The Null-Bundle of an Einstein-Riemann Space I: General Theory

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Communicated by V. HLAVATY

1. Introduction. Several years ago, Kerr and Schild [1] obtained a general solution of Einstein's empty space field equations when the metric has the form

$$(1.1) \quad h_{AB} = g_{AB} + \ell_A \ell_B .$$

Here, g_{AB} is the metric of Minkowski space referred to Cartesian coordinates and ℓ_A is a null-vector field: $g^{AB} \ell_A \ell_B = 0$. The method used by Kerr and Schild relied heavily on the fact that g_{AB} was the metric of a flat space, and only the empty space field equations were considered.

This series of papers is devoted to generalizing Kerr and Schild's method of approach to the general case in which the space with metric tensor g_{AB} is not flat and the full Einstein field equations are used. We shall also introduce several alternative methods that naturally arise out of the theory of normed null bases which we shall develop. This will allow us to exhibit a number of new exact solutions.

2. The null-bundle of a hyperbolic-normal metric space. Let g_{AB} denote the metric tensor of a 4-dimensional, hyperbolic-normal metric space E of signature -2 , and let ℓ_A denote the components of a null-vector field in E :

$$(2.1) \quad g^{AB} \ell_A \ell_B = 0 .$$

Definition 2.1. A symmetric tensor field h_{AB} is said to form a *null-argumentation* of g_{AB} if and only if there exists a null-vector field (ℓ_A) on E such that

$$(2.2) \quad h_{AB} = g_{AB} + \ell_A \ell_B .$$

Theorem 2.1. Let the tensor field h^{AB} be defined by

$$(2.3) \quad h^{AB} = g^{AB} - \ell^A \ell^B ,$$

where

$$(2.4) \quad \ell^A = g^{AB} \ell_B .$$

We have

$$(2.5) \quad h^{AB}h_{BC} = \delta_C^A$$

and ℓ_A is a null-vector of h^{AB} :

$$(2.6) \quad h^{AB}\ell_A\ell_B = 0.$$

Proof. It follows from (2.2) and (2.3) that

$$h^{AB}h_{BC} = \delta_C^A - \ell^A\ell^B g_{BC} + g^{AB}\ell_B\ell_C - \ell^A\ell^B\ell_B\ell_C.$$

Since the last term on the right-hand side vanishes by (2.1), while (2.4) shows that the second and third terms are equal but opposite in sign, we establish (2.5). Equation (2.6) follows immediately from (2.1), (2.2), and (2.4), since

$$h^{AB}\ell_A\ell_B = g^{AB}\ell_A\ell_B - (\ell^A\ell_A)^2.$$

Theorem 2.1 tells us that the tensor h_{AB} possesses an inverse for any null-vector field ℓ_A , and hence h_{AB} is nonsingular. The field h_{AB} may thus serve as a metric tensor for a new space. This motivates the following definition.

Definition 2.2. A space E_i is said to be a *null-augmentation* of the space E if and only if there exists a null vector field (ℓ_A) on E such that the metric tensor of E_i can be written in the form

$$(2.7) \quad h_{AB} = g_{AB} + \ell_A\ell_B.$$

Definition 2.3. The totality $\mathfrak{N}$ of all spaces obtained from E by null-augmentation is said to constitute the *null-bundle* of E ; the individual spaces comprising $\mathfrak{N}$ will be termed elements of $\mathfrak{N}$.

In order to simplify future considerations, we make the following agreement.

Agreement 2.1. The space E is considered to be an element of its own null-bundle.

Theorem 2.2. All elements of the null-bundle of E have the same signature and the same determinant:

$$(2.8) \quad \det(h_{AB}) = \det(g_{AB}).$$

Proof. Under the substitution $\ell_A = \alpha\ell_A^*$, for α an arbitrary parameter, (2.2) gives

$$h_{AB}(\alpha) = g_{AB} + \alpha^2\ell_A^*\ell_B^*.$$

It then follows that $\det(h_{AB}(\alpha))$ is a quartic polynomial in the variable α^2 . By Theorem 2.1, h_{AB} is nonsingular for all null-vectors ℓ_A , and hence $h_{AB}(\alpha)$ is nonsingular for all values of α . Thus, $\det(h_{AB}(\alpha)) \neq 0$ for all values of α , and hence $\det(h_{AB}(\alpha))$ is a constant function of α by the fundamental theorem of algebra. We thus have $\det(h_{AB}(\alpha)) = \det(h_{AB}(0)) = \det(g_{AB})$ which estab-

lishes (2.8). The invariance of signature under null-augmentation is easily proved in a similar fashion.

Corollary 2.1. *Every element of $\mathcal{N}$ is a hyperbolic-normal metric space.*

3. The affine connection on the null-bundle. In this section we shall compute the affine connection of any null-augmentation of the space E in terms of the affine connection of E and the field ℓ_A and its derivatives. Throughout this paper we shall use ∇_A as the symbol for covariant differentiation formed on the Christoffel symbols γ_{BC}^A based on g_{AB} and a semicolon followed by a subscript to denote covariant differentiation in E_t . The comma will be reserved to denote coordinate partial differentiation.

We first establish several lemmas that will be required later on.

Lemma 3.1. *Let the vector L_A be defined by*

$$(3.1) \quad L_A = \ell^C \nabla_C \ell_A .$$

We have

$$(3.2) \quad \ell^B \nabla_C \ell_B = 0$$

and

$$(3.3) \quad \ell^B L_B = 0 .$$

Proof. Equation (3.2) follows directly from covariant differentiation of (2.1) in E . Equation (3.3) then follows from contracting (3.2) with ℓ^C and using (3.1).

Lemma 3.2. *Either L_A is a null-vector, in which case*

$$(3.4) \quad L_A = \rho \ell_A ,$$

or L_A forms a spacelike vector field.

Proof. In a 4-dimensional hyperbolic-normal metric space, a vector L_B orthogonal to a null-vector is either spacelike or proportional to that null-vector. When this is combined with (3.3), the result follows.

Let E_t be a generic element of the null bundle of E . Since Ricci's lemma holds in E_t , we have

$$0 = h_{AB;C} = g_{AB;C} + \ell_{A;C} \ell_B + \ell_A \ell_{B;C} .$$

If we expand the covariant derivative in terms of partial derivatives and the components of affine connection Γ_{BC}^A of E_t , we obtain

$$(3.5) \quad 0 = g_{AB,C} - \Gamma_{CA}^E g_{EB} - \Gamma_{CB}^E g_{EA} + \ell_{A,C} \ell_B + \ell_A \ell_{B,C} - \Gamma_{CA}^E \ell_E \ell_B + \ell_{B,C} \ell_A - \Gamma_{CB}^E \ell_E \ell_A .$$

In E , on the other hand, we have

$$(3.6) \quad 0 = \nabla_C g_{AB} = g_{AB,C} - \gamma_{CA}^E g_{EB} - \gamma_{CB}^E g_{EA}$$

and

$$(3.7) \quad \nabla_C \ell_A = \ell_{A,C} - \gamma_{CA}^E \ell_E .$$

Hence, if we set

$$(3.8) \quad T_{CA}^E = \Gamma_{CA}^E - \gamma_{CA}^E ,$$

$$(3.9) \quad Q_{CAB} = \ell_B \nabla_C \ell_A + \ell_A \nabla_C \ell_B ,$$

and eliminate the partial derivatives between (3.5), (3.6), and (3.7), we obtain

$$(3.10) \quad T_{CA}^E h_{EB} + T_{CB}^E h_{EA} = Q_{CAB} .$$

The system of equations (3.10) can be solved for the T 's by Christoffel's elimination method. This yields

$$(3.11) \quad T_{BC}^A = \frac{1}{2} h^{AE} (Q_{BEC} - Q_{ECB} + Q_{CBE}) .$$

Finally, (2.3), (3.1), and (3.9) can be used to evaluate the right-hand side of (3.11) in terms of ℓ_A , $\nabla_A \ell_B$, and L_A . When this is done and (3.2) is used to simplify the result, the following theorem obtains.

Theorem 3.1. *The affine connection Γ_{BC}^A of any element of the null-bundle of E is given by*

$$(3.12) \quad \Gamma_{BC}^A = \gamma_{BC}^A + T_{BC}^A ,$$

where

$$(3.13) \quad T_{BC}^A = \ell^A \{ L_{(B} \ell_{C)} + \nabla_{(B} \ell_{C)} \} + \ell_{(B} \nabla_{C)} \ell^A - \ell_{(B} \nabla^A \ell_{C)} .$$

The following results, which are readily obtained from Theorem 3.1, will be needed later.

Lemma 3.3. *We have*

$$(3.14) \quad 2T_{BC}^A \ell^C = \ell^A L_B + \ell_B L^A ,$$

$$(3.15) \quad 2T_{BC}^A \ell_A = -\ell_C L_B - \ell_B L_C ,$$

$$(3.16) \quad T_{BC}^A \ell^B \ell^C = T_{BC}^A \ell_A \ell^B = T_{BC}^A \ell_A \ell^B \ell^C = T_{AB}^A = 0 .$$

Theorem 3.2. *We have*

$$(3.17) \quad \ell_{A;B} \ell^B = L_A$$

for the element E_i of the null-bundle of E .

Proof. By Theorem 3.1 and the definition of the covariant derivative in any E_i , we have

$$\ell_{A;B} \ell^B = \ell^B \nabla_B \ell_A - T_{BA}^C \ell_C \ell^B .$$

The second term on the right-hand side vanishes by Lemma 3.3. The theorem thus follows when use is made of (3.1).

Theorem 3.3. *The vector field (ℓ_A) is geodesic in the corresponding E_t if and only if it is geodesic in E .*

Proof. A necessary and sufficient condition that (ℓ_A) be geodesic in the corresponding E_t is $\ell_{A;B}\ell^B = \rho\ell_A$ for some scalar function ρ . By Theorem 3.2, we thus have $\rho\ell_A = L_A = \ell^B\nabla_B\ell_A$, which is both necessary and sufficient for (ℓ_A) to be geodesic in E .

Corollary 3.1. *A necessary and sufficient condition for (ℓ_A) to be geodesic in the corresponding E_t is that L_A be a null vector in E ; in which case we have*

$$(3.18) \quad L_A = \rho\ell_A.$$

4. Curvature. The evaluation of Γ_{BC}^A given in Theorem 3.1, together with the well known evaluation of the curvature tensor for such a connection (see [2], p. 141) allows an immediate evaluation of the curvature tensor on the null bundle of E .

Theorem 4.1. *Let $K_{ABC}{}^D$ denote the curvature tensor of E and let $R_{ABC}{}^D$ denote the curvature tensor of a typical element of $\mathcal{N}$. We have*

$$(4.1) \quad R_{ABC}{}^D = K_{ABC}{}^D + 2\nabla_{[A}T_{B]C}^D + 2T_{[A}^DT_{B]C}{}^E.$$

Theorem 4.2. *Set*

$$(4.2) \quad R_{BC} = R_{DBC}{}^D;$$

then

$$(4.3) \quad R_{BC} = K_{BC} + \nabla_E T_{BC}^E - T_{BE}^F T_{FC}^E.$$

Proof. The result follows immediately from (4.1) and (4.2) when note is taken that $T_{AB}^A = 0$ by Lemma 3.3.

Theorem 4.2 provides us with a basis for evaluation of $L^A L_A$ in terms of the Ricci curvatures of E and E_t and the vector field (ℓ_A) .

Theorem 4.3. *For any element of the null bundle of E , we have*

$$(4.4) \quad L_A L^A = (K_{BC} - R_{BC})\ell^B \ell^C.$$

Proof. From (4.3), we have

$$(4.5) \quad R_{BC}\ell^B \ell^C = K_{BC}\ell^B \ell^C + \ell^B \ell^C \nabla_E T_{BC}^E - (\ell^B T_{BE}^F)(\ell^C T_{FC}^E).$$

When (3.14) is used to expand the third term on the right-hand side of this relation, and use is made of (2.1) and (3.3), the result is

$$4(\ell^B T_{BE}^F)(\ell^C T_{FC}^E) = (\ell^F L_E + \ell_E L^F)(\ell^E L_F + \ell_F L^E) = 0.$$

If we substitute the expression given by (3.13) into the second term on the right-hand side of (4.5), and if we make use of (2.1), (3.1), and (3.3) to simplify the resulting expression, we obtain

$$2\ell^B\ell^C\nabla_E T_{BC}^E = \ell^B\ell^C\ell^A\nabla_A(\nabla_B\ell_C + \nabla_C\ell_B).$$

The following calculation now results with the aid of (2.1), (3.1), (3.2), and (3.3):

$$\begin{aligned}\ell^C\ell^B\ell^A\nabla_A\nabla_B\ell_C &= \ell^C\ell^A\nabla_A(\ell^B\nabla_B\ell_C) - (\ell^A\nabla_A\ell^B)\ell^C\nabla_B\ell_C \\ &= \ell^C\ell^A\nabla_AL_C \\ &= \ell^A\nabla_A(\ell^CL_C) - (\ell^A\nabla_A\ell^C)L_C \\ &= -L^CL_C.\end{aligned}$$

We thus have

$$(4.6) \quad 2\ell^B\ell^C\nabla_E T_{BC}^E = -2L^CL_C,$$

from which the theorem follows.

Theorem 4.4. *We have*

$$(4.7) \quad (K_{BC} - R_{BC})\ell^B\ell^C \geq 0,$$

with equality holding if and only if (ℓ_A) is geodesic in both E and the corresponding E_t .

Proof. The result is an immediate consequence of Lemma 3.2, Corollary 3.1, and Theorem 4.3.

The following theorem is the result of a lengthy but straightforward calculation based on Theorems 3.1 and 4.2, and involves extensive use of the relations (2.1), (3.1), (3.2), and (3.3).

Theorem 4.5. *We have*

$$\begin{aligned}(4.8) \quad 2(R_{BC} - K_{BC}) &= 2N_{BC} = L_B L_C - \ell_B\ell_C(L^E L_E - \tfrac{1}{2}\Omega^{EF}\Omega_{EF}) \\ &\quad + \ell_B[(U_E^E + D)L_C + \nabla^E\Omega_{CE} - L^E U_{EC}] \\ &\quad + \ell_C[(U_E^E + D)L_B + \nabla^E\Omega_{BE} - L^E U_{EB}] \\ &\quad + (U_E^E + D)\mathfrak{L}_{BC} + U_{\cdot C}^E\Omega_{BE} + U_{\cdot B}^E\Omega_{CE},\end{aligned}$$

where

$$(4.9) \quad U_{AB} = \nabla_A\ell_B, \quad \Omega_{AB} = 2U_{[AB]}, \quad \mathfrak{L}_{AB} = 2U_{(AB)},$$

and where D denotes the differential operator $\ell^A\nabla_A$.

The results collected below are immediate consequences of the definitions given by (4.9).

Lemma 4.1. *We have*

$$(4.10) \quad 2U_{AB} = \mathfrak{L}_{AB} + \Omega_{AB},$$

$$(4.11) \quad \ell^A\mathfrak{L}_{AB} = \ell^A\Omega_{AB} = L_B,$$

$$(4.12) \quad g^{AB}\mathfrak{L}_{AB} = 2U_E^E.$$

Remark. The notation introduced by (4.9) is to a specific purpose since Ω_{AB} is twice the rotation of the field ℓ_A , while $\mathfrak{L}_{AB}$ is the Lie derivative of g_{AB} with respect to the field ℓ_A . We shall thus be able to characterize the various elements of the null-bundle of E in terms of the rotation of ℓ_A and the deformation of the metric of E under dragging along by ℓ_A . This will significantly facilitate our obtaining certain solutions.

5. The Ricci class of E . An interesting question is suggested by the form of (4.8), namely, when will an element of the null-bundle of E have the same Ricci tensor as E . The answer to this question is the basis by which new solutions of the vacuum Einstein equations may be constructed.

Definition 5.1. The elements of the null-bundle of E that have the same Ricci tensor as E are said to constitute the *Ricci class* of E .

Theorem 5.1. *There exists an element of the null-bundle of E with the same Ricci tensor as E if and only if there exists a null vector field (ℓ_A) on E that satisfies the system of partial differential equations.*

$$(5.1) \quad \nabla_E T_{BC}^E - T_{BE}^F T_{FC}^E = 2N_{BC} = 0.$$

The set of all functionally independent null-vectors on E that are solutions of (5.1) defines the set of all spaces E_i that are members of the Ricci class of E .

Proof. The result is an immediate consequence of Theorem 4.2 and Definition 5.1.

Theorem 5.2. *If E_i is an element of the Ricci class of E , then (ℓ_A) is geodesic in both E and E_i .*

Proof. A straightforward combination of Theorems 4.4, 4.5, and 5.1 leads to the conclusion stated.

Theorem 5.3. *The system (5.1) is equivalent to the system*

$$(5.2) \quad L_A = \rho \ell_A,$$

$$(5.3) \quad 0 = \ell_B \ell_C (\rho^2 + \frac{1}{2} \Omega_{EF}^E \Omega_{EF} + 2(U_E^E + D)\rho) \\ + (U_E^E + D)\mathfrak{L}_{BC} + (U_C^E + \ell_C \nabla^E)\Omega_{BE} + (U_B^E + \ell_B \nabla^E)\Omega_{CE}.$$

Proof. By Theorem 5.1, E_i is an element of the Ricci class of E if and only if (5.1) is satisfied, and hence (ℓ_A) is geodesic in E by Theorem 5.2. It thus follows from corollary (3.1) that (5.2) holds. A combination of equations (4.3) and (4.8) shows that the system (5.1) is equivalent to setting the right-hand side of (4.8) equal to zero. When (5.2) is used in conjunction with (2.1), (3.1), and (4.9) to simplify the resulting equations, the system (5.3) is obtained.

Theorem 5.4. *If E_i is an element of the Ricci class of E , then*

$$(5.4) \quad (U_E^E + D)(\rho + U_F^F) = 0.$$

Proof. By Theorem 5.1 and 5.3, the hypothesis is satisfied if and only if (ℓ_A) satisfies (5.2) and (5.3). If we contract (5.3) with g^{BC} and use (2.1) and (4.9), we obtain

$$0 = 2\ell^B \nabla^E \Omega_{BE} - \Omega^{EF} \Omega_{EF} + g^{BC}(U_E^E + D)\mathfrak{L}_{BC}.$$

The result then follows on noting that

$$(5.5) \quad g^{BC}(U_E^E + D)\mathfrak{L}_{BC} = 2(U_E^E + D)U_F^F$$

and

$$(5.6) \quad \ell^B \nabla^E \Omega_{BE} = (U_E^E + D)\rho + \frac{1}{2}\Omega^{EF} \Omega_{EF},$$

which are easily established with use of (3.2), (4.11), and Lemma 4.1.

Theorem 5.5. *If E_t is an element of the Ricci class of E , then*

$$(5.7) \quad \rho^2 + \frac{1}{2}\Omega^{EF} \Omega_{EF} + 2(U_E^E + D)\rho = 0.$$

Proof. By the same reasoning as in the proof of Theorem 5.4, (ℓ_A) must satisfy (5.2) and (5.3). If we contract (5.3) with ℓ^B and use (2.1) and (3.2), the result is

$$0 = \ell_C \ell^B \nabla^E \Omega_{BE} + \ell^B(U_E^E + D)\mathfrak{L}_{BC} + U_E^E \ell^B \Omega_{BE}.$$

When (3.1), (3.2), (4.9), (5.2), and Lemma 4.1 are used, we easily establish that

$$(5.8) \quad \ell^B(U_E^E + D)\mathfrak{L}_{BC} = \ell_C(U_E^E + D)\rho,$$

$$(5.9) \quad U_E^E \ell^B \Omega_{BE} = \rho^2 \ell_C,$$

$$(5.10) \quad \ell^B \nabla^E \Omega_{BE} = (U_E^E + D)\rho + \frac{1}{2}\Omega^{EF} \Omega_{EF},$$

from which the result follows.

Theorem 5.6. *If E_t is a member of the Ricci class of E , then*

$$(5.11) \quad 2(U_E^E + D)U_F^F - \frac{1}{2}\Omega^{EF} \Omega_{EF} \geq 0$$

and

$$(5.12) \quad \rho = (2(U_E^E + D)U_F^F - \frac{1}{2}\Omega^{EF} \Omega_{EF})^{1/2}.$$

Proof. The result follows immediately from (5.4) and (5.7) and the fact that ρ is a real scalar function.

Theorem 5.7. *The space E_t is a member of the Ricci class of E if and only if*

$$(5.13) \quad L_A = \rho \ell_A,$$

$$(5.14) \quad (\ell_B \nabla^E + U_{\cdot B}^E)\Omega_{CE} + (\ell_C \nabla^E + U_{\cdot C}^E)\Omega_{BE} + (U_E^E + D)\mathfrak{L}_{BC} = 0.$$

Proof. Theorems 5.1 and 5.3 show that the system (5.2) and (5.3) constitutes both necessary and sufficient conditions for E_t to be an element of the Ricci class of E . When Theorem 5.5 is used, it is seen that the system (5.3)

reduces to the system (5.14). Thus, since the system (5.14) is easily seen to imply the condition (5.4), on contraction with g^{bc} , and to imply (5.7) when contracted with ℓ^b , the result is established.

We now inquire into the connectedness of the Ricci class of E in the null-bundle of E .

Definition 5.2. The set of all elements of the null bundle of E whose metric tensors are of the form

$$(5.15) \quad h_{AB}(\lambda) = g_{AB} + \lambda^2 \ell_A \ell_B,$$

where λ is a real number and (ℓ_A) is a fixed null-vector field in E , is referred to as a λ -ray.

Theorem 5.8. *If E_i is an element of the Ricci class of E , then every element of the λ -ray that contains E_i is also an element of the Ricci class of E .*

Proof. Let the metric tensor of E_i be $h_{AB} = g_{AB} + \ell_A \ell_B$. The null-vector field defining the λ -ray containing E_i may then be identified by the requirement $h_{AB}(1) = h_{AB}$. With (ℓ_A) thus fixed for the λ -ray, define the null-vector field (ℓ_A^*) by the relations $\ell_A^* = \lambda \ell_A$, and denote all quantities obtained from (ℓ_A^*) by a superimposed asterisk. Since E_i is an element of the Ricci class of E by hypothesis, Theorem 5.5. shows that ρ is a homogeneous operator of degree one on the (ℓ_A) 's, i.e., $\rho(\lambda \ell_A) = \lambda \rho(\ell_A)$. It then follows from the definition of L_A that $L_A - \rho \ell_A$ and the left-hand side of (5.14) are homogeneous operators on (ℓ_A) of degree two. Consequently, the quantities $\ell_A^* = \lambda \ell_A$ and $\rho^* = \lambda \rho$ constitute a solution of the system (5.13) and (5.14) if ℓ_A and ρ do. It then follows from Theorem 5.7 that the element of the null-bundle of E with the metric tensor $h_{AB}^* = g_{AB} + \ell_A^* \ell_B^* = h_{AB}(\lambda)$ is an element of the Ricci class of E for any λ .

Remark. It is of interest to note that the expression N_{BC} given by (4.8) is not a homogeneous system of operators on the (ℓ_A) 's. It is only when we equate N_{BC} to zero that the system can be decomposed into an equivalent system, each of whose elements is a homogeneous operator on the (ℓ_A) 's.

Corollary 5.1. *The Ricci class of E is comprised solely of λ -rays in the null bundle of E .*

Theorem 5.9. *If E_i is an element of the Ricci class of E , then the tensor T_{BC}^A that serves to define the affine connection in E_i by (3.12) is given by*

$$(5.16) \quad T_{BC}^A = \ell^A \{ \rho \ell_B \ell_C + \nabla_{(B} \ell_{C)} \} - \ell_{(B} \nabla^A \ell_{C)} + \ell_{(B} \nabla_{C)} \ell^A.$$

Proof. A direct combination of (3.13) and (5.13) gives (5.16).

We conclude this section with the following result that will be required later on.

Theorem 5.10. *If E_i is an element of the Ricci class of E , then ℓ_A is a simultaneous eigenvector of $\mathfrak{L}_{AB}$ and Ω_{AB} with eigenvalue ρ :*

$$(5.17) \quad \ell^A \mathfrak{L}_{AB} = \rho \ell_B = \ell^A \Omega_{AB}.$$

Proof. Under the hypothesis, Theorem 5.7 gives $L_A = \rho \ell_A$. The result then follows from Lemma 4.1.

6. The field equations. Throughout the remainder of this paper, we shall assume that the Einstein field equations hold over the null-bundle of E . We consequently have

$$(6.1) \quad K_{AB} - \frac{1}{2}K g_{AB} = \kappa T_{AB}, \quad K = K_{AB} g^{AB},$$

in E , where T_{AB} is the associated momentum-energy tensor of E . Since the Einstein field equations hold on the null bundle of E , the momentum-energy tensor T_{AB}^* associated with an element E_i of $\mathfrak{N}$ is consequently defined by the relations

$$(6.2) \quad R_{AB} - \frac{1}{2}R h_{AB} = \kappa T_{AB}^*, \quad R = R_{AB} h^{AB}.$$

Theorem 6.1. *We have*

$$(6.3) \quad \begin{aligned} \kappa T_{AB}^* = \kappa T_{AB} + N_{AB} - \frac{1}{2}(N_C^C - \kappa T_{CD} \ell^C \ell^D - N_{CD} \ell^C \ell^D) g_{AB} \\ + \frac{1}{2}(\kappa T + \kappa T_{CD} \ell^C \ell^D - N_C^C + N_{CD} \ell^C \ell^D) \ell_A \ell_B, \end{aligned}$$

where

$$(6.4) \quad T = T_{AB} g^{AB}.$$

Proof. Combining (6.2) with (2.2) and (2.3), we have

$$\kappa T_{AB}^* = R_{AB} - \frac{1}{2}R_{CD}(g^{CD} - \ell^C \ell^D)(g_{AB} + \ell_B \ell_C).$$

If we eliminate the R_{CD} 's by means of equations (4.8) and then express the K_{CD} 's in terms of the T_{CD} 's by means of (6.1), the result follows. Since N_{AB} is given explicitly in terms of (ℓ_A) and its derivatives by Theorem 4.5, the above result establishes the desired explicit representation of T_{AB}^* in terms of T_{AB} , g_{AB} , ℓ_A and its derivatives.

7. The energy class of E . We now inquire as to the conditions under which an element of the null-bundle of E will have the same momentum-energy tensor as E . These will be of use in constructing nonvacuum solutions of the Einstein field equations.

Definition 7.1. The elements of the null-bundle of E that have the same momentum-energy tensor as E are said to constitute the *energy class* of E .

Theorem 7.1. *There exists an element of the null bundle of E with the same momentum-energy tensor as E if and only if there exists a null vector field (ℓ_A) on E that satisfies the system of partial differential equations*

$$(7.1) \quad \begin{aligned} 0 = N_{AB} - \frac{1}{2}(N_C^C - \kappa T_{CD} \ell^C \ell^D - N_{CD} \ell^C \ell^D) g_{AB} \\ + \frac{1}{2}(\kappa T + \kappa T_{CD} \ell^C \ell^D - N_C^C + N_{CD} \ell^C \ell^D) \ell_A \ell_B. \end{aligned}$$

Proof. The result is an immediate consequence of Theorem 6.1.

The following results are required in order to obtain inferences from the above theorem.

Lemma 7.1. *We have*

$$(7.2) \quad N_{CD}\ell^C\ell^D = -L_C L^C,$$

$$(7.3) \quad 2N_C^C = 2\nabla^E L_E - L_C L^C + 2(U_E^E + D)U_F^F,$$

$$(7.4) \quad 2N_{BC}\ell^C = \ell_B(\nabla^E L_E - L_C L^C + \frac{1}{2}\Omega^{EF}\Omega_{EF}) + (U_E^E + D)L_B - L^C U_{BC}.$$

Proof. The results follow from (4.8) when (3.2), (3.3), (4.4), and (4.9) are used.

Theorem 7.2. *Necessary and sufficient conditions that E_i be an element of the energy class of E is that the null vector field (ℓ_A) satisfy the system of partial differential equations*

$$(7.5) \quad \begin{aligned} 2\kappa T_{CD}\ell^C\ell^D &= N_C^C - 2N_{CD}\ell^C\ell^D \\ &= \frac{3}{2}L_C L^C + \nabla^E L_E + (U_E^E + D)U_F^F, \end{aligned}$$

$$(7.6) \quad \begin{aligned} 0 &= \frac{1}{2}\ell_A(\nabla^E L_E - \frac{3}{2}L_C L^C + \Omega^{EF}\Omega_{EF} - (U_E^E + D)U_F^F) \\ &\quad + (U_E^E + D)L_A - L^C U_{AC}, \end{aligned}$$

$$(7.7) \quad 0 = N_{AB} + \frac{1}{2}\kappa T\ell_A\ell_B - \frac{1}{4}N_C^C(g_{AB} + \ell_A\ell_B).$$

Proof. The condition (7.5) is obtained by contracting (7.1) with g^{AB} and using (2.1), (3.2), (3.3), (4.4) and (4.9) in conjunction with the results of Lemma 7.1. Similarly, (7.6) is obtained from (7.1) when this equation is contracted with ℓ^B . The system (7.7) then results when (7.5) and (7.6) are used to simplify (7.1).

Theorem 7.3. *The system (7.6) implies*

$$(7.8) \quad L_C L^C = 0,$$

and hence we have

$$(7.9) \quad L_C = \rho\ell_C.$$

Proof. A contraction of (7.6) with ℓ^A and use of (2.1), (3.1), and (4.9) gives the relation $\ell^A(U_E^E + D)L_A - L_C L^C = 0$. Now,

$$\ell^A(U_E^E + D)L_A = \ell^A D L_A = D(\ell^A L_A) - L_A L^A = -L_A L^A$$

by (3.1) and (3.3), and hence we obtain (7.8). The relations (7.9) follow directly from (7.8) and Lemma 3.2.

Theorem 7.4. *Necessary and sufficient conditions for E_i to be an element of the energy class of E are*

$$(7.10) \quad L_A = \rho \ell_A,$$

$$(7.11) \quad 2\kappa T_{CD} \ell^C \ell^D = (U_{E^{\cdot}}^E + D)(\rho + U_{F^{\cdot}}^F),$$

$$(7.12) \quad 0 = 2\rho^2 + \Omega^{EF} \Omega_{EF} + (U_{E^{\cdot}}^E + D)(3\rho - U_{F^{\cdot}}^F),$$

$$(7.13) \quad 0 = \kappa T \ell_A \ell_B - \frac{1}{2} g_{AB} (U_{E^{\cdot}}^E + D)(\rho + U_{F^{\cdot}}^F) + (U_{E^{\cdot}}^E + D) \mathfrak{L}_{AB} \\ + (\ell_A \nabla^E + U_{E^{\cdot}}^E) \Omega_{BE} + (\ell_B \nabla^E + U_{E^{\cdot}}^E) \Omega_{AE}.$$

Proof. We have established (7.10) in Theorem 7.3. When (7.10) is substituted into (7.5) and (2.1), (3.2), (4.4), and (4.9) are used, we obtain (7.11). The relation (7.12) is obtained in the same way when (7.10) is substituted into (7.6) and note is taken that ℓ_A is assumed to have at least one nonvanishing component. We next note that the right-hand side of (5.3) is just the explicit evaluation of N_{AB} under the condition (7.10). Hence, substituting (7.10) into (7.7) and using this evaluation, we establish (7.13).

Theorem 7.5. *If E_i is a member of the energy class of E , then*

$$(7.14) \quad (U_{E^{\cdot}}^E + D)(2U_{F^{\cdot}}^F - 3\kappa T_{CD} \ell^C \ell^D) - \frac{1}{2} \Omega^{EF} \Omega_{EF} \geq 0,$$

and

$$(7.15) \quad \rho = ((U_{E^{\cdot}}^E + D)(2U_{F^{\cdot}}^F - 3\kappa T_{CD} \ell^C \ell^D) - \frac{1}{2} \Omega^{EF} \Omega_{EF})^{1/2}.$$

Proof. The result follows from Theorem 7.4 when we combine (7.11) and (7.12) and note that ρ is a real-valued scalar function.

We next establish the connectivity of the elements of the energy class of E .

Theorem 7.6. *If E_i is an element of the energy class of E , then every element of the λ -ray containing E_i is also a member of the energy class of E .*

Proof. The proof follows the lines of the proof of Theorem 5.8 when (7.15) is used to show that the system of equations (7.10) through (7.13) is defined in terms of homogeneous operators on the (ℓ_A) 's.

8. Intersection of the Ricci and energy classes. We now investigate the conditions under which an element of the null-bundle of E can be a member of both the Ricci class and of the energy class of E .

Theorem 8.1. *Let E_i be a member of the Ricci class of E . Necessary and sufficient conditions that E_i is also a member of the energy class of E are*

$$(8.1) \quad \kappa T = -K = 0,$$

$$(8.2) \quad \kappa T_{CD} \ell^C \ell^D = K_{CD} \ell^C \ell^D = 0.$$

Proof. If E_i is an element of the Ricci class of E , then Theorem 5.1 shows that the null-vector field (ℓ_A) satisfies the system $N_{AB} = 0$. Theorem 7.2 then leads to the results $T = T_{CD} \ell^C \ell^D = 0$, from which the corresponding curvature conditions are obtained by (6.1) and (2.1).

Theorem 8.2. *Let E_i be a member of the energy class of E . Necessary and sufficient conditions that E_i be a member of the Ricci class of E are*

$$(8.3) \quad K = K_{cD} \ell^c \ell^D = 0.$$

Proof. The result is an obvious combination of Theorems (5.4), (5.5), (5.6), and (7.4).

Theorem 8.3. *If E_i is a member of both the Ricci class and of the energy class of E , then every element of the λ -ray containing E_i also satisfies these conditions.*

Proof. An immediate combination of Theorems 5.8, 7.6, 8.1, and 8.2 establishes the result.

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Date Communicated: JANUARY 31, 1966