The Tallest Column

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1. Introduction. If a tall column is made from a given volume of material, it must be made thin, but a tall thin column may buckle under its own weight. It seems clear that by properly shaping the column—*e.g.* by tapering it at the top—it may be made taller without buckling. Therefore we seek that shape which yields the tallest column, stable against buckling, which can be made with a given volume of material, assuming that the shape of the cross section and the Young's modulus of the material are also given. We consider this problem both for an unloaded column, and for one with a given load at its top. The only limitation upon the height which we consider is lateral buckling. This problem is a generalization of one which was previously treated by J. B. Keller [1] and I. Tadjbakhsh & J. B. Keller [2] in which the weight of the column was neglected. Some of the analysis and the method of numerical solution are the same as those of F. I. Niordson [3], who considered a similar problem.

We have solved the problem and found that the tallest unloaded column is approximately twice as tall as the column of uniform cross section made of the same material and having the same cross-sectional shape. For a loaded column, the ratio of the maximum height to the height of the uniform column diminishes from 2.034 at zero load to $(4/3)^{1/4} = 1.075$ when the load is infinite compared to the column weight. The result for the latter case follows from [1]. We also find the shape of the tallest column. As the top is approached, its cross-sectional area vanishes like the cube of the distance from the top in the unloaded case and like the two-thirds power of this distance in the loaded case. Thus there is a cusp at the top in the unloaded case and a round top in the loaded case.

Our analysis is based upon the Bernoulli–Euler theory, according to which the bending moment of a bent column is proportional to its curvature. We use this theory to obtain the linear equation for the determination of the deflection

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and buckling load of the column. By performing a variational analysis on this equation, we are led to a boundary value problem for a non-linear integro-differential equation for the shape of the tallest column. We have solved this problem numerically by an iteration method for the unloaded case. We have also deduced the behavior of the column near its top and bottom. In addition we have converted the problem to one for a first ordinary differential equation which permits us to show that the problem has one and only one solution.

Previous investigators of the buckling of columns who have taken the column weight into account, have always assumed that the shape of the column was given. Greenhill [4] first solved the buckling problem for an unloaded uniform column. He also considered various special forms for the variation of cross-sectional area with height such that the equation for the deflection was a Bessel equation. An account of some of his work, together with references to other related works, is given by Timoshenko [5].

From a mathematical point of view, our problem consists in maximizing the lowest eigenvalue of a linear second order ordinary differential equation with homogeneous boundary conditions. A coefficient function in the differential equation is to be varied subject to an integral side condition. Problems of this type are called isoperimetric problems in the calculus of variations. Our result can be formulated as an isoperimetric inequality which gives an upper bound on the lowest eigenvalue of the differential equation, but we shall not formulate it in that way. We actually determine only a shape which yields a stationary eigenvalue. Methods such as those used in [2] could be used to prove that it is really a maximum.

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2. Formulation. Let us consider a vertical column of height H and volume V made of material of mass density ρ and Young's modulus E . Let $A(z)$ be the area and $I(z)$ the moment of inertia, about an axis perpendicular to the plane of bending, of the cross section at height z . We shall assume that all cross sections are similar and similarly oriented. Then $I(z) = \alpha A^2(z)$ where α is a constant determined by the shape of the cross section and the plane of bending. Let P be a vertical load at the top of the column and g be the acceleration of gravity so that ρg is the weight density of the column. The top of the column at $z = H$ will be assumed to be free, while the bottom at $z = 0$ will be assumed to be clamped. Finally let $y(z)$ be the lateral deflection of the column from the vertical in some fixed vertical plane, which we call the plane of bending.

We shall attempt to determine the shape of the column, characterized by $A(z)$, in such a way as to maximize the height H . The limitation on the height is assumed to be lateral buckling due to instability. Therefore we shall assume that the column is just in its lowest mode of instability. Then by the Bernoulli-Euler theory, the deflection $y(z)$ satisfies the following equation of equilibrium, obtained by equating EIy_{zz} to the bending moment due to the load and the

weight of the column:

$$(1) \quad E\alpha A^2 y_{zz} = \int_z^H \rho g A(z') [y(z') - y(z)] dz' + P[y(H) - y(z)], \quad 0 \leq z \leq H.$$

The condition that the column is clamped at $z = 0$ implies that

$$(2) \quad y(0) = y_s(0) = 0.$$

The condition that the column volume is V is

$$(3) \quad \int_0^H A(z) dz = V.$$

The mathematical problem we consider can now be formulated. It is to find the largest possible value of H , and the corresponding column shape $A(z) \geq 0$ satisfying (3), such that (1) and (2) have a non-trivial solution $y(z)$ with no internal nodes. The condition on internal nodes is imposed to make certain that $y(z)$ is the lowest mode of buckling.

Before attempting to solve the problem, it is convenient to introduce the dimensionless variables x , $a(x)$, $\eta(x)$, λ and k defined by

$$(4) \quad \begin{aligned} x &= z/H, & a(x) &= HA(xH)/V, & \eta(x) &= y(xH)/H, \\ \lambda &= \rho g H^4 / \alpha E V, & k &= P / \rho g V. \end{aligned}$$

Upon introducing these quantities into (1)–(3) we obtain

$$(5) \quad a^2(x) \eta_{xx} = \lambda \left\{ \int_x^1 a(x') [\eta(x') - \eta(x)] dx' + k[\eta(1) - \eta(x)] \right\}, \quad 0 \leq x \leq 1,$$

$$(6) \quad \eta(0) = \eta_s(0) = 0,$$

$$(7) \quad \int_0^1 a(x) dx = 1.$$

Our problem is now to find the largest value of λ and the corresponding $a(x) \geq 0$ satisfying (7) such that (5) and (6) have a non-trivial solution $\eta(x)$ with no internal nodes.

The equation (5) becomes more familiar if it is differentiated with respect to x and written as an equation for the slope $\varphi(x)$ defined by

$$(8) \quad \varphi(x) = \eta_x(x).$$

Then (5) yields

$$(9) \quad (a^2 \varphi_x)_x + \lambda \left\{ \int_x^1 a(x') dx' + k \right\} \varphi = 0, \quad 0 \leq x \leq 1.$$

The second condition in (6) is

$$(10) \quad \varphi(0) = 0.$$

Upon taking the limit of (5) as x tends to 1 we obtain another condition

$$(11) \quad \lim_{x \rightarrow 1} a^2(x) \varphi_x(x) = 0.$$

Equation (9) together with (10) and (11) form an eigenvalue problem for the eigenfunction φ and eigenvalue λ . Thus our problem becomes that of finding a column shape $a(x) \geq 0$ satisfying (7) such that the lowest eigenvalue of the problem (9)–(11) is as large as possible.

3. Variational equation. Let us obtain the Rayleigh quotient for λ by multiplying (2.9) by φ and integrating with respect to x from 0 to 1. This yields

$$(1) \quad \int_0^1 \varphi(a^2 \varphi_x)_x dx + \lambda \int_0^1 \left\{ \int_x^1 a(x') dx' + k \right\} \varphi^2 dx = 0.$$

Integration by parts in the first integral yields

$$(2) \quad \varphi a^2 \varphi_x \Big|_0^1 - \int_0^1 a^2 \varphi_x^2 dx + \lambda \int_0^1 \left\{ \int_x^1 a(x') dx' + k \right\} \varphi^2 dx = 0.$$

From (2.10) and (2.11) it seems that $\varphi a^2 \varphi_x$ vanishes at $x = 0$ and $x = 1$. We shall tentatively assume this to be the case, and ultimately verify that it is true. Then (2) can be solved for λ , which is found to be

$$(3) \quad \lambda = \int_0^1 a^2 \varphi_x^2 dx \Big/ \int_0^1 \left\{ \int_x^1 a(x') dx' + k \right\} \varphi^2 dx.$$

The expression on the right side of (3) is the Rayleigh quotient associated with the eigenvalue problem (2.9)–(2.11).

It is well known that the lowest eigenvalue λ is the minimum value which the Rayleigh quotient attains when φ is taken to be any piecewise differentiable function satisfying the boundary conditions (2.10) and (2.11). We wish to find that function $a(x)$ for which λ is a maximum. Thus the value of λ which we seek is given by the maximum with respect to $a(x)$, of the minimum with respect to $\varphi(x)$ of the Rayleigh quotient:

$$(4) \quad \lambda = \max_a \min_{\varphi} \int_0^1 a^2 \varphi_x^2 dx \Big/ \int_0^1 \left\{ \int_x^1 a(x') dx' + k \right\} \varphi^2 dx.$$

The admissible functions $a(x)$ are non-negative and satisfy (2.7).

From the variational characterization of λ in (4) we can obtain two Euler or variational equations, one by varying φ and the other by varying a . The equation obtained by varying φ is just the equation (2.9) from which we started. In varying a we should take account of the dependence of φ on a , but, as in the work of F. I. Niordson [3], the vanishing of the variation with respect to φ makes this unnecessary. To obtain the Euler equation with respect to $a(x)$ we first subtract

$$\mu_0 \left[\int_0^1 a(x) dx - 1 \right]$$

from the Rayleigh quotient (3), where μ_0 is a Lagrange multiplier. Then taking the variation with respect to $a(x)$ of the resulting expression we obtain

$$(5) \quad \frac{2a\varphi_x^2}{\int_0^1 \left\{ \int_x^1 a(x') dx' + k \right\} \varphi^2 dx} - \frac{\int_0^1 a^2 \varphi_x^2 dx' \int_0^x \varphi^2 dx'}{\left[\int_0^1 \left\{ \int_x^1 a(x') dx' + k \right\} \varphi^2 dx \right]^2} - \mu_0 = 0.$$

By using (3) we can simplify (5) to

$$(6) \quad \frac{2\lambda a\varphi_x^2}{\int_0^1 a^2 \varphi_x^2 dx} - \frac{\lambda^2 \int_0^x \varphi^2 dx}{\int_0^1 a^2 \varphi_x^2 dx} - \mu_0 = 0.$$

Solving for a yields

$$(7) \quad a = \frac{\lambda}{2} \varphi_x^{-2} \left[\int_0^x \varphi^2 dx + \mu_0 \lambda^{-2} \int_0^1 a^2 \varphi_x^2 dx \right].$$

We now introduce the new constant

$$\mu = \mu_0 \lambda^{-2} \int_0^1 a^2 \varphi_x^2 dx$$

and then (7) becomes

$$(8) \quad a = \frac{\lambda}{2} \varphi_x^{-2} \left(\mu + \int_0^x \varphi^2 dx \right).$$

Upon using (8) to eliminate a from (2.9) and (2.11) we obtain

$$(9) \quad \left[\varphi_x^{-3} \left(\mu + \int_0^x \varphi^2 dx' \right)^2 \right]_x + \left\{ 2 \int_x^1 \varphi_x^{-2} \left(\mu + \int_0^{x'} \varphi^2 dx'' \right) dx' + \frac{4k}{\lambda} \right\} \varphi = 0,$$

$$(10) \quad \lim_{x \rightarrow 1} \frac{\lambda^2}{4} \varphi_x^{-3} \left(\mu + \int_0^x \varphi^2 dx \right)^2 = 0.$$

Equation (10) implies that

$$(11) \quad \varphi_x(1) = \infty,$$

and (2.10) is still

$$(12) \quad \varphi(0) = 0.$$

When (8) is used in (2.7) it becomes

$$(13) \quad \int_0^1 \varphi_x^{-2} \left(\mu + \int_0^x \varphi^2 dx' \right) dx = 2/\lambda.$$

From (9) and (11)–(13) we must determine $\varphi(x)$, μ and λ . Then (8) yields $a(x)$.

It is to be noted that if φ , μ and λ is a solution of (9)–(13) then so is $\gamma\varphi$, $\gamma^2\mu$

and λ where γ is any non-zero constant. Thus we may choose $\gamma = \mu^{-1/2}$ and obtain a solution in which μ is replaced by unity. Therefore setting $\mu = 1$ corresponds to choosing the normalization of φ . Then we must find $\varphi(x)$ and λ satisfying (9) and (11)–(13) in which $\mu = 1$.

4. Behavior near the top and bottom. In order to study the behavior of $\varphi(x)$ near $x = 1$, which represents the top of the column, let us seek φ of the form

$$(1) \quad \varphi(x) = b_0 + b(1-x)^c + \dots$$

Here b_0 , b and c are constants to be determined. From (3.11) it is necessary that $c < 1$. Let us first suppose that $c > -\frac{1}{2}$ so that φ^2 is integrable at $x = 1$. Then (3.9) becomes

$$(2) \quad -3(c-1)(bc)^{-3}(1-x)^{-3(c-1)-1} \left(\mu + \int_0^1 \varphi^2 dx \right)^2 \\ + \left\{ -2 \left(\mu + \int_0^1 \varphi^2 dx \right) (bc)^{-2} \frac{(1-x)^{-2(c-1)+1}}{-2(c-1)+1} + \frac{4k}{\lambda} \right\} \\ \cdot [b_0 + b(1-x)^c] + \dots = 0.$$

The three lowest powers of $(1-x)$ in (2) for c in the allowed range $-\frac{1}{2} < c < 1$ are 0, c and $2-3c$. For a non trivial solution two of these exponents must be equal and less than or equal to the third. Thus either $c = 0 \leq 2-3c$ or $2-3c = 0 \leq c$ or $2-3c = c \leq 0$. The first two alternatives lead to $c = 0$ and $c = \frac{2}{3}$ respectively while the last yields no consistent value of c . The value $c = 0$ is excluded because then the two terms in (1) would have the same power of $1-x$ and therefore not be distinct. Thus $c = \frac{2}{3}$ and (1) becomes

$$(3) \quad \varphi(x) = b_0 + b(1-x)^{2/3} + \dots$$

In obtaining the result (3), one of the leading powers of $(1-x)$ was taken from the term multiplied by k . In the special case $k = 0$, in which there is no load at the top of the column, this term is absent. Then the above analysis shows that there is no value of $c > -\frac{1}{2}$ for which (2) can hold. In this case we seek a value of $c < -\frac{1}{2}$, so we may omit the b_0 in (1). Then φ^2 is not integrable at $x = 1$ and we have

$$(4) \quad \int_0^x \varphi^2 dx = -b^2 \frac{(1-x)^{2c+1}}{2c+1} + \dots$$

Now (3.9) becomes

$$(5) \quad \left\{ [(bc)^{-3}(1-x)^{-3(c-1)}] \left[b^4 \frac{(1-x)^{2(2c+1)}}{(2c+1)^2} \right] \right\}_x \\ + 2 \left\{ \int_x^1 (bc)^{-2}(1-x)^{-2(c-1)} \frac{(-b^2)}{2c+1} (1-x)^{2c+1} dx \right\} b(1-x)^c + \dots = 0.$$

The two leading powers of $(1 - x)$ in the two terms of (5) are both $c + 4$. Therefore their coefficients must add up to zero. By performing the differentiation and integration in (5) we obtain in this way

$$(6) \quad \frac{c^{-3}(c+5)}{(2c+1)^2} - \frac{2c^{-2}}{(2c+1)4} = 0.$$

After clearing fractions, this becomes the quadratic equation $c^2 - c/2 = 5$ which has the solution $c = 5/2$ and $c = -2$. Only the second solution satisfies the condition $c < -\frac{1}{2}$. Thus when $k = 0$ we have

$$(7) \quad \varphi(x) = b(1-x)^{-2} + \dots, \quad (k = 0).$$

The result (3), when used in (3.8) yields

$$(8) \quad a(x) = a_1(1-x)^{2/3} + \dots, \quad k \neq 0.$$

Here a_1 is a constant. This is the same behavior near the top as that found by I. Tadjbakhsh & J. B. Keller [2] in the corresponding problem in which the weight of the column was neglected, as is to be expected. The result (7), when used in (3.8) yields instead

$$(9) \quad a(x) = \frac{\lambda}{24}(1-x)^3 + \dots, \quad k = 0.$$

In this case the column has a cusp at the top whereas when $k \neq 0$ the top is round.

At the bottom $x = 0$, φ vanishes according to (3.12). Therefore we shall seek φ near $x = 0$ in the form

$$(10) \quad \varphi(x) = \beta x^c + \delta x^{2c} + \gamma x^{3c} + \dots.$$

Upon using (10) in (3.9) we find that $c = 1$ or $c = \frac{1}{2}$. When $c = 1$ then $\delta = 0$ while γ is determined in terms of β and (10) becomes

$$(11) \quad \varphi(x) = \beta x + \frac{2(k+1)\beta^5}{9\lambda\mu^2}x^3 + \dots.$$

When (11) is used in (3.8) it yields

$$(12) \quad a(x) = \frac{\lambda\mu}{2\beta^2} - \frac{2(k+1)\beta^2}{3\mu}x^2 + \dots.$$

The case $c = \frac{1}{2}$ must be excluded because it makes the integral (3.13) diverge. The result (12) is the same behavior of $a(x)$ near the bottom as that found by I. Tadjbakhsh & J. B. Keller [2] in the corresponding problem in which the weight of the column was neglected.

Let us now use the results of this section to verify the assumption that $\varphi a^2 \varphi_x$ vanishes at $x = 1$ and $x = 0$, which was made in going from (3.2) to (3.3). First for $k \neq 0$ we have from (3) and (8) that $\varphi a^2 \varphi_x = 2/3b_0a_1^2b(1-x) + \dots$ which does vanish at $x = 1$. Next for $k = 0$ we find from (7) and (9) that

$$\varphi a^2 \varphi_x = \frac{2b^2 \lambda^2}{(24)^2} (1-x) + \dots,$$

which also vanishes at $x = 1$. Near $x = 0$ for all k we see from (11) and (12) that $\varphi a^2 \varphi_x = (\lambda \mu / 2\alpha)^2 x + \dots$ which vanishes at $x = 0$. Thus the assumption is verified.

5. Solution by iteration. To determine $\varphi(x)$, λ and μ we must solve (3.9) and (3.11)–(3.13). To do so, we integrate (3.9) with respect to x from $x = 1$, and use (3.11) to obtain

$$(1) \quad \varphi_x^{-3} \left(\mu + \int_0^x \varphi^2 dx' \right)^2 = \int_x^1 \left\{ 2 \int_{x'}^1 \varphi_x^{-2} \left(\mu + \int_0^{x''} \varphi^2 dx''' \right) dx'' + \frac{4k}{\lambda} \right\} \varphi dx'.$$

Now we solve (1) for φ_x and integrate again from $x = 0$, using (3.12), to obtain

$$(2) \quad \varphi(x) = \int_0^x \left[\int_{x'}^1 \left\{ 2 \int_{x''}^1 \varphi_x^{-2} \left(\mu + \int_0^{x'''} \varphi^2 dx'''' \right) dx''' + \frac{4k}{\lambda} \right\} \varphi dx'' \right]^{-1/3} \cdot \left(\mu + \int_0^{x'} \varphi^2 dx'' \right)^{2/3} dx'.$$

We also solve (3.13) for λ , obtaining

$$(3) \quad \lambda = 2 \left[\int_0^1 \varphi_x^{-2} \left(\mu + \int_0^{x'} \varphi^2 dx'' \right) dx' \right]^{-1}.$$

Next we multiply (3.9) by φ and integrate it with respect to x from 0 to 1. After integrating by parts we can solve the resulting equation for μ and obtain

$$(4) \quad \mu = \left\{ \int_0^1 \left[\varphi_x^{-2} \left(\int_0^x \varphi^2 dx' \right)^2 dx + 4k\lambda^{-1} \varphi^2 \right] dx / \int_0^1 \varphi_x^{-2} dx \right\}^{1/2}.$$

The steps can be reversed to show that a solution of (2), (3) and (4) also satisfies (3.9) and (3.11)–(3.13).

Let us denote the right sides of (2), (3), and (4) by $F(\varphi, \mu, \lambda)$, $G(\varphi, \mu)$, and $H(\varphi, \mu, \lambda)$ respectively. Then (2)–(4) can be rewritten in the form

$$(5) \quad \varphi = F(\varphi, \mu, \lambda),$$

$$(6) \quad \lambda = G(\varphi, \mu),$$

$$(7) \quad \mu = H(\varphi, \lambda).$$

We now formulate an iteration scheme, following F. I. Niordson [3], by choosing φ_0 , μ_0 and λ_0 arbitrarily and defining

$$(8) \quad \varphi_{n+1} = F(\varphi_n, \mu_n, \lambda_n),$$

$$(9) \quad \lambda_{n+1} = G(\varphi_n, \mu_n),$$

$$(10) \quad \mu_{n+1} = H(\varphi_n, \lambda_n).$$

In the case $k = 0$ of an unloaded column, F and H do not involve λ . Then we use the following simpler iteration scheme, involving only φ and μ :

$$(11) \quad \mu_n = H(\varphi_n),$$

$$(12) \quad \varphi_{n+1} = F(\varphi_n, \mu_n).$$

We have used the iteration scheme (11), (12) for the unloaded column successfully starting with $\varphi_0 = x$. The calculation was performed by introducing $m + 1$ equally spaced points in the interval $0 \leq x \leq 1$ with separation $h = 1/m$, and defining the iterates φ_n at these points. For a given value of h , the iteration was continued until two successive iterates satisfied the condition

$$|\varphi_{n+1}(x) - \varphi_n(x)|/|\varphi_n(x)| < 10^{-6}$$

at all $m + 1$ values of x . Then from the final iterate $\varphi_{n+1}(x)$, λ was computed using (3). This calculation was repeated with successively larger values of m and correspondingly smaller values of h . Then the results were extrapolated to $h = 0$ by Newton's formula

$$(13) \quad \lambda = \lambda_1 - \Delta\lambda_1 + \Delta^2\lambda_1 - \Delta^3\lambda_1 + \cdots.$$

The values of m employed were $m = 51, 68, 101$ and 204 . The corresponding values of λ are shown in the table, together with the tabular differences. When these results are used in (13) they yield for the unloaded column

$$(14) \quad \begin{aligned} \lambda &= 132.6094 + 1.5387 + 0.0411 + 0.0027 \\ &= 134.19. \end{aligned}$$

For a uniform column, $\lambda = 7.833$. Thus the ratio of the maximum height $H_{\max}$ to that of the corresponding uniform column, H_{uniform} is according to the definition (2.4) of λ

$$(15) \quad \frac{H_{\max}}{H_{\text{uniform}}} = \left(\frac{134.19}{7.833} \right)^{1/4} = 2.034.$$

Thus by appropriately shaping the column, it can be made slightly more than twice as tall as an untapered column on the same material.

Table

m	$204h$	λ	$\Delta\lambda$	$\Delta^2\lambda$	$\Delta^3\lambda$
204	1	132.6094			
			-1.5387		
102	2	131.0707		0.0411	
			-1.4976		-0.0027
68	3	129.5731		0.0384	
			-1.4592		
51	4	128.1139			

6. Reduction to a nonlinear first order ordinary differential equation. The method of the last section has been used successfully to obtain numerical solutions of the problem for $\varphi(x)$ and λ . However it is not convenient for assessing all the solutions of the nonlinear equation and verifying that there is one and only one solution satisfying the boundary conditions. For this purpose we have introduced a number of transformations which finally convert the problem into one for a nonlinear first order ordinary differential equation. This appears to be a considerable simplification of the problem, at least for theoretical purposes and possibly also for numerical ones. It permits us to prove that the problem has one and only one solution in the unloaded case. A more detailed analysis of this equation would probably yield a similar result for the loaded case.

In order to simplify (3.9) let us first multiply it by φ^{-1} and then differentiate with respect to x . This yields

$$(1) \quad \left\{ \varphi^{-1} \left[\varphi_x^{-3} \left(\mu + \int_0^x \varphi^2 dx \right)^2 \right] \right\}_x - 2\varphi_x^{-2} \left(\mu + \int_0^x \varphi^2 dx \right) = 0.$$

We now introduce the new variables s and $u(s)$ defined by

$$(2) \quad s = \mu + \int_0^x \varphi^2 dx', \quad u(x) = \varphi^3(x).$$

In terms of them we have

$$(3) \quad \frac{d}{dx} = \varphi^2 \frac{d}{ds}, \quad \varphi_x = \varphi^2 \frac{d\varphi}{ds} = \frac{1}{3} \frac{d\varphi^3}{ds} = \frac{1}{3} u_s.$$

Upon introducing these variables into (1) and simplifying the result we obtain

$$(4) \quad u[s^2 u_s^{-3}]_{ss} - u_{ss} s^2 u_s^{-3} = 0.$$

This is an equation of third order for $u(s)$.

The value $x = 0$ corresponds to $s = \mu$ as we see from (2), and $x = 1$ corresponds to $s = s_0$ defined by

$$(5) \quad s_0 = \mu + \int_0^1 \varphi^2 dx.$$

When $k \neq 0$, s_0 is finite but when $k = 0$, $s_0 = \infty$ as we see from (4.7). The conditions (3.11)–(3.13) now become

$$(6) \quad u_s(s_0) = \infty,$$

$$(7) \quad u(\mu) = 0,$$

$$(8) \quad \int_{\mu}^{s_0} u^{-2/3} u_s^{-2} s ds = 2/9\lambda.$$

To obtain a third boundary condition on $u(s)$, which is necessary since (4) is of third order, we set $x = 0$ in (3.9) and use (3.13) to evaluate the integral. Then we introduce u and s and find

$$(9) \quad u^{1/3}(s^3 u_s^{-3})_{,s} \big|_{s=\mu} = -4(1+k)/27\lambda.$$

Equation (4) together with (6), (7) and (9) constitute a problem for the determination of $u(s)$ in the interval $\mu \leq s \leq s_0$ and then (8) is an equation for λ .

Once $u(s)$ and λ are found, a can be computed from (3.8) which becomes, when (2) is used,

$$(10) \quad a = 9\lambda s/2u_s^2.$$

This equation yields a as a function of s . To obtain it as a function of x , we must determine x in terms of s by integrating (3) which yields

$$(11) \quad x = \int_{\mu}^s u^{-2/3} ds.$$

Now (10) and (11) give $a(x)$ parametrically. The condition that $x = 1$ when $s = s_0$ yields the following equation, which may be considered an equation for s_0

$$(12) \quad 1 = \int_{\mu}^{s_0} u^{-2/3} ds.$$

The constant $\mu > 0$ is arbitrary.

The behavior of $u(s)$ near $s = s_0$ can be found from that of $\varphi(x)$ near $x = 1$. From (2) and (4.3) we have when $k \neq 0$,

$$(13) \quad s = s_0 - b_0^2(1-x) + \dots, \quad k \neq 0,$$

$$(14) \quad u(s) = b_0^3 + 3bb_0^{2/3}(s_0 - s)^{2/3} + \dots, \quad k \neq 0.$$

Similarly when $k = 0$ we have from (2) and (4.7)

$$(15) \quad s = \frac{b^2}{3}(1-x)^{-3} + \dots, \quad k = 0,$$

$$(16) \quad u(s) = \frac{9}{b}s^2 + \dots, \quad k = 0.$$

Near $s = \mu$ we find from (2) and (4.11) for any value of k

$$(17) \quad s = \mu + \beta^2 x^3/3 + \dots, \quad u(s) = 3\beta(s - \mu) + \dots.$$

The equation for $u(s)$ can be simplified by the introduction of the new variables t and $w(t)$ defined by

$$(18) \quad s = e^t, \quad u(s) = u_0 \exp\left(\int_{t_0}^t w(t') dt'\right).$$

When these variables are introduced into (4) it becomes the second order equation

$$(19) \quad 3ww_{,t} + (27w - 14w^2)w_t - 12w_t^2 - 8w^4 + 26w^3 - 20w^2 = 0.$$

The range of t is now $\log \mu \leq t \leq \log s_0$.

To find the behavior of $w(t)$ near the endpoint $t = \log s_0$ we use (14) and (18)

in the case $k \neq 0$ to obtain

$$(20) \quad b_0^3 + 3bb_0^{2/3}(s_0 - e^t)^{2/3} + \dots = u_0 \exp \left(\int_{t_0}^t w(t') dt' \right).$$

Differentiating (20) yields, for t near $\log s_0$,

$$(21) \quad \begin{aligned} w(t) &= -2bb_0^{-7/3}(s_0 - e^t)^{-1/3}e^t + \dots \\ &= -2bb_0^{-7/3}s_0^{2/3}(\log s_0 - t)^{-1/3} + \dots, \quad k \neq 0. \end{aligned}$$

Thus in this case $w(t)$ tends to $-\infty$ as t tends to $\log s_0$. When $k = 0$, (16) and (18) yield near $t = \infty$, which corresponds to $s = \infty$,

$$(22) \quad u(s) = \frac{9}{b}e^{2t} + \dots = u_0 \exp \left(\int_{t_0}^t w dt' \right).$$

Thus

$$(23) \quad w(\infty) = 2, \quad k = 0.$$

To find the behavior of $w(t)$ near the endpoint $t = \log \mu$ we have from (17) and (18) that

$$(24) \quad 2^{3/4}\beta^{3/2}(s - \mu)^{3/4} + \dots = u_0 \exp \left(\int_{t_0}^t w(\tau) d\tau \right).$$

Taking logarithms in (24) and differentiating with respect to t yields, for t near $\log \mu$,

$$(25) \quad w(t) = e^t(e^t - \mu)^{-1} + \dots = (t - \log \mu)^{-1} + \dots.$$

Thus for any value of k , $w(t)$ tends to $+\infty$ as t tends to $\log \mu$ from above.

The problem for $w(t)$ is to solve (19) in the range $\log \mu \leq t \leq \log s_0$ with the behavior (25) near $t = \log \mu$ and either the behavior (21) near $t = \log s_0$ when $k \neq 0$ or the behavior (23) at $t = \infty$ when $k = 0$. The constants $\mu > 0$ and $t_0 > \log \mu$ can be chosen arbitrarily while u_0 and, in the case $k \neq 0$, s_0 are to be found. The conditions (8), (9) and (12) can be used to find u_0 , s_0 and λ . In terms of w and t they become

$$(26) \quad \int_{\log \mu}^{\log s_0} w^{-2}(t) \exp \left(4t - \frac{8}{3} \int_{t_0}^t w(\tau) d\tau \right) dt = 2u_0^{8/3}/9\lambda,$$

$$(27) \quad \exp \left(-t + \frac{1}{3} \int_{t_0}^t w(\tau) d\tau \right) \left(w^{-3} \exp \left(5t - 3 \int_{t_0}^t w(\tau) d\tau \right) t \right) \Big|_{t=\log \mu} \\ = -4(1 + k)u_0^{8/3}/27\lambda,$$

$$(28) \quad u_0^{2/3} = \int_{\log \mu}^{\log s_0} \exp \left(t' - \frac{2}{3} \int_{t_0}^{t'} w(\tau) d\tau \right) dt'.$$

Once $w(t)$, λ and u_0 are found, a and x are given by (10) and (11) which become

$$(29) \quad a = 9\lambda e^{3t}/2w^2(t)u_0^2 \exp \left(2 \int_{t_0}^t w(\tau) d\tau \right),$$

$$(30) \quad x = u_0^{-2/3} \int_{\log \mu}^t \exp \left(t' - \frac{2}{3} \int_{t_0}^{t'} w(\tau) d\tau \right) dt'.$$

A simplification of the equation (19) for $w(t)$ can be made by observing that (19) does not involve t explicitly. Therefore we can reduce it to a first order equation by introducing $p(w)$ defined by

$$(31) \quad p(w) = w_t.$$

Then $w_{tt} = p_t = p_w w_t = p_w p$ and (19) becomes

$$(32) \quad 3wpp_w + (27w - 14w^2)p - 12p^2 - 8w^4 + 26w^3 - 20w^2 = 0.$$

It is convenient to rewrite (32) in the form

$$(33) \quad p_w = (8w^4 - 26w^3 + 20w^2 + (14w^2 - 27w)p + 12p^2)/3pw.$$

From (25) we find that for any value of k , as t tends to $\log \mu$, w tends to $+\infty$ and p tends to $-\infty$. As t tends to $\log s_0$ we find from (21) that for $k \neq 0$, w tends to $-\infty$ and

$$(34) \quad p(w) = +\text{const } w^4 + \dots \quad \text{as } w \rightarrow -\infty, \quad k \neq 0.$$

For $k = 0$, (23) shows that $p(w)$ tends to zero as w tends to 2 so

$$(35) \quad p(2) = 0, \quad k = 0.$$

Thus for $k = 0$ we seek a solution $p(w)$ of (33) which satisfies (35) and tends to $-\infty$ as w tends to $+\infty$. For $k \neq 0$ we seek a solution $p(w)$ of (33) which tends to $+\infty$ as w tends to $-\infty$ and to $-\infty$ as w tends to $+\infty$.

Once $p(w)$ is found, $w(t)$ can be obtained by integrating (31) which yields

$$(36) \quad t = \log \mu - \int_w^\infty dw/p(w).$$

In the case $k \neq 0$, t tends to $\log s_0$ as w tends to $-\infty$. Thus from (36) $\log s_0$ is given by

$$(37) \quad \log s_0 = \log \mu - \int_{-\infty}^\infty dw/p(w), \quad k \neq 0.$$

To obtain u_0 we use $w(t)$ and s_0 , given by (36) and (37), in (28) and then (27) yields λ . Finally $a(x)$ can be found from (29) and (30). The constants μ and t_0 in (27)–(30) can be chosen arbitrarily subject to the conditions $\mu > 0$ and $\log \mu < t_0 < \log s_0$. Thus the solution of the problem has finally been reduced to solving the first order ordinary differential equation (33) and then evaluating some integrals.

To examine the solutions of (33) it is convenient to consider the direction field it determines in the p, w plane. This field has singularities at the points where the numerator and denominator of the right side of (33) vanish simultaneously. There are three such points all on the axis $p = 0$ at $w = 0$, $w = 5/4$

and $w = 2$. The point $p = 0$, $w = 2$ can be seen, upon closer examination, to be a saddle point with two trajectories through it. Of these just one tends to $p = -\infty$ as w tends to $+\infty$. Thus in the unloaded case (33) has one and only one solution satisfying (35) and behaving correctly at infinity. Therefore in the unloaded case the original problem (3.9) and (3.11)–(3.13) has just one solution for $\varphi(x)$ and λ , and therefore for $a(x)$. Presumably a more detailed examination of the direction field would lead to a similar result for the loaded case.

REFERENCES

- [1] KELLER, J. B., The Shape of the Strongest Column, *Archiv. Rat. Mech. and Anal.*, **5** (1960) 275–285.
- [2] TADJBAKHSI, I., & KELLER, J. B., Strongest Columns and Isoperimetric Inequalities for Eigenvalues, *J. Appl. Mech.*, **29E** (1962) 159–164.
- [3] NIORDSON, F. I., On the Optimal Design of a Vibrating Beam, *Quart. App. Math.*, **23** (1965) 47–53.
- [4] GREENHILL, A. G., *Proc. Cambridge Phil. Soc.*, **4** (1881).
- [5] TIMOSHENKO, S., *Theory of Elastic Stability*, McGraw-Hill, New York, 1936, p. 115.

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