Transformations Having Continuous Spectrum

R. V. CHACON*

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The square root problem can be completely settled for transformations having discrete spectrum with the aid of the discrete spectrum theorem. One of the important problems in ergodic theory is to investigate to what extent the implications of the discrete spectrum theorem remain in the general case. Along these lines, Halmos [2] has asked whether every transformation with continuous spectrum has a square root.

The purpose of this paper is to give a negative answer to the question. Indeed, the counterexample which is constructed does more than is required of it. It has no square roots in the class of non-singular transformations. The transformation also satisfies the additional condition of not being strongly mixing. The latter additional condition is of some interest in view of the fact that the shift transformations are strongly mixing, and is obtained by combining the ideas of [1] with those which are needed for the square root problem by itself. We remark that this does not lengthen the proof appreciably since the constructions are compatible.

The methods given can be applied also to obtain certain other results. We mention in particular the n -th root problem and various topological approximation problems. We intend to carry out these applications in a later paper.

Let (X, F, μ) be an interval, the Lebesgue sets and Lebesgue measure, respectively. Since we are interested in equivalence classes, modulo sets of measure zero, of transformations of X onto X , our equations are understood to hold almost everywhere. In particular, transformations need be defined only almost everywhere. The (invertible) transformation τ of X onto X is measurable if $A \in F$ implies $\tau(A) \in F$ and $\tau^{-1}(A) \in F$. A transformation τ of X onto X is non-singular if it is measurable and if $A \in F$ and $\mu(A) = 0$ implies that $\mu(\tau(A)) = \mu(\tau^{-1}(A)) = 0$. A transformation τ is measure preserving if it is measurable and if $A \in F$ implies $\mu(A) = \mu(\tau(A)) = \mu(\tau^{-1}(A))$. Measure preserving transformations τ induce unitary operators T_τ by setting $T_\tau f(x) = f(\tau^{-1}(x))$ for $f \in L_2$. The transforma-

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tion τ is said to have discrete spectrum if it is measure preserving and if its induced unitary operator has a complete orthonormal set of eigenfunctions. The transformation τ is said to have continuous spectrum if it is measure preserving and if its induced unitary operator has for its sole eigenfunction the function identically equal to one almost everywhere.

Our result may be stated as follows:

Theorem. *There exists an explicit construction of an invertible and measure preserving transformation τ of the unit interval such that i) τ has continuous spectrum, ii) τ is not strongly mixing and iii) τ has no non-singular square roots.*

(It is not hard to see that a non-singular square root of an invertible, ergodic and measure preserving transformation is also measure preserving; we do not use this fact in the sequel.)

1. Definition of the transformation. Given a sequence $\{j(k), k = 1, 2, \dots\}$ of integers with $j(k) \geq 1, k \geq 1$, we shall define an invertible measure preserving transformation $\tau = \tau(\{j(k)\})$ having the desired properties. For convenience the transformation τ is defined on the union of the unit interval with another interval N disjoint from it; the length of N depends on the sequence $\{j(k)\}$.

The transformation is defined inductively. For $r = k$, we suppose that the transformation is defined on part of the space and we extend the domain of definition for $r = k + 1$.

For $r = k$, we suppose the transformation given as follows: Let

$$m(k) = 3m(k-1)j(k) + 4, \quad m(0) = 2,$$

and suppose that there exist $m(k)$ intervals of equal length $A_1(k), \dots, A_{m(k)}(k)$, each a subinterval of the unit interval or of N , and that τ maps $A_{i-1}(k)$ linearly onto $A_i(k)$, $i = 2, \dots, m(k)$.

For $r = k + 1$, we need to extend the definition of τ in such a way that it admits a representation as in the preceding paragraph, with k replaced by $k + 1$. This is accomplished in two steps.

The first step is the following. Divide $A_i(k)$, ($1 \leq i \leq m(k)$), into $j(k+1)$ consecutive subintervals of equal length denoting them

$$B_{i,1}(k+1), \dots, B_{i,j(k+1)}(k+1), \quad (1 \leq i \leq m(k)).$$

Since $A_{i-1}(k)$ is mapped linearly onto $A_i(k)$, ($2 \leq i \leq m(k)$), τ also maps $B_{i-1,j}(k+1)$ linearly onto $B_{i,j}(k+1)$, ($2 \leq i \leq m(k), 1 \leq j \leq j(k+1)$). In this first step we extend τ by mapping $B_{m(k),j}(k+1)$ linearly onto $B_{1,j+1}(k+1)$, ($1 \leq j \leq j(k+1) - 1$). There are $m(k)$ $A_i(k)$'s and each $A_i(k)$ is divided into $j(k+1)$ $B_{i,j}(k+1)$'s so that there are $m(k) \cdot j(k+1)$ $B_{i,j}(k+1)$'s altogether. We write these sets with a single subscript $B_i(k+1)$, ($i = 1, \dots, m(k) \cdot j(k+1)$), in such a way that τ as thus far extended maps $B_{i-1}(k+1)$ linearly onto $B_i(k+1)$, ($2 \leq i \leq m(k)j(k+1)$).

The second step in the extension of τ is obtained by dividing $B_i(k+1)$ into three consecutive subintervals of equal length $B_i^1(k+1), B_i^2(k+1), B_i^3(k+1)$.

The transformation τ as already defined maps $B_{i-1}^u(k+1)$ linearly onto $B_i^u(k+1)$, ($2 \leq i \leq m(k)j(k+1)$, $1 \leq u \leq 3$). We take from that subinterval of N where τ is as yet undefined four consecutive subintervals of equal length, equal to the length of the $B_i^u(k+1)$, and denote these four subintervals of N by $E_i(k+1)$, ($1 \leq j \leq 4$), (we suppose that N is just big enough so that at each stage there is an interval left over). Then extend τ so that the sequence of $3m(k)j(k+1) + 4$ sets (with $(k+1)$ deleted for simplicity), with $s = m(k)j(k+1)$,

$$E_1, B_1^1, \dots, B_s^1, E_2, E_3, B_1^2, \dots, B_s^2, E_4, B_1^3, \dots, B_s^3$$

has the property that τ maps each set, except for the last, linearly onto the next set. We denote this sequence by $A_i(k+1)$, ($1 \leq i \leq m(k+1)$), where $m(k+1) = 3m(k)j(k+1) + 4$.

To complete the induction it remains to define τ for $r = 0$. We set $A_1(0) = (0, \frac{1}{2})$ and $A_2(0) = (\frac{1}{2}, 1)$ and define τ initially as the linear map of $A_1(0)$ onto $A_2(0)$. An intuitive description of the transformation is the following.

At the k th stage there is a stack of $m(k)$ intervals of equal length and τ maps each point to the one directly above, so that the points of the top interval are not yet mapped anywhere.

The first step of the extension of τ is obtained as follows. The original stack is split into $j(k+1)$ equal and consecutive substacks, and these are stacked in order. This has the effect of mapping a $j(k+1) - 1/j(k+1)$ part of the top interval of the original stack into the bottom interval of the original stack (the symmetric difference of the image under τ of the top interval and the bottom interval has measure which tends to zero as $j(k+1)$ tends to infinity, therefore).

In the second step of the extension, we take the resulting stack, which is clearly composed of $m(k)j(k+1)$ intervals, and divide it into three consecutive and equal substacks. We then take four consecutive and equal subintervals of N where τ is as yet undefined, of length equal to the length of each of the intervals of the three stacks, and put one under the first stack, two under the second stack, and one under the third stack. The thus modified three stacks are then stacked in order.

Note that if we regard the union of the first interval of each of the modified stacks as a set, then this set is mapped upward until it gets to the top, which is one third covered by a further subinterval. At the next step at least one third returns to the bottom, and the part covered by the further subinterval returns to the bottom in two steps. We could have obtained the same effect by putting an extra interval instead of two under the second stack, and no extra intervals under the first and third stacks. The reasons for the more complicated procedure will become clear.

2. Properties of the transformation. The transformation is defined inductively as we have seen. In what follows we do not suppose that the measure space is normalized. We deal with statements depending on $\{j(k)\}$ and on r

and usually concerning the transformation $\tau = \tau(\{j(k)\})$ of the following sort: The statement will be shown to be true for $r = 0$ and the truth of the statement for $r = 0, \dots, k-1$, with a certain choice of $j(1), \dots, j(k-1)$, will imply that there exists a $K = K(j(1), \dots, j(k-1))$ such that if $j(k) \geq K$ then the statement is true for $r = k$ with that choice of $j(1), \dots, j(k)$. This clearly means that there exists a sequence $\{j(k)\}$ such that the statement is true for all r . It is furthermore clear that if there is a finite number of such statements then there is a single sequence $\{j(k)\}$ such that all the statements will be true for all r . We shall refer to this state of affairs by saying that a certain statement holds for all r if $\{j(k)\}$ tends to infinity sufficiently rapidly. It is clear from the discussion that if a finite number of statements hold for separate $\{j(k)\}$ tending to infinity sufficiently rapidly, then there exists a single sequence $\{j(k)\}$ tending to infinity sufficiently rapidly so that all the statements are true for all r .

For the sake of simplicity, we say that the set A equals the set B with an error of δ if the measure of the symmetric difference of A and N is no more than δ , and we write $A = B + E(\delta)$. We note that we have used the symbol Σ interchangeably with $\cup$ in denoting unions of sets.

Lemma 1. *Let $\{E_{i,k}, 1 \leq j \leq r(k), 1 \leq k < +\infty\}$ generate the Borel field and suppose that for each $k, 1 \leq k < +\infty$, the sets $\{E_{i,k}, 1 \leq j \leq r(k)\}$ are pairwise disjoint. Suppose also that each $E_{i,k}$ is the union of a finite number of sets from the class $\{E_{i,k+1}, 1 \leq j \leq r(k+1)\}$. Then for each $\epsilon > 0$ and A in the Borel field there exists a set of indices $G(\epsilon, A)$ and an integer $k(\epsilon, A)$ such that A equals $\bigcup_{j \in G(\epsilon, A)} E_{j, k(\epsilon, A)}$ with an error of ϵ , and*

$$\mu(A \cap E_{j, k(\epsilon, A)}) \geq (1 - \epsilon)\mu(E_{j, k(\epsilon, A)}),$$

for $j \in G(\epsilon, A)$.

Proof. This is a straightforward result in measure theory.

Lemma 2. *The sets $\{A_i(k), 1 \leq j \leq m(k), 1 \leq k < +\infty\}$ generate the Borel field; furthermore each $A_i(k)$ is the union of $3j(k+1)$ sets of the class $\{A_i(k+1), 1 \leq j \leq m(k+1)\}$.*

Proof. Follows at once from the construction.

Lemma 3. *The sets $\{B_i(k), 1 \leq j \leq m(k-1)j(k), 1 \leq k < +\infty\}$ generate the Borel field, furthermore each $B_i(k)$ is the union of $3 \cdot j(k+1)$ sets of the class $\{B_i(k+1), 1 \leq j \leq m(k)j(k+1)\}$.*

Proof. Also follows at once from the construction.

Lemma 4. *Given $\{\delta(k)\} \downarrow 0$ and if $\{j(k)\}$ tends to infinity sufficiently rapidly, then for all k ,*

$$\text{i) } \mu\left(\bigcup_{i=1}^{m(k)} A_i(k)\right) \geq (1 - \delta(k))\mu(X + N),$$

- ii) $\mu(\tau A_{m(k)}(k) \triangle A_1(k)) \leq (1 - \delta(k))\mu(A_1(k)),$
 iii) $m(k)$ depends only on $\delta(0), \dots, \delta(k-1)$.

Proof. By $A \triangle B$ we mean the symmetric difference of the sets A and B . The lemma follows at once from the construction, since $m(k) = 3m(k-1)j(k) + 4$.

Lemma 5. *Let $\{j(k)\}$ be given. Then $\tau(\{j(k)\})$ has continuous spectrum.*

Proof. Suppose the contrary. Then there exists a constant λ of absolute value one and a function f of absolute value one almost everywhere such that (τ is clearly ergodic),

$$(4.1) \quad T_\tau f = \lambda f, \quad \lambda \neq 1.$$

It follows from lemmas 1 and 3 that given $\epsilon_1, \epsilon_2 > 0$ there exists a set

$$B = B_{j(\epsilon_1, \epsilon_2)}(k(\epsilon_1, \epsilon_2))$$

such that it has a subset A with the property that

$$(4.2) \quad \mu(A) \geq (1 - \epsilon_1)\mu(B),$$

and that, for $x \in A$,

$$(4.3) \quad f(x) = \exp[i(\theta + \delta(x))], \quad |\delta(x)| \leq \epsilon_2.$$

In extending τ from $k = k(\epsilon_1, \epsilon_2)$ to $k+1$ we split, according to the construction, the stack composed of $B_j(k)$, $j = 1, \dots, m(k)j(k+1)$, into three substacks and put one extra interval at the beginning of the first substack, two extra intervals at the beginning of the second substack and one extra interval at the beginning of the third substack, and then stack these modified substacks in order. This means that the interval $B_j(k)$ is split into three intervals of equal length $B_j^1(k)$, $B_j^2(k)$, $B_j^3(k)$ such that

$$(4.4) \quad \begin{aligned} \tau^{m(k)+2} B_j^1(k) &= B_j^2(k) \subset B_j(k), \\ \tau^{m(k)+1} B_j^2(k) &= B_j^3(k) \subset B_j(k). \end{aligned}$$

It follows from (4.1) that

$$(4.5) \quad \begin{aligned} T_\tau^{-(m(k)+2)} f(x) &= \lambda^{-(m(k)+2)} f(x), \\ T_\tau^{-(m(k)+1)} f(x) &= \lambda^{-(m(k)+1)} f(x). \end{aligned}$$

Putting (4.2), (4.3), (4.4) and (4.5) together we see that if ϵ_1 is small enough then there exists two sets of positive measure $B^1 = B_j^1(k) \cap A$, $B^2 = B_j^2(k) \cap A$ such that, for $x \in B^1$,

$$(4.6) \quad \exp[i(\theta + \delta(x))] = \lambda^{m(k)+2} \exp[i(\theta + \delta(\tau^{m(k)+2} x))],$$

and such that for $x \in B^2$,

$$(4.7) \quad \exp[i(\theta + \delta(x))] = \lambda^{m(k)+1} \exp[i(\theta + \delta(\tau^{m(k)+1} x))].$$

Equations (4.6), (4.7) and (4.3), writing $\lambda = e^{i\psi}$, give us that, modulo 2π , $|\psi| \leq 4\epsilon_2$. Since ϵ_2 can be chosen arbitrarily small, this implies that $\lambda = 1$.

Lemma 6. *The transformation $\tau(\{j(k)\})$ is not strongly mixing.*

Proof. Let $B = B_1(k+1)$. We have from the construction that

$$r(B \cap \tau^{m(r)j(r)+2}B) \geq \frac{1}{3}r(B),$$

for $r \geq k$. Since $r(B_1(k+1))$ tends to zero as k tends to infinity, the normalized measure of $B_1(k+1)$ also has this property. The proof is complete if we take k sufficiently large since strong mixing implies that

$$\lim_{i \rightarrow \infty} \frac{r(B \cap \tau^i B)}{r(X+N)} = \left[\frac{r(B)}{r(X+N)} \right]^2.$$

In the sequel it will be convenient to define $1 \cdot A = A$, $0 \cdot A = \phi$, where A is a set, and ϕ is the empty set.

Lemma 7. *Let $A = (\frac{1}{2}, 1)$, and let $\{j(k)\}$ be given. Then for each k there exists a sequence $\{x_i(k)\}$, $1 \leq i \leq m(k)$ of zeros and ones such that*

$$A = \sum_{i=1}^{m(k)} x_i(k) \cdot A_i(k).$$

Furthermore we have

$$\tau^j A = \sum_{i=1}^{m(k)} x_{i-j} A_i(k) + E(\eta(j(k+1))),$$

for $j = 1, \dots, m(k)$, where $\eta(j(k+1))$ tends to zero as $j(k+1)$ tends to infinity, where the subscripts are taken modulo $m(k)$.

Proof. The first part of the lemma follows at once from the construction. The second part follows since τ maps $A_{m(k)}(k)$ onto $A_1(k)$ with error tending to zero as $j(k+1)$ tends to infinity.

It is convenient in what follows to define (a, β) , where $\beta = (a_1, \dots, a_n)$, to be $(a, a_1, \dots, a_n)$, with analogous meaning for similar expressions.

Lemma 8. *Let*

$$\alpha(k) = (x_1(k), \dots, x_{m(k)}(k))$$

where the sequences $(x_1(k), \dots, x_{m(k)}(k))$ are those of lemma 7. Then the $\alpha(k)$ are given inductively by

$$\begin{aligned} \alpha(0) &= (0, 1) \\ \alpha(1) &= (0, \overbrace{\alpha(0), \dots, \alpha(0)}^{j(1)}, 0, 0, \overbrace{\alpha(0), \dots, \alpha(0)}^{j(1)}, 0, \overbrace{\alpha(0), \dots, \alpha(0)}^{j(1)}) \\ &\vdots \\ \alpha(k) &= (0, \overbrace{\alpha(k-1), \dots, \alpha(k-1)}^{j(k)}, 0, 0, \overbrace{\alpha(k-1), \dots, \alpha(k-1)}^{j(k)}, 0, \overbrace{\alpha(k-1), \dots, \alpha(k-1)}^{j(k)}). \end{aligned}$$

Proof. This again follows from the construction of the transformation.

We note that subscripts on the elements of $\alpha(k)$ are understood to be modulo $m(k)$, so that letting $i - j = r$ modulo $m(k)$ then $x_{i-j}(k) = x_r(k)$. Furthermore, note that $m(k)$, $k = 0, 1, 2, \dots$, are even numbers so that with the convention we use $x_N(k)$ has an even subscript if and only if N is even, for each k .

Recall the definition of $\alpha(k)$ in terms of $\alpha(k+1)$:

$$\alpha(k) = \underbrace{(0, \overbrace{\alpha(k-1), \dots, \alpha(k-1)}^{j(k)}, 0, 0, \overbrace{\alpha(k-1), \dots, \alpha(k-1)}^{j(k)}, 0, \overbrace{\alpha(k-1), \dots, \alpha(k-1)}^{j(k)})}_{3j(k)m(k-1)+4 \text{ elements}},$$

and introduce the vector $\beta(k)$:

$$(9.1) \quad \beta(k) = \underbrace{(\overbrace{\alpha(k-1), \dots, \alpha(k-1)}^{j(k)}, \overbrace{0, \dots, 0}^{2j(k)m(k-1)+4})}_{3j(k)m(k-1)+4 \text{ elements}},$$

so that we may write

$$(9.2) \quad \alpha = S\beta + S^{j(k)m(k-1)+3}\beta + S^{2j(k)m(k-1)+4}\beta,$$

where S is the one step shift operator in the space of sequences of length $3 \cdot j(k)m(k-1) + 4$ (we deleted (k) following α and β in (9.2)).

Lemma 9. *Let M be an integer and let $N = \langle M \rangle$, $r(1) = \langle N + m(k-1)j(k) + 1 \rangle$, and $r(2) = \langle N + 2m(k-1)j(k) + 3 \rangle$, where $\langle A \rangle$ is that integer equal to A modulo $m(k)$ and which satisfies $-m(k)/2 \leq \langle A \rangle < m(k)/2$. Then there exist three positive numbers K_1, K_2, K_3 (depending on $j(k)$) such that*

$$(9.3) \quad \lim_{j(k) \rightarrow \infty} \frac{3(K_1 + K_2 + K_3)m(k-1)}{m(k)} = 1,$$

and such that, for $t = 0, 1$,

$$(9.4) \quad \begin{aligned} & \sum_{i=1}^{m(k)} x_i(k)x_{i-M-t}(k) \leq 3K_1 \sum_{i=1}^{m(k-1)} x_i(k-1)x_{i-N-t}(k-1) \\ & + K_2 \left[2 \sum_{i=1}^{m(k-1)} x_i(k-1)x_{i-r(1)-t}(k-1) + \sum_{i=1}^{m(k-1)} x_i(k-1)x_{i-r(1)-1-t}(k-1) \right] \\ & + K_3 \left[2 \sum_{i=1}^{m(k-1)} x_i(k-1)x_{i-r(2)-t}(k-1) + \sum_{i=1}^{m(k-1)} x_i(k-1)x_{i-r(2)+1-t}(k-1) \right]. \end{aligned}$$

Proof. Fix the value of k , and using the notation of (9.1) and (9.2) we see that

$$(9.5) \quad \sum_{i=1}^{m(k)} x_i(k)x_{i-M}(k) = \alpha \cdot S^N \alpha,$$

and that,

$$(9.6) \quad \alpha \cdot S^N \alpha = (S\beta + S^{j(k)m(k-1)+3}\beta + S^{2j(k)m(k-1)+4}\beta) \cdot (S^{N+1}\beta + S^{N+j(k)m(k-1)+3}\beta + S^{N+2j(k)m(k-1)+4}\beta).$$

Using the fact that $\gamma \cdot S\delta = S^{-1}\gamma \cdot \delta$ and that $S^r = S^{3j(k)m(k-1)+4+r}$, it follows from (9.6) that

$$(9.7) \quad \begin{aligned} \alpha \cdot S^N \alpha &= \beta \cdot S^N \beta + \beta \cdot S^{N+2m(k-1)j(k)+2}\beta + \beta \cdot S^{N+m(k-1)j(k)+1}\beta \\ &\quad + \beta \cdot S^{N+m(k-1)j(k)+2}\beta + \beta \cdot S^N \beta + \beta \cdot S^{N+2m(k-1)j(k)+3}\beta \\ &\quad + \beta \cdot S^{N+2m(k-1)j(k)+3}\beta + \beta \cdot S^{N+m(k-1)j(k)+1}\beta + \beta \cdot S^N \beta \\ &= \beta \cdot S^N \beta + \beta \cdot S^{r(2)-1}\beta + \beta \cdot S^{r(1)}\beta \\ &\quad + \beta \cdot S^{r(1)+1}\beta + \beta \cdot S^N \beta + \beta \cdot S^{r(2)}\beta \\ &\quad + \beta \cdot S^{r(2)}\beta + \beta \cdot S^{r(1)}\beta + \beta \cdot S^N \beta. \end{aligned}$$

We now denote the elements of β as follows

$$(9.8) \quad \beta(k) = (y_1(k), \dots, y_{m(k)}(k)).$$

It can then be obtained easily from (9.7) and (9.8) that

$$(9.9) \quad \begin{aligned} \alpha \cdot S^N \alpha &= 3 \sum_{i=1}^{m(k)} y_i y_{i-N} + 2 \sum_{i=1}^{m(k)} y_i y_{i-r(1)} + \sum_{i=1}^{m(k)} y_i y_{i-r(1)-1} \\ &\quad + 2 \sum_{i=1}^{m(k)} y_i y_{i-r(2)} + \sum_{i=1}^{m(k)} y_i y_{i-r(2)+1}. \end{aligned}$$

For the fixed value of k we are working with we define

$$\bar{\alpha}(k) = (\overbrace{0, \bar{\alpha}(k-1), \dots, \bar{\alpha}(k-1)}^{j(k)}, \overbrace{0, 0, \bar{\alpha}(k-1), \dots, \bar{\alpha}(k-1)}^{j(k)}, \overbrace{0, \bar{\alpha}(k-1), \dots, \bar{\alpha}(k-1)}^{j(k)}),$$

where, however,

$$\bar{\alpha}(k-1) = (\overbrace{1, 1, \dots, 1}^{m(k-1)}).$$

All the previous formulas hold with bars on the various terms if the bar vectors are defined in the obvious way. Our purpose in introducing them is to make certain estimates, as follows: We let G_1 be the set of indices where both $\bar{y}_i$ and $\bar{y}_{i-N}$ are equal to one; G_2 the set of indices where both $\bar{y}_i$ and $\bar{y}_{i-r(1)}$ are equal to one; G_3 the set where both $\bar{y}_i$ and $\bar{y}_{i-r(1)-1}$ are equal to one; G_4 the set where both $\bar{y}_i$ and $\bar{y}_{i-r(2)}$ are equal to one and finally, G_5 the set where $\bar{y}_i$ and $\bar{y}_{i-r(2)+1}$ are both equal to one. It follows from (9.9) with bars that

$$(9.10) \quad \lim_{j(k) \rightarrow \infty} \frac{3n(G_1) + 2n(G_2) + n(G_3) + 2n(G_4) + n(G_5)}{m(k)} = 1,$$

when $n(G_i)$ is the number of elements in G_i . Furthermore, in (9.9) without bars the sums can be taken respectively over $G_1, \dots, G_5$ since $y_s = 1$ implies that $\bar{y}_s = 1$. We note also that for each i the set G_i is a set of consecutive integers, and that G_2 and G_3 as well as G_4 and G_5 differ by at most one element. This implies that

$$\sum_{i=1}^{m(k)} y_i(k) y_{i-N-t}(k) \leq K_1^* \sum_{i=1}^{m(k-1)} x_i(k-1) x_{i-N-t}(k-1), \quad t = 0, 1,$$

where K_1^* is not larger than 2 plus the whole number of times that $m(k-1)$ goes into the number of elements in G_1 (we had denoted the number of elements in G_1 by $n(G_1)$).

Similarly, we have that

$$\sum_{i=1}^{m(k)} y_i(k) y_{i-r(1)-t}(k) \leq K_2^* \sum_{i=1}^{m(k-1)} x_i(k-1) x_{i-r(1)-t}(k-1),$$

and that

$$\sum_{i=1}^{m(k)} y_i(k) y_{i-r(1)-1-t}(k) \leq K_3^* \sum_{i=1}^{m(k-1)} x_i(k-1) x_{i-r(1)-1-t}(k-1),$$

$t = 0, 1$, where K_2^* is not larger than 2 more than the number of times $m(k-1)$ goes into $n(G_2)$, and K_3^* is not larger than 2 more than the number of times $m(k-1)$ goes into $n(G_3)$.

In an analogous way

$$\sum_{i=1}^{m(k)} y_i(k) y_{i-r(2)-t}(k) \leq K_4^* \sum_{i=1}^{m(k-1)} x_i(k-1) x_{i-r(2)-t}(k-1),$$

and also

$$\sum_{i=1}^{m(k)} y_i(k) y_{i-r(2)+1-t}(k) \leq K_5^* \sum_{i=1}^{m(k-1)} x_i(k-1) x_{i-r(2)+1-t}(k-1),$$

$t = 0, 1$, where K_4^* and K_5^* are defined as in the previous cases and are not larger, respectively than 2 plus the number of times that, again as in the previous cases, $m(k-1)$ goes into $n(G_4)$ and $n(G_5)$. We see from the remarks following (9.10) that we may take $K_1 = K_1^*$, $K_2 = \max(K_2^*, K_3^*)$ and $K_3 = \max(K_4^*, K_5^*)$.

Before proceeding to the next lemma it is convenient to introduce the following notation:

$$M(k) = \frac{1}{3m(k)} \sup \left[2 \sum_{i=1}^{m(k)} x_i(k) x_{i-j(1)}(k) + \sum_{i=1}^{m(k)} x_i(k) x_{i-j(2)}(k) \right],$$

where the sup is taken over $j(1), j(2)$ such that $j(1) - j(2) = 1$. Note that $M(0) = 1/3$.

Lemma 10. *Let $j(1), \dots, j(k-1)$ be given so that*

$$M(r) < \frac{1}{3} + \epsilon, \quad r = 1, \dots, k-1.$$

Then there exists N so that $j(k) \geq N$ implies that

$$M(k) < \frac{1}{3} + \epsilon.$$

Proof. The lemma is an immediate consequence of lemma 9.

Lemma 11. Given $\epsilon > 0$, then if $\{j(k)\}$ tends to infinity sufficiently rapidly we have for all n that

$$\frac{1}{m(n)} \sum_{i=1}^{m(n)} x_i(n) x_{i-j}(n) < \frac{1}{3} + \epsilon,$$

for all j an odd integer.

Proof. The result holds for $n = 0$, as we see easily. We suppose that $j(1), \dots, j(k-1)$ have been chosen so that the result holds for $n = 1, \dots, k-1$ and also so that

$$(11.1) \quad M(n) < \frac{1}{3} + \epsilon,$$

holds also for $n = 1, \dots, k-1$. It then follows from lemma 9 that

$$(11.2) \quad \begin{aligned} \sum_{i=1}^{m(k)} x_i(k) x_{i-j}(k) &\leq 3K_1 \sum_{i=1}^{m(k-1)} x_i(k-1) x_{i-j}(k-1) \\ &+ K_2 \left[2 \sum_{i=1}^{m(k-1)} x_i(k-1) x_{i-r(1)}(k-1) + \sum_{i=1}^{m(k-1)} x_i(k-1) x_{i-r(1)-1}(k-1) \right] \\ &+ K_3 \left[2 \sum_{i=1}^{m(k-1)} x_i(k-1) x_{i-r(2)}(k-1) + \sum_{i=1}^{m(k-1)} x_i(k-1) x_{i-r(2)+1}(k-1) \right]. \end{aligned}$$

Applying the induction hypothesis and (11.1) to the right side of (11.2) we obtain that for $j(k) \geq N(j(1), \dots, j(k-1))$ it is less than or equal to

$$3K_1 m(k-1) \left(\frac{1}{3} + \epsilon \right) + 3K_2 m(k-1) \left(\frac{1}{3} + \epsilon \right) + 3K_3 m(k-1) \left(\frac{1}{3} + \epsilon \right).$$

If we now divide through by $m(k)$ and apply the first part of lemma 9, we see that if $j(k)$ is chosen sufficiently large the conclusion of the lemma holds for $n = k$. This concludes the proof of the lemma.

Lemma 12. Given $\epsilon > 0$ then if $\{j(k)\}$ tends to infinity sufficiently rapidly we have that

$$\mu(A \cap \tau^j(A)) < \frac{1}{3} + \epsilon,$$

for all j and odd integer.

Proof. This follows directly from lemma 11 and from lemmas 7 and 8.

As we have remarked earlier, $\tau A_{m(k)}(k)$ equals $A_1(k)$ with error tending to zero as $j(k)$ tends to infinity (recall that this means that the measure of the symmetric difference of the sets tends to zero). Before proceeding to the next lemma it is convenient to introduce further notation: Let $G_j(k)$, $1 \leq k \leq m(k)$, be that subset of $A_j(k)$ which is mapped by $\tau^{m(k)-j+1}$ into $A_1(k)$. We have then

that $G_j(k) = \tau^{j-m(k)} G_{m(k)}(k)$. We also let

$$G^*(k) = \bigcup_{i=1}^{m(k)} G_i(k), \quad A^*(k) = \bigcup_{i=1}^{m(k)} A_i(k).$$

It follows from our preceding remarks that $G^*(k)$, which is obviously contained in $A^*(k)$, equals $A^*(k)$ with error tending to zero as $j(k+1)$ tends to infinity. Next, let $A = (\frac{1}{2}, 1)$ as in lemma 7. We have then as before that

$$(13.1) \quad A = \sum_{i=1}^{m(k)} x_i(k) A_i(k),$$

and that for any transformation σ ,

$$(13.2) \quad \sigma(A) = \sum_{i=1}^{m(k)} x_i(k) \sigma(A_i(k)).$$

Recall that we write $A = B + E(\delta)$ for two sets whose symmetric difference has measure less than δ .

Lemma 13. *Let σ be an arbitrary measurable transformation which commutes with τ and let $A = (\frac{1}{2}, 1)$. We have then that*

$$(13.3) \quad \sigma(A \cap G^*(k)) = \sum_{i=1}^{m(k)} x_i(k) \tau^{i-1}(\sigma(G_1(k)) \cap G^*(k)) + E(\delta),$$

where δ tends to zero uniformly in σ as $j(k+1)$ tends to infinity (and where the $x_i(k)$ are given as in lemma 7).

Proof. First we establish that

$$(13.4) \quad G^*(k) = X + N + E(\eta),$$

where η tends to zero as $j(k+1)$ tends to infinity. To see this note that we may write

$$cG^*(k) = cA^*(k) + \sum_{j=0}^{m(k)} \tau^{-j} [A_{m(k)}(k) - G_{m(k)}(k)]$$

and consequently that

$$\begin{aligned} r(cG^*(k)) &\leq r(cA^*(k)) + m(k)r(A_{m(k)}(k) - G_{m(k)}(k)) \\ &\leq r(cA^*(k)) + \frac{m(k)r(A_{m(k)}(k))}{j(k+1)}. \end{aligned}$$

This implies (13.4) as we may readily see from the construction.

Next we note that (13.4) implies

$$(13.5) \quad \sigma(G_1(k)) \cap G^* = \sigma(G_1(k)) + E(\delta/m(k)),$$

where δ tends to zero uniformly in σ as $j(k+1)$ tends to infinity. It follows from the definition that

$$(13.6) \quad A \cap G^*(k) = \sum_{i=1}^{m(k)} x_i(k) G_i(k) = \sum_{i=1}^{m(k)} x_i \tau^{i-1} G_1(k).$$

Equations (13.5) and the fact that τ is measure preserving imply that

$$(13.7) \quad \tau^{i-1} \sigma(G_1(k)) = \tau^{i-1} (\sigma(G_1(k)) \cap G^*) + E(\delta/m(k)).$$

Equations (13.6) and (13.7) together with the fact that σ and τ commute yield

$$(13.8) \quad \sigma(A \cap G^*(k)) = \sum_{i=1}^{m(k)} x_i \tau^{i-1} (\sigma(G_1(k)) \cap G^*(k)) + E(\delta),$$

where δ tends to zero uniformly in σ as $j(k+1)$ tends to infinity.

Lemma 14. *If σ is a measurable square root of τ , then for each k there exist $\{H_i(k), 1 \leq i \leq m(k)\}$, pairwise disjoint subsets of $G_1(k)$, such that*

$$(14.1) \quad G^*(k) \cap \sigma(G_1(k)) = \sum_{i=1}^{m(k)} \tau^{i-1} H_i(k),$$

$$H_i(k) = \tau^{1-i} (G_i(k) \cap \sigma(G_1(k))).$$

Proof. If we define the sets by the second formula of (14.1) it is clear that they satisfy the first. It only remains to show that the sets are pairwise disjoint. In order to see this, suppose that there exist $i(1) < i(2)$ such that there exists a point $x \in H_{i(1)} \cap H_{i(2)}$. It follows from the definitions that this implies that there exist two points x_1 and x_2 in $G_1(k)$ such that $\sigma(x_1) = \tau^{i(2)-i(1)} \sigma(x_2)$. Applying σ to both sides of the equation we obtain that x belongs to $G_{i(2)-i(1)}(k)$, which yields a contradiction.

In the following lemma, as in lemma 13, the $x_i(k)$ are given as in lemma 7.

Lemma 15. *If σ is a measurable square root of τ then there exist $\{J_i(k), 1 \leq i \leq m(k)\}$ pairwise disjoint subsets of $A_1(k)$ such that*

$$\sigma(A \cap G^*(k)) = \sum_{i=1}^{m(k)} \sum_{j=1}^{m(k)} x_i(k) \tau^{(i+j-2) \bmod m(k)} J_j(k) + E(\delta),$$

where δ tends to zero uniformly in σ as $j(k+1)$ tends to infinity.

Proof. We note that as the $j(k)$ are changed the transformation τ is changed, and consequently so are its square roots. The uniformity condition given by the lemma is with respect to this changing class of square roots.

We have from lemmas 13 and 14, with $J_i(k) = H_i(k)$ that

$$(15.1) \quad \sigma(A \cap G^*(k)) = \sum_{i=1}^{m(k)} \sum_{j=1}^{m(k)} x_i(k) \tau^{i-1} \tau^{j-1} J_j(k) + E(\delta_1),$$

where δ_1 tends to zero uniformly in σ as $j(k+1)$ tends to infinity. Note also that, as we shall show,

$$(15.2) \quad \tau^{m(k)} J_i(k) \cap J_j(k) = \phi,$$

if $i \neq j$, $1 \leq i, j \leq m(k)$. If not there exist $i(1)$ and $i(2)$ such that

$$\tau^{m(k) - (i(1)-1) + (i(2)-1)} \tau^{i(1)-1} J_{i(1)} \cap \tau^{i(2)-1} J_{i(2)} \neq \phi.$$

This implies that there exist points x_1 and x_2 in $G_1(k)$ such that

$$(15.3) \quad \tau^{m(k) - (i(1)-1) + (i(2)-1)} x_1 = x_2.$$

Equation (15.3) and the fact that the points are in $G_1(k)$ implies that the exponent in (15.3) is equal to zero modulo $m(k)$. This is impossible, and we have (15.2).

Equation (15.2) implies that $\tau^i J_i = \tau^{i \bmod m(k)} J_i + E(\delta_i)$, $0 \leq j \leq 2m(k)$, where the sum of the δ_i is bounded by the measure of $cG_1(k) \cap A_1(k)$. This in turn implies that

$$(15.4) \quad \sum_{j=1}^{m(k)} \sum_{i=1}^{m(k)} x_i(k) \tau^{i-1} \tau^{j-1} J_i(k) \\ = \sum_{j=1}^{m(k)} \sum_{i=1}^{m(k)} x_i(k) \tau^{(i+j-2) \bmod m(k)} J_i(k) + E[m(k)r(cG_1(k) \cap A_1(k))],$$

which together with (15.1) yields the lemma, since the measure of the set $cG_1(k) \cap A_1(k)$ tends to zero independently of σ as $j(k+1)$ tends to infinity.

The next lemma is purely combinatorial as are lemmas 8 through 11. It will enable us to make estimates on the size of the intersection of the set $A = (\frac{1}{2}, 1)$ with odd translates (that is, translates under τ^{2n-1}) of itself, under the assumption that a non-singular square root exists. The estimate will be that the size of the intersection of A with certain of its odd translates is nearly as big as A . This of course is sufficient to prove the theorem. The sequences we refer to in the lemma which follows need not be given as in lemma 7, although it will only be applied to such sequences. Finally we remark that we chose the set A to be equal to $(\frac{1}{2}, 1)$ for convenience. It could equally well have been chosen to be $(0, \frac{1}{2})$ for example. For a set G we again let $n(G)$ be the number of elements in it.

Lemma 16. Let $\alpha_i = (x_{i1}, \dots, x_{in})$, $i = 1, \dots, k$ be k sequences of zeros and ones of length n . Let $\{b_i, i = 1, \dots, k\}$ be non-negative numbers having sum equal to one, and suppose that there is a subset G of $\{1, \dots, n\}$ such that

$$(16.1) \quad \sum_{i=1}^k b_i x_{ij} \geq 1 - \eta, \quad \text{for all } j \text{ in } G.$$

Then there exists an integer w , $1 \leq w \leq k$, such that,

$$\sum_{j \text{ in } G} \sum_{i=1}^k b_i x_{ij} x_{wi} \geq n(G)(1 - 2\eta).$$

Proof. To see that we may assume without loss of generality that $b_i = 1/k$, we proceed as follows. First note that we can certainly suppose them to be rational numbers. Writing these numbers over a common denominator we have

$b_i = m_i/m$. Then consider the problem for the $\bar{\alpha}$'s obtained by taking each α_i m_i times; this reduces the lemma to the case where $\bar{k}$ equals the sum of the m_i 's and where all the $\bar{b}$ are equal to $1/m$.

Let $B(i)$ be the subset of G such that for j in $B(i)$

$$(16.2) \quad x_{ij} = 0.$$

Let

$$(16.3) \quad \delta = \inf_{1 \leq i \leq k} \frac{n(B(i))}{n(G)},$$

where $n(B(i))$ and $n(G)$ stand for the number of elements in $B(i)$ and G respectively. By changing the order of summation we have from (16.3) that

$$(16.4) \quad \sum_{j \text{ in } G} \left[\sum_{i=1}^k \psi_{B(i)}(j) \right] \geq k \cdot n(G) \cdot \delta.$$

It then follows easily from (16.4) that for at least one value of j , say $j(0)$,

$$(16.5) \quad \sum_{i=1}^k \psi_{B(i)}(j(0)) \geq k \delta.$$

Equation (16.5) implies that $[k\delta]$ of the sets B_i intersect (here $[k\delta]$ means the least integer greater than or equal to $k\delta$), say $B_{i(1)}, B_{i(2)}, \dots, B_{i([k\delta])}$, and $j(0) \in C = \bigcap_{i=1}^{[k\delta]} B_{i(i)}$. We then have

$$(16.6) \quad \sum_{i=1}^k x_{ij(0)} = \sum_{j \in (i(1), \dots, i([k\delta]))} x_{ij(0)} + \sum_{j \notin (i(1), \dots, i([k\delta]))} x_{ij(0)}.$$

It follows from the definition of $B_{i(r)}$ that $x_{ij(0)} = 0$ for $j \in (i(1), \dots, i([k\delta]))$, and consequently (16.6) implies that

$$(16.7) \quad \sum_{i=1}^k x_{ij(0)} \leq k - [k\delta] \leq k(1 - \delta).$$

Since we have assumed that $b_i = 1/k$, (16.7) implies

$$\sum_{i=1}^k b_i x_{ij(0)} \leq (1 - \delta),$$

which together with (16.1) and (16.3) yields that

$$(16.8) \quad \inf_{1 \leq i \leq n} n(B(i))/n(G) = \delta \leq \eta.$$

Equation (16.8) in turn yields that there exists an integer w such that

$$(16.9) \quad n(B(w))/n(G) \leq \eta.$$

Now,

$$\begin{aligned}
 \sum_{j \in G} \sum_{i=1}^k x_{ij} x_{wj} &= \sum_{i=1}^k \sum_{j \in G} x_{ij} x_{wj} \\
 &= \sum_{i=1}^k \sum_{j \in G-B(w)} x_{ij} x_{wj} + \sum_{i=1}^k \sum_{j \in B(w)} x_{ij} x_{wj} \\
 &= \sum_{i=1}^k \sum_{j \in G-B(w)} x_{ij} x_{wj} = \sum_{i=1}^k \sum_{j \in G-B(w)} x_{ij} ,
 \end{aligned}$$

so that

$$\begin{aligned}
 \sum_{j \in G} \sum_{i=1}^k x_{ij} x_{wj} &= \sum_{i=1}^k \sum_{j \in G-B(w)} x_{ij} + \sum_{i=1}^k n(B(w)) - \sum_{i=1}^k n(B(w)) \\
 &\geq \sum_{i=1}^k \sum_{j \in G} x_{ij} - k \cdot n(B(w)).
 \end{aligned}$$

Keeping in mind that $b_i = 1/k$, this implies

$$\begin{aligned}
 \sum_{j \in G} \sum_{i=1}^k b_i x_{ij} x_{wj} &\geq \sum_{j \in G} \sum_{i=1}^k b_i x_{ij} - n(B(w)) \\
 &\geq n(G)(1 - \eta) - n(B(w)) \\
 &= n(G) \left\{ 1 - \eta - \frac{n(B(w))}{n(G)} \right\} \\
 &= n(G) \{ 1 - 2\eta \}.
 \end{aligned}$$

Lemma 17. *If $\{j(k)\}$ tends to infinity sufficiently rapidly and if $\tau(j(k))$ has a non-singular square root σ , then there exists an integer $N(\eta)$ such that*

$$(17.1) \quad \mu(\sigma(A) - \tau^N(A)) \leq \eta$$

for each $\eta > 0$.

Proof. If $\{j(k)\}$ tends to infinity sufficiently rapidly, then there exists a sequence $\{\delta(k)\} \downarrow 0$ such that

$$(17.1) \quad \lim_{k \rightarrow \infty} \delta(k)/\mu(A_1(k)) = 0,$$

$$(17.2) \quad \sigma(A \cap G^*(k)) = \sum_{i=1}^{m(k)} \sum_{i=1}^{m(k)} x_{i-i+1}(k) D_{i,i}(k) + E(\delta(k)),$$

where $D_{i,i}(k) = \tau^{i-1} J_i(k)$ and consequently where $D_{i,i}(k) \subset A_i(k)$, $1 \leq i$, $j \leq m(k)$, and $\tau D_{i,i}(k) = D_{i+1,i}(k)$, $1 \leq j \leq m(k) - 1$, as we may see from lemma 15. It follows from lemma 7 that we may also satisfy at the same time the conditions

$$(17.3) \quad G^*(k) = X + N + E(\delta(k))$$

and

$$(17.4) \quad \int \left| \left[\sum_{i=1}^{m(k)} \psi_{A_i} x_{i-j} \right] - \psi_{rjA} \right| \leq \delta(k), \quad j = 0, 1, \dots, m(k).$$

We have from lemmas 1 and 2 that for $\lambda > 0$ and $M > 0$ there exists an $n \geq M$ and an index set G such that

$$(17.5) \quad \sigma(A) = \bigcup_{i \in G} A_i(n) + E(\lambda),$$

$$(17.6) \quad \sigma(A) \cap A_i(n) = A_1(n) + E(\lambda \mu(A_i(n))), \quad i \in G.$$

We obtain from (17.2), (17.3) and (17.6) that

$$(17.7) \quad \sum_{j=1}^{m(k)} x_{i-j+1}(n) D_{i,j}(n) = A_i(n) + E(2 \delta(n) + \lambda \mu(A_i(n))), \quad i \in G.$$

We then define $b_j = \mu(D_{1,i}) / \mu(\bigcup_{i=1}^{m(k)} D_{1,i})$ and note that it follows from (17.7) that

$$\sum_{i=1}^{m(k)} x_{i-j+1} b_j \geq 1 - \left[\frac{2 \delta(n)}{\mu(A_i(n))} + \lambda \right], \quad i \in G.$$

Since the sequence $\{\delta(k)\}$ has property (17.1) we have that if M is sufficiently large then

$$(17.8) \quad \sum_{j=1}^{m(k)} x_{i-j+1} b_j \geq 1 - 2\lambda, \quad i \in G.$$

We are now in a position to apply lemma 16 with $\alpha_i = \{x_{i-j+1}, 1 \leq j \leq m(k)\}$ to obtain that there exists a w such that

$$(17.9) \quad \sum_{i \in G} \sum_{j=1}^{m(k)} b_j x_{i-j+1} x_{i-w+1} \geq n(G)(1 - 4\lambda).$$

In order to evaluate the left side of equation (17.9) we keep in mind the definition of the constants x_i and b_i , and the fact that the sets $A_i(k)$, $i = 1, 2, \dots, m(k)$ are pairwise disjoint and that $D_{i,i}(k) \subset A_i(k)$. We see that it is given by

$$(17.10) \quad \frac{1}{\mu(\sum_{j=1}^{m(k)} D_{1,j})} \int \left\{ \sum_{i=1}^{m(k)} \psi_{A_i} x_{i-w+1} \right\} \left\{ \sum_{i \in G} \sum_{j=1}^{m(k)} x_{i-j+1} \psi_{D_{i,j}} \right\}.$$

Further it follows from (17.7) that

$$\sum_{j=1}^{m(k)} D_{1,j}(n) = A_1(n) + E(2 \delta(n) + \lambda \mu(A_1(n))),$$

and thus that for M large enough

$$(17.11) \quad \mu\left(\sum_{j=1}^{m(k)} D_{1,j}\right) \geq \mu(A_1)(1 - \lambda).$$

Formulas (17.5) and (17.11) imply that

$$(17.12) \quad \mu\left(\sum_{j=1}^{m(k)} D_{1,j}\right) \cdot n(G) \cdot (1 - 4\lambda) \geq (\mu(\sigma(A)) - \lambda)(1 - \lambda)(1 - 4\lambda).$$

We have furthermore that (17.2), (17.3) and (17.5) imply that

$$\sigma(A) = \sum_{i \in G} \sum_{j=1}^{m(k)} x_{i-j+1} D_{i,j}(n) + E(\lambda + 2\delta(n)),$$

and consequently that, with M sufficiently large,

$$(17.13) \quad \int \left| \sum_{i \in G} \sum_{j=1}^{m(k)} x_{i-j+1} \psi_{D_{i,j}} - \psi_{\sigma(A)} \right| \leq \lambda + 2\delta(n) \leq 2\lambda.$$

Putting (17.9), (17.10), (17.12), (17.4) and (17.13) together we obtain that $\int \psi_{\tau^{w-1}(A)} \psi_{\sigma(A)} \geq \mu(\sigma(A)) + \varepsilon(\lambda)$, where $\varepsilon(\lambda)$ tends to zero as λ tends to zero, and we see that the lemma follows by taking λ sufficiently small, with $N = w - 1$.

3. Proof of the theorem. To see that the theorem holds we choose a sequence $\{j(k)\}$ tending to infinity sufficiently rapidly that lemmas 5, 6, 12, and 17 hold. We have then a transformation having continuous spectrum which is not strongly mixing and which satisfies the condition that

$$i) \quad r(A \cap \tau^j A) < \frac{1}{3} + \epsilon,$$

for all odd j , for $A = (\frac{1}{2}, 1)$. Furthermore if the transformation has a non-singular square root, then for each $\eta > 0$ there is an $N = N(\eta)$ such that

$$ii) \quad r(\sigma(A) - \tau^N(A)) \leq \eta.$$

Part ii) means that $\sigma(A) \subset \tau^N(A) + F$, where the measure of F is less than or equal to η , and thus

$$\begin{aligned} \sigma \cdot \sigma(A) &\subset \tau^N(\sigma(A)) + \sigma(F) \\ &\subset \tau^N(\tau^N(A) + F) + \sigma(F) \\ &= \tau^{2N}(A) + \tau^N(F) + \sigma(F), \end{aligned}$$

since σ is non-singular and τ is measure preserving, this implies that

$$A = \tau^{2N-1}(A) + E(\eta) + E(\varepsilon(\eta)),$$

where $\varepsilon(\eta)$ tends to zero as η tends to zero, contradicting i), if ϵ is sufficiently small and η is sufficiently small.

BIBLIOGRAPHY

- [1] CHACON, R. V., Weakly mixing transformations which are not strongly mixing, to appear.
- [2] HALMOS, P. R., *Lectures on ergodic theory*, Math. Soc. Japan, 1956.

Ohio State University

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