ν -axis, we will now study their existence for equations (9) for arbitrarily large modulus. For this purpose, we use the formulae (Cochran, 1964)

$$(16) \quad J_n(x) = (12\pi n)^{-1/2} \cdot \exp \left[n - n \log \left(\frac{2n}{x} \right) \right] \cdot [1 + O(n^{-1})]$$

and

$$(17) \quad Y_n(x) = \left(\frac{\pi n}{2} \right)^{-1/2} \cdot \exp \left[n \log \left(\frac{2n}{x} \right) - n \right] \cdot [1 + O(n^{-1})]$$

for $n \rightarrow \infty$. These equations yield

$$(18) \quad J_{n-1}(x) = \left(\frac{2n}{x} \right) J_n(x) [1 + O(n^{-1})],$$

and

$$(19) \quad Y_{n-1}(x) = \left(\frac{x}{2n} \right) Y_n(x) [1 + O(n^{-1})].$$

Applying the appropriate recursion relations (Watson 1962) we find

$$(20) \quad J'_n(x) = -\frac{n}{x} J_n(x) + J_{n-1}(x),$$

and

$$(21) \quad Y'_n(x) = -\frac{n}{x} Y_n(x) + Y_{n-1}(x).$$

In combination with equations (16), (17), (18) and (19) we obtain for our expression (9) approximately

$$(22) \quad \begin{aligned} & J_n(r) Y'_n(kz) - J'_n(kz) Y_n(r) \\ & \approx \frac{1}{\sqrt{2\pi n}} \cdot \left(\frac{\pi n}{2} \right)^{-1/2} \left[\frac{-2n^2(kz) + (kz)^2}{2n(kz)} \right] \cdot e^{n \log(2n/kz)} \cdot [1 + O(n^{-1})]^2 \\ & + \frac{n}{kz \sqrt{2\pi n}} \cdot \left(\frac{2n}{r} \right) \sqrt{\frac{2}{\pi n}} \cdot [e^{2n - n \log(2n/kz)}] \cdot [1 + O(n^{-1})]^2 \\ & = \left[-\frac{1}{\pi} \left(\frac{2n}{kz} \right)^n - \left(\frac{b}{a} \right)^n \cdot \frac{e^{2n}}{\pi kz} \right] [1 + O(n^{-1})] \approx n^n [1 + O(n^{-1})]. \end{aligned}$$

From the asymptotic expression (22) it is clear that there are no real ν -zeros of equation (9) with arbitrary large modulus. The same result has been obtained by Cochran 1964 for the other two sets of boundary conditions.

4. The nature of the ν -zeros for real argument. With the aid of the well known relations

$$\begin{aligned}
 J_\nu(x) &= \frac{1}{2}[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)], & J'_\nu(x) &= \frac{1}{2}[H_\nu^{(1)'}(x) + H_\nu^{(2)'}(x)] \\
 Y_\nu(x) &= \frac{1}{2i}[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)], & Y'_\nu(x) &= \frac{1}{2i}[H_\nu^{(1)'}(x) - H_\nu^{(2)'}(x)] \\
 J_{-\nu}(x) &= \frac{1}{2}[H_{-\nu}^{(1)}(x) + H_{-\nu}^{(2)}(x)], & J'_{-\nu}(x) &= \frac{1}{2}[H_{-\nu}^{(1)'}(x) + H_{-\nu}^{(2)'}(x)] \\
 Y_{-\nu}(x) &= \frac{1}{2i}[H_{-\nu}^{(1)}(x) - H_{-\nu}^{(2)}(x)], & Y'_{-\nu}(x) &= \frac{1}{2i}[H_{-\nu}^{(1)'}(x) - H_{-\nu}^{(2)'}(x)]
 \end{aligned}
 \tag{23}$$

and

$$H_{-\nu}^{(1)}(x) = e^{i\nu\pi} \cdot H_\nu^{(1)}(x), \quad H_{-\nu}^{(2)}(x) = e^{-i\nu\pi} \cdot H_\nu^{(2)}(x) \tag{24}$$

we find that

$$\begin{aligned}
 &J_\nu(r)Y'_\nu(kz) - J'_\nu(kz)Y_\nu(r) \\
 &= J_{-\nu}(r)Y'_{-\nu}(kz) - J'_{-\nu}(kz)Y_{-\nu}(r).
 \end{aligned}
 \tag{25}$$

Hence the left side of equation (9) is an even function of ν . Again the identical result was obtained by Cochran for his cases.

In exactly the same way as Cochran proved his results we could prove in our cases the simplicity of all non-vanishing ν -zeros with real arguments $l\rho$ and r . Also that the origin of $\nu = 0$ is at most a double ν -zero. Hence the origin can appear then only as a zero of precisely the second order.

In conclusion we can state that the only discernible difference between a cylindrical annulus with Dirichlet's condition on one and Neumann's condition on the second boundary, and the case of Dirichlet's conditions and Neumann's conditions the same on both boundaries, related to imaginary ν -zeros, will be a difference in the distribution of these zeros. There will be a shift of those zeros by $\pm\pi/2$ along the imaginary ν -axis. There will also be a difference in the distribution and location of real ν -zeros between the Dirichlet and Neumann case on one hand and the mixed cases considered above on the other hand. However, no method has yet been devised to analyze this difference so far other than by a numerical analysis.

REFERENCES

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Teledyne Industries
Alexandria, Virginia.

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