Change of Velocity in Flows

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1. Introduction. Our purpose in this paper is to investigate the connection between the spectrum of a flow and its velocity. If the velocity of a flow is changed it is necessary, in general, to change the measure to one invariant with respect to the new flow. Our principal result for ergodic flows is that it is always possible to change the velocity of the flow and the measure to an equivalent invariant measure in such a way that the resulting spectrum is continuous. We obtain first, a discrete version of this result which was formulated by Professor S. Kakutani and which introduces some of the ideas needed in the continuous case. The discrete result is that there exists a set such that if the discrete flow is stopped by one time unit whenever it reaches the set, then the resulting flow has continuous spectrum.

2. Preliminaries. Let $(X, \mathfrak{F}, \mu)$ be the unit interval, the Lebesgue sets, and the Lebesgue measure.

Definition 2.1. A measurable tower T is an ordered class of measurable sets $\{A_{i,k}, j = 1, \dots, n; k = 1, \dots, m(j)\}$ which form a partition $P(T)$ of X and which have the property that $\mu(A_{i,k}) = \mu(A_{i,1}), 1 \leq j \leq n, 1 \leq k \leq m(j)$. The sets $\{A_{i,k}, 1 \leq k \leq m(j)\}$ are called the j -th column of the tower, and the set $A_{i,k}$ is called the k -th set of the j -th column. The base of the tower is defined as the set

$$B(1, T) = \bigcup_{i=1}^n A_{i,1}$$

and the top of the tower is defined as the set

$$B(2, T) = \bigcup_{i=1}^n A_{i,m(i)}.$$

Finally, the width of the tower is defined as the number $W(T) = \mu(B(1, T)) = \mu(B(2, T))$. Note that the sets of each column have equal measure.

Definition 2.2. The transformation τ_T induced by the tower $T = \{A_{i,k}, 1 \leq j \leq n; 1 \leq k \leq m(j)\}$ is defined by mapping $A_{i,k}$ linearly onto $A_{i,k+1}$,

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$1 \leq k \leq m(j) - 1$, $1 \leq j \leq n$. Note that the domain of definition of τ_T is the set $X - B(2, T)$ and that its range is the set $X - B(1, T)$.

If we regard the j -th column as a stack of sets with $A_{j,k}$ the k -th set in the stack counting from the bottom, then τ_T maps each point to the one directly above in the tower, and τ_T is undefined on the top of the tower.

Definition 2.3. We say that the tower $T(2)$ is finer than the tower $T(1)$ if $P(T(2))$ is finer than $P(T(1))$ and if $\tau_{T(2)}$ is an extension of $\tau_{T(1)}$.

Definition 2.4. A nested sequence of towers $\{T(r)\}$ is one for which

i) $T(r)$ is finer than $T(r - 1)$, $r \geq 2$,

and

ii) $\lim_{r \rightarrow \infty} W(T(r)) = 0$.

The transformation $\lim_{r \rightarrow \infty} \tau_{T(r)}$, which clearly exists if $\{T(r)\}$ is a nested sequence, is defined to be $\tau_{T(r)}$, the transformation induced by the sequence $\{T(r)\}$.

Definition 2.5. If $T = \{A_{j,k}, 1 \leq j \leq n, 1 \leq k \leq m(j)\}$ is a measurable tower and if $B_{j(0),k(0)}$ is a measurable subset of $A_{j(0),k(0)}$, then the tower $S(B_{j(0),k(0)}, T)$ generated by $B_{j(0),k(0)}$ and the tower T is defined by setting

$$S = \{C_{j,k}, 1 \leq j \leq n + 1, 1 \leq k \leq m(S, j)\},$$

where

$$m(S, j) = \begin{cases} m(j), & 1 \leq j \leq j(0) - 1, \\ m(j(0)), & j = j(0), \quad j = j(0) + 1, \\ m(j - 1), & j(0) + 2 \leq j \leq n + 1, \end{cases}$$

and where

$$C_{j,k} = \begin{cases} A_{j,k}, & 1 \leq j \leq j(0), \quad 1 \leq k \leq m(S, j), \\ \tau^{k-k(0)} B_{j(0),k(0)}, & j = j(0), \quad 1 \leq k \leq m(S, j(0)), \\ \tau^{k-k(0)} (A_{j(0),k(0)} - B_{j(0),k(0)}), & j = j(0) + 1, \quad 1 \leq k \leq m(S, j(0) + 1), \\ A_{j-1,k}, & j(0) + 2 \leq j \leq n + 1, \quad 1 \leq k \leq m(S, j). \end{cases}$$

Definition 2.6. If $T = \{A_{j,k}, 1 \leq j \leq n, 1 \leq k \leq m(j)\}$ is a measurable tower and if B is a measurable set we define the tower $S(B, T)$ generated by B and the tower T by setting $B_{j,k} = A_{j,k} \cap B$ and applying Definition 2.5 inductively with the sets $B_{j,k}$ (arranged in any fixed order; it is clear that the resulting tower is independent of their order). Note that the domains of definition of τ_T and of $\tau_{S(B,T)}$ coincide and that the transformations are equal.

Definition 2.7. If P is a partition, $\mathfrak{F}(P)$ is the Borel field generated by P .

Definition 2.8. We say that a measurable set A equals a measurable set B with an error of δ if the measure of the symmetric difference of A and B is no more than δ , and write $A = B + E(\delta)$.

Definition 2.9. Let τ be an invertible measure preserving transformation on X . We call $\{\tau^n\}$, $n = 0, \neq 1, \neq 2, \dots$, a *discrete flow* on X .

Lemma 2.1. ([5]). *If τ is anti-periodic then there exists a partition of X consisting of three measurable sets A_1, A_2, A_3 , such that*

$$(1.1) \quad \tau(A_1) = A_2$$

$$(1.2) \quad \tau((A_2 - \tau^{-1}(A_3)) + A_3) = A_1.$$

Proof. Let $\mathfrak{A}$ be the class of measurable sets A such that $\mu(A \cap \tau(A)) = 0$. Order $\mathfrak{A}$ by set inclusion and apply Zorn's lemma to obtain a maximal set A_1 in the class (we have used the fact that L_1 is a complete lattice). Next define $A_2 = \tau(A_1)$ and $A_3 = \tau^{-1}(A_1) - A_2$. That (1.1) and (1.2) are satisfied follows from the maximality of A_1 .

Theorem 2.1 ([5]) *If τ is anti-periodic then there exists a nested sequence $\{T(r)\}$ such that $\tau = \tau_{\{T(r)\}}$ and such that $W(T(r)) \leq 1/2^r$.*

Proof. Apply Lemma 2.1 to obtain $T(1)$ defined by

$$T(1) = \{A_{i,k}(1), 1 \leq j \leq 2, 1 \leq k \leq m(1, j)\},$$

where $m(1, 1) = 2$, $m(1, 2) = 3$ and the sets $A_{i,k}$ are defined by $A_{21}(1) = \tau^{-2}A_3$, $A_{22}(1) = \tau^{-1}A_3$, $A_{23}(1) = A_3$, $A_{11}(1) = A_1 - A_{21}(1)$, and finally $A_{12}(1) = \tau A_{11}(1)$. Note that on its domain of definition $\tau_{T(1)}$ coincides with τ and that $W(T(1)) \leq \frac{1}{2}$.

Next, consider the transformation induced by τ on the base of $T(1)$ and apply Lemma 2.1 as in the preceding step. Taking into account the nature of the induced transformation, we obtain a tower $T(2)$ with the properties that $T(2)$ is finer than $T(1)$, that $\tau_{T(2)}$ coincides, on its domain of definition, with τ , and with $W(T(2)) \leq 1/2^2$.

Repeating the argument inductively on the transformation induced on the bases we obtain the theorem.

Definition 2.10. Let $\epsilon > 0$ and let B be a measurable set. We say that a function f is *constant within ϵ on B* if there exists a constant α and a measurable subset E of the set B such that $|f - \alpha| \leq \epsilon$ on $B - E$ and such that $\mu(E) \leq (1 - \epsilon)\mu(B)$. Let $\mathfrak{A} = \{B_1, \dots, B_n\}$ be a class of pairwise disjoint measurable sets. We say that a function f is *simple within ϵ on $\mathfrak{A}$* if it is constant within ϵ on each set B_i , $1 \leq i \leq n$ (note that the n constants are not necessarily equal).

Lemma 2.2. ([5]). *Let $P(r) = \{A_{i,k}(r), 1 \leq j \leq n(r), 1 \leq k \leq m(r, j)\}$ $r = 1, 2, \dots$, be a sequence of partitions such that $P(r)$ is finer than $P(r - 1)$, $r \geq 2$, and such that the Borel field is generated by the sets in $P(r)$, $r \geq 1$. Then if $\epsilon > 0$ and if B is a measurable set then there exists a set G of pairs of indices and*

an integer $r(\epsilon)$ such that B equals $\bigcup_{(j,k) \in G} A_{j,k}(r(\epsilon))$ an error of ϵ , and such that

$$\mu(B \cap A_{j,k}(r(\epsilon))) \geq (1 - \epsilon)\mu(A_{j,k}(r(\epsilon))),$$

for $(j, k) \in G$.

Proof. Readily obtained from standard arguments in measure theory.

Lemma 2.3. *Let $P(r) = \{A_{j,k}(r), 1 \leq j \leq n(r), 1 \leq k \leq m(r, j)\}$ $r=1, 2, \dots$, be a sequence of partitions such that $P(r)$ is finer than $P(r-1)$, $r \geq 2$, and such that the Borel field is generated by the sets in $P(r)$, $r \geq 1$, as in Lemma 1.2. Then if $\epsilon > 0$ and if f is a measurable function then there exists a set G of pairs of indices and an integer $r(\epsilon)$ such that f is simple within ϵ on $\mathfrak{A} = \{A_{j,k}(r(\epsilon)), (j, k) \in G\}$, where $\mu\{\bigcup_{(j,k) \in G} A_{j,k}(r(\epsilon))\} \geq (1 - \epsilon)$.*

Proof. Follows at once from Lemma 2.2.

Lemma 2.4. *Let $\{P(r)\}$ be a sequence of partitions such that for each r , $P(r) = \{A_{j,k}(r), 1 \leq j \leq n(r), 1 \leq k \leq m(r, j)\}$ is a partition of the measurable set $M(r)$ and satisfying the conditions*

(a) *that for each $\delta > 0$ and each set $A_{j,k}(r)$ there exists an index $r^* = r^*(\delta, j, k, r)$ such that $A_{j,k}(r)$ equals the union of certain of the sets in each $P(r)$ satisfying $r \geq r^*$ with an error of $\delta/\mu(A_{j,k}(r))$,*

(b) *that the Borel field is generated by the sets in $P(r)$, $r \geq 1$.*

Then if $\epsilon > 0$ and if f is a measurable function then there exists a set G of pairs of indices and an integer $r(\epsilon)$ such that f is simple within ϵ on $\mathfrak{A} = \{A_{j,k}(r(\epsilon)), (j, k) \in G\}$ where $\mu\{\bigcup_{(j,k) \in G} A_{j,k}(r(\epsilon))\} \geq (1 - \epsilon)$.

Proof. Follows as Lemma 2.3 from a corresponding version of Lemma 2.2.

Lemma 2.5. *If B is a measurable set and T is a measurable tower then there exists a measurable tower $T(1)$ finer than T such that the domains of definition of τ_T and of $\tau_{T(1)}$ are equal.*

Proof. Let $T(1) = S(B, T)$. That it has the required properties follows from Definition 2.6.

3. An example having continuous spectrum. In this section we prove the following theorem.

Theorem 3.1. *There exists a construction of an invertible measure preserving transformation τ which has continuous spectrum and which is not strongly mixing.*

That transformations of this type exist follows from a category theorem of Halmos [6] and from a construction due to Kakutani-von Neumann [7] (as well as from some results in Gaussian processes due to Maruyama). The construction given here is an adaptation of the construction given for one of the lemmas in [4]. Our purpose in giving a proof of Theorem 3.1 is to introduce the ideas which are used to obtain the main results of the paper. We note that the con-

struction is sufficiently general to show that the transformations satisfying the theorem are dense in the strong topology in the anti-periodic transformations. To contrast the constructions, we obtain a transformation which is isomorphic to that of [7].

Construction of the transformation. We say that a *double tower* is a measurable tower of the form $T = \{I_{1,1}, \dots, I_{1,m(1)}; I_{2,1}\}$ where the $I_{j,k}$, $1 \leq j \leq 2$, $1 \leq k \leq m(j)$, are intervals of the same length. Recall that τ_T maps $I_{1,k}$ linearly onto $I_{1,k+1}$, $1 \leq k \leq m(1) - 1$, and that τ_T is undefined on $I_{1,m(1)}$ and on $I_{2,1}$. Each double tower T gives rise to a double tower $D(T)$ as follows: Write $I_{j,k}$ as the sum of two disjoint and consecutive intervals of the same length $I_{j,k}^1, I_{j,k}^2$ and set $D(T) = \{I_{1,1}(1), \dots, I_{1,2m(1)+1}(1), I_{2,1}(1)\}$ where

$$I_{1,k}(1) = \begin{cases} I_{1,k}^1, & 1 \leq k \leq m(1), \\ I_{1,k-m(1)}^2, & m(1) + 1 \leq k \leq 2m(1), \\ I_{2,1}^1, & k = 2m(1) + 1, \end{cases}$$

and where $I_{2,1}(1) = I_{2,1}^2$. In terms of the description of T in terms of columns, T is composed of two columns, the first of height $m(1)$ and the second of height 1, both columns being composed of intervals all of the same length. The tower $D(T)$ is obtained from T as follows: Split the first column into two equal and consecutive subcolumns, and also split the second column, which is composed of a single interval, into two equal and consecutive subintervals. Next place the second subcolumn above the first subcolumn and then the first subinterval above that to form the first column of $D(T)$, leaving the second subinterval to form the second column of $D(T)$. The width of the intervals of $D(T)$ is half the width of the intervals of T .

The transformation τ is defined by setting $\tau = \tau_{\{T(r)\}}$ where $\{T(r)\}$ is defined inductively by putting $T(1) = \{[0, 1/3], [1/3, 2/3]; [2/3, 1]\}$ and $T(r) = D(T(r-1))$, $k \geq 2$.

Properties of the transformation. Write $T(r) = \{I_{1,1}(r), \dots, I_{1,m(r)}(r); I_{2,1}(r)\}$ where $m(1) = 2$ and $m(r) = 2m(r-1) + 1$, $r \geq 2$. Note that $\tau^{m(r)-1}I_{1,1}(r) = I_{1,m(r)}(r)$ and that $\tau I_{1,m(r)}^1(r) = I_{1,1}^2(r)$, implying that $\tau^{m(r)}I_{1,1}^1(r) = I_{1,1}^2(r)$. Note also that $\tau I_{1,m(r)}^2(r) = I_{2,1}^2(r) = I_{1,m(r+1)}(r+1)$ and that $\tau I_{1,m(r+1)}^1(r+1) = I_{1,1}^2(r+1)$ where $I_{1,1}^2(r+1)$ is the last quarter of $I_{1,1}(r)$. This remark, with $\tau^{m(r)-1}I_{1,1}(r) = I_{1,m(r)}(r)$ implies that, if $B(r)$ is the third quarter of $I_{1,1}(r)$, $\tau^{m(r)+1}B(r) \subset I_{1,1}(r)$ (in fact $\tau^{m(r)+1}B(r)$ is the second quarter of $I_{1,1}(r)$). We have therefore shown that $I_{1,1}(r)$ has a subset $A(r) = I_{1,1}^1(r)$, of measure half the measure of $I_{1,1}(r)$, such that

$$(3.1) \quad \tau^{m(r)}A(r) \subset I_{1,1}(r),$$

and that $I_{1,1}(r)$ has a subset $B(r)$, of measure a quarter the measure of $I_{1,1}(r)$ such that

$$(3.2) \quad \tau^{m(r)+1}B(r) \subset I_{1,1}(r),$$

which imply the corresponding statements for $I_{1,j}(r)$, $1 \leq j \leq m(r)$.

To see that τ has continuous spectrum, suppose to the contrary. Then there exists a constant λ and a measurable function $f \neq 0$ such that

$$(3.3) \quad f(\tau x) = \lambda f(x) \quad \text{for almost all } x.$$

Since the sequence $\{T(k), k \geq k(0)\}$ satisfies the conditions of Lemma 1.3, then there exists $\delta(f) > 0$ such that for $\epsilon > 0$ there exist $k(\epsilon)$ and $j(\epsilon)$ such that f is constant within ϵ on $I_{1,j(\epsilon)}(k(\epsilon))$, where the constant $\alpha(\epsilon)$ approximating f on $I_{1,j(\epsilon)}(k(\epsilon))$ satisfies

$$(3.4) \quad |\alpha(\epsilon)| \geq \delta(f) > 0.$$

It follows from (3.1) and (3.2) for $j = j(\epsilon)$ that if $\epsilon > 0$ is sufficiently small then

$$(3.5) \quad |\lambda^{m(k)}\alpha(\epsilon) - \alpha(\epsilon)| \leq 2\epsilon,$$

$$(3.6) \quad |\lambda^{m(k)+1}\alpha(\epsilon) - \alpha(\epsilon)| \leq 2\epsilon.$$

Equations (3.5), (3.6) and (3.4) imply that $\lambda = 1$. Since τ is clearly ergodic, this completes the proof. The argument can be simplified by noting that ergodicity implies that $|f| \equiv 1$, but we will apply the argument later on when this condition is not satisfied.

That τ is not strongly mixing may be seen on noting that

$$\mu(A_{1,1}(r) \cap \tau^{m(k)} A_{1,1}(r)) = \frac{1}{2}\mu(A_{1,1}(r)),$$

$k \geq r$, and that

$$\lim_{r \rightarrow \infty} \mu(A_{1,1}(r)) = 0.$$

4. Change of velocity in discrete flows. As we remarked earlier, we call $\{\tau^n\}$ a discrete flow, where τ is an invertible and measure preserving transformation. Given a measurable set B we shall define an invertible and measure preserving transformation τ_B so that the discrete flow $\{\tau_B^n\}$ may be regarded as the flow $\{\tau^n\}$ stopped by one time unit whenever it reaches the set B .

Definition 4.1. Let $(X(1), \mathfrak{F}(1), \mu(1, \cdot))$ be another copy of $(X, \mathfrak{F}, \mu)$ and let e be the identity mapping of X onto $X(1)$. We define *the measure space enlarged by B* to be the measure space which is the direct sum of the measure space $(X, \mathfrak{F}, \mu)$ and the measure space $(X(1) \cap e(B), \mathfrak{F}(1) \cap e(B), \mu_{e(B)}(1, \cdot))$, where $\mu_{e(B)}(1, \cdot)$ is the measure $\mu(1, \cdot)$ restricted to $\mathfrak{F}(1) \cap e(B)$.

Definition 4.2. The invertible and measure preserving transformation τ_B is defined on the measure space enlarged by B as follows:

$$\tau_B(x) = \begin{cases} \tau(x), & x \notin B, \\ e(x), & x \in B, \\ \tau(e^{-1}(x)), & x \in e(B). \end{cases}$$

It is helpful to think of the measure space enlarged by B to be given on X

with a roof or layer over the set B . We may then regard the transformation τ_B as mapping points not under the roof in the same way as τ , mapping points under the roof to those points directly above on the roof, and finally as mapping points on the roof in the same way that τ maps their pre-images. If we observe the action of τ_B restricted to X we note that whenever a point reaches B it remains in that position for an extra time unit.

Professor S. Kakutani has also shown (unpublished) that if τ has discrete spectrum then there exists a measurable set B such that τ_B has continuous spectrum, and conjectured that the result holds in the general case. The main result of this section is a proof of this conjecture.

Theorem 4.1. *Let τ be an anti-periodic invertible and measure preserving transformation and let A be a measurable set of positive measure. Then A contains a measurable subset B such that $f(\tau_B x) = \lambda f(x)$ a.e., where $f(x) \not\equiv 0$ a.e. on A , implies that $\lambda = 1$.*

Proof. Let $\{\delta(r)\}$ be a monotone sequence of positive numbers less than or equal to one which will be specified later. It follows by Theorem 1.1 (choosing a subsequence) and Lemma 1.5 that there exists a sequence of towers $\{T(r)\} = \{[A_{i,k}(r), 1 \leq j \leq n(r), 1 \leq k \leq m(r, j)]\}$ such that

$$(4.1) \quad \tau = \tau_{(T(r))},$$

$$(4.2) \quad B, \text{ the Borel field, is generated by } \{P(T(r)) \mid r \geq 1\},$$

$$(4.3) \quad A \in \mathfrak{F}(P(T(r))), \quad r \geq 1,$$

$$(4.4) \quad \frac{W(T(r))}{W(T(r-1))} \leq \delta(r).$$

It follows by (4.3) that the sets of the towers are (up to sets of measure zero) either disjoint from, or else contained in, the set A . Denote by $G_i(r)$ the set of indices k for which $A_{i,k}(r)$, $1 \leq k \leq m(r, j)$, is contained in A , and by $\alpha_i(r)$ the number of elements in $G_i(r)$, so that it is the number of sets in the stack $\{A_{i,k}(r), 1 \leq k \leq m(r, j)\}$ which are contained in A . For each integer N it is clear that

$$(4.5) \quad A = \bigcup_{i: \alpha_i(r) \geq N} \bigcup_{k \in G_i(r)} A_{i,k}(r) + E(\eta_1(r)),$$

where $\eta_1(r)$ tends to zero as r tends to infinity. Another way of stating (4.5) is that as r tends to infinity most of A will be found among those stacks having at least N layers contained in the set A . We denote by $B_i(r)$ the $[\alpha_i(r)/2]$ -th set contained in A of the column $\{A_{i,k}(r), 1 \leq k \leq m(r, j)\}$. It is clear from (4.4) that the sequence $\{\delta(r)\}$ can be chosen so that

$$(4.6) \quad B_i(r) = B_i(r) - \bigcup_{i \geq r+1} \bigcup_{1 \leq l \leq n(i)} B_l(i) + E\left\{\frac{\eta_2(r)}{\mu(B_i(r))}\right\},$$

where $\eta_2(r)$ tends to zero as r tends to infinity, since the width $w(T(t))$ equals the measure of the set $\bigcup_{1 \leq l \leq n(t)} B_l(t)$. Let $C_i(r)$ be an arbitrary but fixed subset of $B_i(r) - \bigcup_{t \geq r+1} \bigcup_{1 \leq l \leq n(t)} B_l(t)$ of measure equal to one half the measure of $B_i(r)$. It follows from (4.6) that such a set exists for r sufficiently large. By renumbering we may assume without loss of generality that such a set exists for all r . By an inductive application of Lemma 2.4 and renumbering we see that we may also assume without loss of generality that

$$C_i(r) \in \mathfrak{F}\{P(T(r+1))\},$$

on noting that $\bigcup_{1 \leq l \leq n(t)} B_l(t)$ remains unchanged on applying Lemma 1.4 since it consists of dividing the stacks of the tower into more stacks without further stacking of the layers.

We are now in a position to define the subset B of the set A such that τ_B has continuous spectrum:

$$B = \bigcup_{r \geq 1} \bigcup_{1 \leq l \leq n(r)} C_l(r).$$

Let $f \neq 0$ be such that $f(\tau x) = \lambda f(x)$ a.e. for a constant λ . It follows from Lemma 2.3 applied with $X = A$ and (4.5) that for $\epsilon > 0$ there exist indices $r(\epsilon)$, $j(\epsilon)$, $k(1, \epsilon)$ and $k(2, \epsilon)$ such that

$$(4.7) \quad \begin{aligned} & f \text{ is constant within } \epsilon \text{ on } A_{j(\epsilon), k(1, \epsilon)}(r(\epsilon)) \text{ and} \\ & A_{j(\epsilon), k(2, \epsilon)}(r(\epsilon)), \text{ where } A_{j(\epsilon), k(1, \epsilon)}(r(\epsilon)), A_{j(\epsilon), k(2, \epsilon)}(r(\epsilon)) \\ & \text{are contained in } A, \text{ and} \end{aligned}$$

$$(4.8) \quad \begin{aligned} & A_{j(\epsilon), k(1, \epsilon)}(r(\epsilon)) \text{ is below and} \\ & A_{j(\epsilon), k(2, \epsilon)}(r(\epsilon)) \text{ is above the set } B_j(r(\epsilon)), \end{aligned}$$

since for N sufficiently large about half of those layers in the column which belong to A will be above B_j and about half of those layers in the column which belong to A will lie below B_j . Since $\tau_{T(r)}$ equals τ on the domain of definition of $\tau_{T(r)}$ it follows that

$$(4.9) \quad \tau^{k(2, \epsilon) - k(1, \epsilon) - s} A_{j(\epsilon), k(1, \epsilon)}(r(\epsilon)) = A_{j(\epsilon), k(2, \epsilon) - s}(r(\epsilon)),$$

$s = 0, \dots, k(2, \epsilon) - k(1, \epsilon)$. It follows from (4.9) and the definition of $\tau_{C_j}(r)$ that $A_{j(\epsilon), k(i, \epsilon)}(r(\epsilon))$, $i = 1, 2$, can be written as the disjoint union of two sets of equal measure, that is

$$A_{j(\epsilon), k(i, \epsilon)}(r) = A_{j(\epsilon), k(i, \epsilon)}^1(r) + A_{j(\epsilon), k(i, \epsilon)}^2(r),$$

$i = 1, 2$, such that

$$(4.10) \quad \begin{aligned} & \tau_{C_j(r)}^{k(0) - k(1)} A_{j(\epsilon), k(1, \epsilon)}^1 = A_{j(\epsilon), k(2, \epsilon)}^2, \\ & \tau_{C_j(r)}^{k(0) - k(1) + 1} A_{j(\epsilon), k(1, \epsilon)}^2 = A_{j(\epsilon), k(2, \epsilon)}^1. \end{aligned}$$

To see that (4.10) holds, note that if in the tower $T(r(\epsilon))$ we add the additional layer $e(C_j(r(\epsilon)))$ between $C_j(r(\epsilon))$ and $\tau(C_j(r(\epsilon)))$ for $1 \leq j \leq m(j, r(\epsilon))$, we obtain a tower T^* of the measure space enlarged by $C_j(r(\epsilon))$ such that τ_{T^*}

coincides on its domain of definition with $\tau_{C_i(r(\epsilon))}$. Since on the column

$$\{A_{j(\epsilon),k}(r(\epsilon)), 1 \leq k \leq m(r(\epsilon), j(\epsilon))\}$$

the transformation $\tau_{C_i(r(\epsilon))}$ coincide with

$$\tau[\bigcup_{i=1}^{n(r(\epsilon))} C_i(r(\epsilon))]$$

we see that (4.10) holds with the second transformation in place of the first. This and the fact that we have supposed without loss of generality that $C_i(t) \in \mathfrak{F}\{P(T(t+1))\}$, applied for $t = 1, \dots, r-1$, implies that there exists an integer $q(\epsilon)$ such that

$$(4.11) \quad \begin{aligned} \tau_{[\bigcup_{i=1}^r \bigcup_{j=1}^{n(t)} C_i(t)]}^{q(\epsilon)} A_{j(\epsilon),k(1,\epsilon)}^1(r(\epsilon)) &= A_{j(\epsilon),k(2,\epsilon)}^1(r(\epsilon)), \\ \tau_{[\bigcup_{i=1}^r \bigcup_{j=1}^{n(t)} C_i(t)]}^{q(\epsilon)+1} A_{j(\epsilon),k(1,\epsilon)}^2(r(\epsilon)) &= A_{j(\epsilon),k(2,\epsilon)}^2(r(\epsilon)), \end{aligned}$$

where, if we let $n(P(T(r)))$ be the number of elements in the partition $P(T(r))$, $0 \leq q(\epsilon) \leq \sum_{t=1}^r n(P(T(r)))$. To see that (4.11) holds, note that if in the tower T^* we add the layers $e(C_i(t))$, $1 \leq t \leq r(\epsilon)$, between $C_i(t)$ and $\tau_{C_i(r(\epsilon))} C_i(t)$, $1 \leq j \leq m(t, j)$, we obtain a tower T^{**} on the measure space enlarged by $\bigcup_{t=1}^r \bigcup_{j=1}^{n(t)} C_i(t)$; the fact that $C_i(t) \in \mathfrak{F}\{P(T(r(\epsilon)))\}$ means that these additional layers do not divide the columns and consequently the columns of T^* merely acquire more layers to yield the columns of T^{**} and thus we see that (4.11) holds on noting that on the domain of definition of $\tau_{T^{**}}$,

$$\tau_{T^{**}} = \tau_{[\bigcup_{i=1}^r \bigcup_{j=1}^{n(t)} C_i(t)]}.$$

We may further suppose without loss of generality that the sequence $\{T(r)\}$ and the sets $\{C_i(r)\}$ are chosen so that

$$(4.12) \quad \tau_B^s A_{j,k}(r) = \tau_{[\bigcup_{i=1}^r \bigcup_{j=1}^{n(t)} C_i(t)]}^s A_{j,k}(r) + E \left\{ \frac{\eta_3(r)}{\mu(A_{j,k}(r))} \right\},$$

where $\eta_3(r)$ tends to zero uniformly in $s = 0, 1, \dots, \sum_{t=1}^r n(P(T(t)))$ as r tends to infinity. To see this it is enough to note that the measure of the set $B - \bigcup_{i=1}^r \bigcup_{j=1}^{n(t)} C_i(t)$ tends to zero, for fixed $T(1), \dots, T(r)$ as $\delta(r+1)$ tends to zero and to note that $\sum_{t=1}^r n(P(T(t)))$ does not depend on $\delta(r+1)$ since it depends only on $T(1), \dots, T(r)$. Equations (4.11) and (4.12) imply that

$$(4.13) \quad \begin{aligned} \tau_B^{q(\epsilon)} A_{j(\epsilon),k(1,\epsilon)}^1(r(\epsilon)) &= A_{j(\epsilon),k(1,\epsilon)}^1(r(\epsilon)) + E \left\{ \frac{\eta_4(r)}{\mu(A_{j(\epsilon),k(1,\epsilon)}^1(r(\epsilon)))} \right\}, \\ \tau^{q(\epsilon)+1} A_{j(\epsilon),k(2,\epsilon)}^2(r(\epsilon)) &= A_{j(\epsilon),k(2,\epsilon)}^2(r(\epsilon)) + E \left\{ \frac{\eta_4(r)}{\mu(A_{j(\epsilon),k(2,\epsilon)}^2(r(\epsilon)))} \right\}, \end{aligned}$$

where $\eta_i(r)$ tends to zero uniformly in $q(\epsilon)$ as r tends to infinity.

Condition (4.7) implies that there exist two constants $\alpha(1, \epsilon)$ and $\alpha(2, \epsilon)$ which approximate f within ϵ on the sets $A_{j(\epsilon),k(1,\epsilon)}(r(\epsilon))$ and $A_{j(\epsilon),k(2,\epsilon)}(r(\epsilon))$

respectively. We may furthermore assume without loss of generality that $|f| \geq \delta > 0$ a.e. on A for some positive number δ (by considering the subset of A where the condition holds). This together with (4.13) implies that for $\epsilon > 0$ sufficiently small

$$\begin{aligned} |\lambda^q \alpha(1) - \alpha(2)| &\leq 2\epsilon, \\ |\lambda^{q+1} \alpha(1) - \alpha(2)| &\leq 2\epsilon. \end{aligned}$$

Since λ and δ do not depend on ϵ , this implies that $\lambda = 1$.

5. Change of velocity in flows. In this section we obtain a result which implies that given an arbitrary ergodic measurable flow $\{\tau t, -\infty < t < +\infty\}$ it is possible to change its velocity and the measure of the space to an invariant measure equivalent to the original measure in such a way that the resulting spectrum is continuous. A change in velocity will, in general, yield a group of measurable transformations under which the original measure is not invariant, and consequently it is necessary to accompany the change of velocity with a change of the measure.

For a general class of measure spaces including $(X, \mathfrak{F}, \mu)$, (the unit interval, the Lebesgue sets, and Lebesgue measure, respectively) Ambrose and Kakutani have shown in [3] that every flow different from the identity on the whole space is isomorphic to a generalized flow built under a function (see Definition 5.5). This is the result we need to construct the continuous towers which have the same function in the argument as the towers of the discrete case.

We follow [3] in giving some of the following definitions. The various measure spaces which are considered are understood to be complete.

Definition 5.1. A flow is a one parameter group $\{\tau_t, -\infty < t < +\infty\}$ of measure preserving transformations of X onto X . A set M is *invariant under the flow* if for each $t, -\infty < t < +\infty$, and x in M , the point $\tau_t x$ is in M . A flow is *ergodic* if there exist no invariant sets except null sets and their complements.

Definition 5.2. A flow $\{\tau_t, -\infty < t < +\infty\}$ is measurable if for any measurable set M the (x, t) -set defined by $\{(x, t): \tau_t x \in M\}$ is measurable in the product space $(X, \mathfrak{F}, \mu) \otimes ((-\infty, +\infty), \mathfrak{M}, \beta)$ where β is the Lebesgue measure defined on the Lebesgue sets $\mathfrak{M}$ of the real line.

Definition 5.3. Let $\{\tau_t(1, \cdot)\}$ and $\{\tau_t(2, \cdot)\}$ be two flows, respectively, on $(X(1), \mathfrak{F}(1), \mu(1, \cdot))$ and $(X(2), \mathfrak{F}(2), \mu(2, \cdot))$. We say that $\{\tau_t(1)\}$ and $\{\tau_t(2)\}$ are *isomorphic* if it is possible to split $X(i)$ into two parts $N(i)$, $X(i) - N(i)$ in such a way that $N(i)$ is a null set invariant under $\{\tau_t(i)\}$, and if it is possible to find a measure preserving transformation of $X(1) - N(1)$ onto $X(2) - N(2)$ which carries $\{\tau_t(1)\}$ onto $\{\tau_t(2)\}$.

Definition 5.4. Let $(X_1, \mathfrak{F}_1, \mu_1)$ be an interval, the Lebesgue sets and the Lebesgue measure, respectively. Let τ be a (invertible) measure preserving transformation of X_1 onto X_1 , and let $f(x_1), x_1 \in X_1$ be a positive and integrable

function such that

$$\sum_{n=0}^{\infty} f(\tau^n x_1) = \sum_{n=0}^{\infty} f(\tau^{-n} x_1) = +\infty,$$

for almost all $x_1 \in X_1$ (note that x_1 is a generic point of X_1). Let $(X, \mathfrak{F}, \mu)$ admit of a representation in the following way: μ is the measure of the product space $(x_1, \mathfrak{F}_1, \mu_1) \otimes ((-\infty, +\infty), \mathfrak{M}, \beta)$ (where $\mathfrak{M}, \beta$ are as in definition 5.2) restricted to the class $\mathfrak{F}$ of measurable subsets of X where X is the ordinate set of f , i.e. the set of points $x = (x_1, u)$ for which $0 \leq u < f(x_1)$ (recall that x_1 stands for a generic point of X_1). Define the flow $\{\tau_t\}$ on $(X, \mathfrak{F}, \mu)$ by setting

$$\tau_t(x_1, u) = \begin{cases} (x_1, u+t), & \text{if } -u \leq t < -u + f(x_1), \\ (\tau^n x_1, u+t-f(x_1)-\cdots-f(\tau^{n-1}x_1)), & \\ & \text{if } -u + \sum_{k=0}^{n-1} f(\tau^k x_1) \leq t < -u + \sum_{k=0}^n f(\tau^k x_1), \\ (\tau^{-n} x_1, u+t+f(\tau^{-1}x_1)+\cdots+f(\tau^{-n}x_1)), & \\ & \text{if } -u - \sum_{k=1}^n f(\tau^{-k}x_1) \leq t < -u - \sum_{k=1}^{n-1} f(\tau^{-k}x_1). \end{cases}$$

We call the flow $\{\tau_t\}$ the flow built under the function f on τ . X_1 is called the base space, τ the base transformation and f the ceiling function.

Definition 5.5. Let $\{\tau_t(r)\}$ be a flow on $(x(r), \mathfrak{F}(r), \mu(r, \cdot))$ built under the function f_r with base space $(x_1(r), \mathfrak{F}_1(r), \mu_1(r, \cdot))$, and base transformation τ_r , $r \geq 1$. Assume that $\sum_{r=1}^{\infty} \mu(r, X(r)) < +\infty$ and put $X = \bigoplus X(r)$, $X_1 = \bigoplus X_1(r)$, $f = \bigoplus f_r$ and $\tau = \bigoplus \tau_r$, where the direct sum of two functions (or transformations) is defined on the direct sum of the domain of definition of each function to equal each function (or transformation) on its own domain. We call the flow $\tau_t = \bigoplus \tau_t(r)$ a generalized flow built under the function f on τ . X_1 is called the base space, τ the base transformation and f the ceiling function.

Definition 5.6. Let $(X_1, \mathfrak{F}_1, \mu_1)$ be a base space and f a ceiling function. We call the pair $\{(x_1, \mathfrak{F}_1, \mu_1), f\}$ a continuous tower T . The (partial) flow $\{\tau_t(T)\}$ induced by the continuous tower T on $(X, \mathfrak{F}, \mu)$ ($(X, \mathfrak{F}, \mu)$ as in definition 5.4) is defined by letting $\tau_t(T, x)$, where $x = (x_1, u)$, be given by

$$\tau_t(T, (x_1, u)) = \begin{cases} (x_1, u+t) & \text{if } 0 \leq u+t < f(u) \\ \text{undefined otherwise,} \end{cases}$$

so that the domain of definition (and the range) depend on t . The number $\mu_1(X_1)$ is the width of the tower T .

Definition 5.7. A flow $\{\tau_t\}$ is proper if every set of positive measure contains a measurable set M such that the measure of $(X - M) \cap \tau_{t(0)}(M)$ is positive for some $t(0)$, $-\infty < t(0) < +\infty$. It is completely improper if every measurable set is equivalent to an invariant set.

Lemma 5.1 ([3]). *Let $\{\tau_t\}$ be a measurable flow. Then $X = X_1 + X_2$ where X_1 and X_2 are disjoint measurable sets and where $\{\tau_t\}$ is proper on X_1 and completely improper on X_2 .*

Lemma 5.2 ([3]). *If $\{\tau_t\}$ is a completely improper flow then there exists an invariant null set N such that $\{\tau_t\}$ is the identity on $X - N$.*

Definition 5.8. A flow $\{\tau_t\}$ is *anti-periodic* if there exists an invariant null set N so that for $x \in X - N$, $x \neq \tau_t x$, $-\infty < t < +\infty$.

Lemma 5.3 ([3]). *A measurable anti-periodic flow is isomorphic to a generalized flow built under a function if and only if it is proper.*

(In [3] the result is obtained for not necessarily anti-periodic flows on general base spaces $(X_1(n), \mathfrak{F}_1(n), \mu_1(n, \cdot))$; that they can be taken to be an interval, the Lebesgue sets and Lebesgue measure, respectively, follows from the fact that they are countably generated, complete and finite measure spaces which are also non-atomic, where the last condition is what is implied by the anti-periodicity.)

Definition 5.9. If $\{\tau_t(1, \cdot)\}$ and $\{\tau_t(2, \cdot)\}$ are flows on $(X, \mathfrak{F}, \mu(1, \cdot))$ and $(X, \mathfrak{F}, \mu(2, \cdot))$, respectively, with the property that for each x the paths $\tau_t(1, x)$ and $\tau_t(2, x)$ are the same, both in the sense of being the same point set and of having these points in the same order for $-\infty < t < +\infty$, then we say that *the flows are obtained from each other by a change of velocity*. Note that $\mu(1, \cdot)$ is invariant with respect to $\tau_t(1, \cdot)$ and that $\mu(2, \cdot)$ is invariant with respect to $\tau_t(2, \cdot)$.

Definition 5.10. A function f is L_2 is an *eigenfunction of the flow $\{\tau_t\}$* if there exists a constant λ , called the *eigenvalue corresponding to f* , such that

$$f(\tau_t x) = \lambda^t f(x), \quad \text{almost everywhere,}$$

for each t , $-\infty < t < +\infty$.

Definition 5.11. Let $(X_1, \mathfrak{F}_1, \mu_1)$ be a base space with ceiling function f . If $a > 0$ and $P = \{B_1, \dots, B_n\}$ is a partition of X_1 , the simple tower S of step a generated by P and f is defined as $S = \{A_{i,k}, 1 \leq j \leq n; 1 \leq k \leq m(j)\}$, where $A_{i,k} = B_i x[a(k-1), ak)$ and where $m(j)$ is the greatest integer l such that $al < f(x)$ for almost all $x \in B_j$. The (partial) flow $\{\tau_t(S)\}$ induced by a simple tower S is the (partial) flow induced by the (continuous) tower $\{(X_1, \mathfrak{F}_1, \mu_1), g\}$ where g is the function defined by $g(x) = m(j)a$ for $x \in B_j$. Note that the ordinate set of g does not equal X but rather is contained in X so that we are in fact using an obvious extension of definition 5.6. The flow induced by the tower $\{(X_1, \mathfrak{F}_1, \mu_1), f\}$ is an extension of the flow induced by the tower $\{(X_1, \mathfrak{F}_1, \mu_1), g\}$.

The main result is the following theorem, which implies the result on ergodic measurable flows mentioned at the beginning of this section.

Theorem 5.1 *If $\{\tau_t\}$ is an anti-periodic measurable flow on $(X, \mathfrak{F}, \mu)$, then there exists a measure $\mu(1, \cdot)$ and an anti-periodic measurable flow $\{\sigma_t\}$ on $(X, \mathfrak{F}, \mu(1, \cdot))$ such that*

$$(5.1) \quad \mu(1, \cdot) \text{ is equivalent to } \mu,$$

$$(5.2) \quad \{\sigma_t\} \text{ is obtained from } \{\tau_t\} \text{ by a change of velocity,}$$

$$(5.3) \quad \{\sigma_t\} \text{ has the constant one as its sole eigenvalue.}$$

Proof. We note first that we may assume without loss of generality that the flow is a generalized flow built under a function, by Lemmas 5.1, 5.2 and 5.3. We may and do further assume without loss of generality that the flow is a flow built under a function, by considering the set where $f \neq 0$ intersected with each of the spaces of the component flows under a function whose direct sum is the generalized flow under a function.

Let the base space be $(X_1, \mathfrak{F}_1, \mu_1)$, the ceiling function f and the base transformation τ . It is of course clear that τ is itself anti-periodic. If we apply Lemma 1.1 inductively first to τ and then to the base transformation of the resulting base space, we obtain a sequence $\{(X_1(r), \mathfrak{F}_1(r), \mu_1(r, \cdot)), f_r, \tau_r\}$ of base spaces, ceiling functions and base transformations, respectively, such that $X_1(r) \subset X_1(r-1)$, $r \geq 1$ and such that the flow $\{\tau_t^{(r)}\}$ built under f_r with base space $(X_1(r), \mathfrak{F}_1(r), \mu_1)$ on τ_r , is, for each r , equal to the flow $\{\tau_t\}$ (where the correspondence between the ordinate sets of $f_{r(1)}$ and $f_{r(2)}$, $r(1) \leq r(2)$ is given by matching the points with the same t along the paths $\tau_t^{(r(1))}x$ and $\tau_t^{(r(2))}x$, $x \in X_1(r(1))$). It follows from Lemma 1.1 that $\mu_1(r, X_1(r)) \leq 1/2^r$ and therefore if we consider the (partial) flow $\{\tau_t(r, \cdot)\}$ induced by the continuous tower $\{(X_1(r), \mathfrak{F}_1(r), \mu_1(r, \cdot)), f_r\}$ it follows that

$$\tau = \lim_{r \rightarrow \infty} \tau_t(r)$$

where the equation is valid on $X - N$ where N is a null invariant set where N depends on $\{\tau_r, r \geq 1\}$, so that we may assume without loss of generality that it holds on X .

Choose a number $a(1) > 0$ and a partition $P_1(1) = \{B_1(1), \dots, B_{n(1)}(1)\}$ of X_1 and let $S(1) = \{A_{j,k}(1), 1 \leq j \leq n(1); 1 \leq k \leq m(1, j)\}$ be the simple tower of step $a(1)$ generated by $P_1(1)$ and f_1 . In each stack $\{A_{j,k}(1), 1 \leq k \leq m(1, j)\}$ pick out the sets $A_{j,k}(1), [m(1, j)/3] < k \leq [(1/a(1)) + [m(1, j)/3]]$ and recall that the set $A_{j,k}(1)$ is the rectangle $B_j(1) \times [a(1)(k-1), a(1)k]$ and that the flow $\{\tau_t\}$ maps the points of these sets upward with velocity equal to one. Let $C_j(1)$ be a subset of B_j with $\mu_1(1, \cdot)$ -measure half the $\mu_1(1, \cdot)$ measure of B_j and change the velocity of the points of $C_{j,k}(1), [m(1, j)/3] < k \leq [(1/a(1)) + [m(1, j)/3]]$ to twice their former velocity, and change the measure of the product space so that it is invariant, and denote the new flow by $\{\tau_t(1)\}$. If we do not normalize the measure space, $\tau_t(1)$ is isomorphic to the flow with base space $(X_1(1), \mathfrak{F}_1(1), \mu_1(1, \cdot))$, ceiling function f_{11} , and base transformation

τ_1 , where $f_{11}(x) = f_1(x) - a(1)(\lfloor(1/a(1)) + [m(1, j)/3]\rfloor - \lfloor m(1, j)/3 \rfloor)$, and where the ordinate set of f_{11} is identified under the isomorphism with X , which is the ordinate set of f_1 , as follows: Consider the ordinate set of f_1 and keeping the relative position of the points compress the rectangles of the ordinate set of the form $C_{i,k}(1)$, $\lfloor m(1, j)/3 \rfloor < k \leq \lfloor (1/a(1)) + [m(1, j)/3] \rfloor$ linearly along the vertical direction so that their height is halved, so that we replace the band composed of those sets by one half as wide and then drop the points above rigidly so that their lower edge rests on the new band. This gives the ordinate set of f_{11} and the desired identification. This change from f_1 to f_{11} induces a change of the ceiling function f_r to the ceiling function $f_{r,1}$ so that the flow with base space $(X_1(r), \mathfrak{F}_1(r), \mu_1(r, \cdot))$, ceiling function $f_{r,1}$ and base transformation $\{\tau_r\}$ is isomorphic to $\{\tau_t(1)\}$. Next we choose $a(2)$ and a partition $P_1(2) = \{B_1(2), \dots, B_{n(2)}(2)\}$ of $X(2)$ and let $S(2) = \{A_{i,k}(2), 1 \leq j \leq n(2); 1 \leq k \leq m(2, j)\}$ be the simple tower generated by $P_1(2)$ and $f_{2,1}$. In each stack $\{A_{i,k}(2), 1 \leq k \leq m(2, j)\}$ choose the sets $A_{i,k}(2)$, $\lfloor m(2, j)/3 \rfloor < k \leq \lfloor (1/a(2)) + [m(2, j)/3] \rfloor$ where $A_{i,k}(2) = B_i(2) \times [a(2)(k-1), a(2)k]$. Let $C_i(2)$ be a subset of $B_i(2)$ of $\mu_1(2, \cdot)$ -measure half the $\mu_1(2, \cdot)$ -measure of $B_i(2)$, let $C_{i,k}(2) = C_i(2) \times [a(2)(k-1), a(2)k]$ and change the velocity of the points of $C_{i,k}(2)$, $\lfloor m(2, j)/3 \rfloor < k \leq \lfloor (1/a(2)) + [m(2, j)/3] \rfloor$ to twice their former velocity (under the flow $\{\tau_t(1)\}$). We obtain a flow $\{\tau_t(2)\}$ which is the flow of $\{(X_1(r), \mathfrak{F}_1(r), \mu_1(r, \cdot)), f_{r,2}, \tau_r\}$ for each $r \geq 2$. Proceeding inductively we obtain a sequence of flows $\{\tau_t(n)\}$, each the flow of $\{(X_1(r), \mathfrak{F}_1(r), \mu_1(r, \cdot)), f_{r,n}, \tau_r\}$ for each $r \geq n$. In particular, $\{\tau_t(n)\}$ is the flow of $\{(X_1(n), \mathfrak{F}_1(n), \mu_1(n, \cdot)), f_{n,n}, \tau_n\}$. It is clear that we may choose the base spaces in such a way that $\mu_1(n, X_1(n))$ tends to zero sufficiently rapidly that

$$\lim_{n \rightarrow \infty} \tau_t(n) = \sigma_t$$

exists and is a flow with invariant measure equivalent to μ , since at the r th step the product measure at that step is changed on a set of product measure at that step bounded by $\mu_1(r, X_1(r))$ so that each of its subsets has measure half the previous measure. We may also suppose that (i) this occurs sufficiently rapidly that each column $\{A_{i,k}(r), 1 \leq k \leq m(r, j)\}$ has a subcolumn $\{D_{i,k}(r), 1 \leq k \leq m(r, j)\}$, where $D_{i,k}(r) = D_i(r) \times [a(r)(\lfloor m(r, j)/2 \rfloor - 1), a(r)m(r, j)]$, $D_i(r) \subset B_i(r)$, and $\mu(D_i(r)) \geq 2\mu(B_i(r))/3$, which is unaffected by changes in the velocity which occur in steps with index greater than r , and that (ii) the sets $\{A_{i,k}(r), 1 \leq j \leq n(r), 1 \leq k \leq m(r, j)\}$ satisfy conditions (a) and (b) of Lemma 2.4, and that (iii) for each N , $\mu(\bigcup_{j: a(r)m(r, j) \geq N} \bigcup_{1 \leq k \leq m(r, j)} A_{i,k}(r))$ tends to one as r tends to infinity.

Suppose that the flow $\{\sigma_t\}$ admits of an eigenfunction f so that

$$(5.4) \quad f(\sigma_t x) = \lambda^t f(x) \text{ almost all } x.$$

It follows from Lemma 1.4 and (iii) that there is a $\delta > 0$ such that for each $\epsilon > 0$ and N there exist indices $r(\epsilon)$, $j(\epsilon)$, $k(1, \epsilon)$ and $k(2, \epsilon)$ such that f is constant within ϵ on $A_{j(\epsilon), k(1, \epsilon)}(r(\epsilon))$ and on $A_{j(\epsilon), k(2, \epsilon)}(r(\epsilon))$ with

$$k(1, \epsilon) < [m(r(\epsilon), j(\epsilon))/3], \quad k(2, \epsilon) > [(1/a(r(\epsilon)) + [m(r(\epsilon), j(\epsilon))/3])]$$

and where, if $\alpha(1)$ is the constant approximating f on $A_{j(\epsilon), k(1, \epsilon)}$ and $\alpha(2)$ is the constant approximating f on $A_{j(\epsilon), k(2, \epsilon)}$, we have $|\alpha(1)| \geq \delta$ and $|\alpha(2)| \geq \delta$, and such that $m(r(\epsilon), j(\epsilon)) \geq N$. If we let $\{D_{j(\epsilon), k}(r(\epsilon)), 1 \leq k \leq m(r(\epsilon), j(\epsilon))\}$ be the subcolumn which is unaffected by changes in the velocity in steps of index greater than or equal to $r(\epsilon)$, we note that, since $\sigma_{a(r(\epsilon))} = \tau_{a(r(\epsilon))}(r(\epsilon) + 1)$ on $D_{j(\epsilon), k}$, $1 \leq k \leq m(r(\epsilon), j(\epsilon))$, that a subset of $A_{j(\epsilon), k(1, \epsilon)}(r(\epsilon))$, of measure at least $1/6$ -th its measure, arrives at the set $A_{j(\epsilon), k(1, \epsilon)}(r(\epsilon))$ after $k(2, \epsilon) - k(1, \epsilon)$ applications of $\tau_{a(r(\epsilon))}(r(\epsilon) + 1)$, and that another subset of $A_{j(\epsilon), k(1, \epsilon)}(r(\epsilon))$, of measure at least $1/12$ -th its measure, arrives at the set $A_{j(\epsilon), k(1, \epsilon)}(r(\epsilon))$ after $k(2, \epsilon) - k(1, \epsilon) - \{[1/a(r(\epsilon)) + [m(r(\epsilon), j(\epsilon))/3]] - [m(r(\epsilon), j(\epsilon))/3]\}/2\}$ applications of $\tau_{a(r(\epsilon))}(r(\epsilon) + 1)$. It is clear that there is no loss of generality in supposing that $a(r)$ is of the form $1/b(r)$ where $b(r)$ is an integer. It is also clear that we may suppose without loss of generality that

$$|a(r(\epsilon))\{[1/a(r(\epsilon))] + [m(r(\epsilon), j(\epsilon))/3] - [m(r(\epsilon), j(\epsilon))/3]\}/2 - \frac{1}{2}\} \leq \epsilon,$$

and consequently we have that, for $\epsilon > 0$ sufficiently small,

$$(5.5) \quad |\lambda^{a(r(\epsilon))(k(2, \epsilon) - k(1, \epsilon))} \alpha(1) - \alpha(2)| \leq 2\epsilon,$$

and that

$$(5.6) \quad |\lambda^{a(r(\epsilon))(k(2, \epsilon) - k(1, \epsilon)) - 1/2 + \eta} \alpha(1) - \alpha(2)| \leq 2\epsilon,$$

where $|\eta| \leq \epsilon$, and $|\alpha(i)| \geq \delta$, where $\delta > 0$ is independent of ϵ . It follows that (5.5) and (5.6) imply that $\lambda = 1$.

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