*Complex Hilbertian Structures on Stable Linear Dynamical Systems**

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The phase space of a linear dynamical system is a real linear space $\mathfrak{M}$, furnished with a skew-symmetric, non-degenerate, bilinear form B , the Lagrange bracket. Motion of the system is represented by a group $\mathcal{G}$ of symplectic transformations. The purpose of this paper is to give necessary and sufficient conditions that there exist in $\mathfrak{M}$ the structure of a complex Hilbert space such that B is the imaginary part of the inner product and $\mathcal{G}$ is unitary.

A linear transformation S is symplectic [1, 2] if and only if it leaves B invariant: $B(\cdot, \cdot) = B(S(\cdot), S(\cdot))$. When $\dim \mathfrak{M}$ is finite there exists a complete structure theory for symplectic transformations [3, 4, 5, 6, 7]. Unlike the analogous theory for orthogonal transformations, symplectics need not be semi-simple, the λ -matrix may have non-linear elementary divisors. But they cannot occur with a stable symplectic, one such that $\bigcup_{-\infty}^{+\infty} S^n X$ is a bounded set for any fixed X .

The structure of stable symplectics on finite dimensional spaces has been studied by Krein [8], Gelfand & Lidskii [9], and Moser [10] in work of considerable practical importance. In this paper will be given a proof (as outlined in a previous announcement [11]) of the main theorem for the infinite dimensional case: If $\mathcal{G}$ is abelian (or compact, or solvable, or, more generally, "amenable" [12]) and stable, a linear transformation j of $\mathfrak{M}$ can be found to serve as square root of minus one, $j^2 = -1$, such that $\langle \cdot, \cdot \rangle = B(j(\cdot), \cdot) + (-1)^{1/2} B(\cdot, \cdot)$ is hermitian definite and the elements of $\mathcal{G}$ are all unitary with respect to the resulting complex structure. Then the spectral theorem for unitary operators is the desired structure theorem. When $\mathcal{G}$ is time-translation, the dynamical system has been broken up into a direct integral of harmonic oscillators, the simplest of stable systems.

The immediately available natural topology is that defined as the weakest with respect to which B is continuous separately in each of its arguments [13, 14], *i.e.*, the weakest topology in which all linear functionals $B(X, \cdot)$ are continuous

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for any fixed X . Then $\mathfrak{M}$ will be a locally convex space in duality with itself in this topology, which will be called the B -topology. Since the weak topology on Hilbert space is determined by the imaginary part of the inner product alone, and since in that topology it is quasi-complete (every bounded, closed subset is complete [13]), we must require that the B -topology be quasi-complete.

But this is still not sufficient for the main theorem. An additional condition must be imposed on B . Its non-degeneracy must be uniform over arbitrarily large finite-dimensional subspaces in a sense to be discussed in the Appendix. Here a definition of uniform non-degeneracy will be given which, though adequate for many applications [11], should actually be a lemma.

Definition 1. B is *uniformly non-degenerate* if and only if there exists some real inner product $\langle \cdot, \cdot \rangle_r$ whose weak topology is at least as strong as the B -topology.

If, as may happen, the weak $\langle \cdot, \cdot \rangle_r$ -topology is properly stronger than a quasi-complete B -topology, then $\langle \cdot, \cdot \rangle_r$ can be replaced by a smaller inner product whose weak topology is exactly the B -topology, as follows:

Complete the pre-Hilbert space $\{\mathfrak{M}, \langle \cdot, \cdot \rangle_r\}$ to the Hilbert space $\{\mathfrak{M}', \langle \cdot, \cdot \rangle_r\}$, with extended $\langle \cdot, \cdot \rangle_r$. Then $B(X, Y) = \langle A_r X, Y \rangle_r$ for some linear transformation A_r with domain $\mathfrak{M}$. Let P_r be the projection of $\mathfrak{M}'$ onto the closure of $A_r \mathfrak{M}$. Since $\langle A_r X, Y \rangle_r = \langle A_r X, P_r Y \rangle_r$, P_r is a homeomorphism between the B -topology on $\mathfrak{M}$ and the weakest topology on $P_r \mathfrak{M}$ such that the functionals $\langle A_r \mathfrak{M}, \cdot \rangle_r$ are all continuous. Hence $P_r \mathfrak{M}$ is quasi-complete in that weakest topology. But on intersections of $P_r \mathfrak{M}$ with closed spheres that topology is identical with the weak topology since $A_r \mathfrak{M}$ is strongly dense in $P_r \mathfrak{M}'$. So those intersections are weakly and therefore strongly complete. $P_r \mathfrak{M}$ is strongly complete and closed in $\mathfrak{M}'$. $\mathfrak{M}$ is dense in $\mathfrak{M}'$, so $P_r \mathfrak{M} = P_r \mathfrak{M}'$.

$A_r \mathfrak{M}$ is dense in $P_r \mathfrak{M}'$ by definition. A_r is a homeomorphism between the B -topology on $\mathfrak{M}$ and the relative topology on $A_r \mathfrak{M}$ induced by the weak topology on $\mathfrak{M}'$. As above, $A_r \mathfrak{M}$ is closed, and hence $A_r \mathfrak{M} = P_r \mathfrak{M}'$.

Therefore, $\langle P_r(\cdot), P_r(\cdot) \rangle_r$ can be taken as the desired, smaller, inner product on $\mathfrak{M}$. So, *we will henceforth assume $\langle \cdot, \cdot \rangle_r$ to be such that its weak topology is the B -topology.* (I am indebted to J. M. Chaiken for a remark which led to the discovery of an error in my first discussion of this point.)

We will also assume that $\mathfrak{M}$ is a phase space:

Definition 2. $\mathfrak{M}$ is a *linear phase space* if and only if B is uniformly non-degenerate and the B -topology is quasi-complete.

$\mathfrak{M}$ can be extended to its complexification $\mathfrak{M}^\circ = \mathfrak{M} \oplus \mathfrak{M}$ with the action of i defined by $i(X \oplus Y) = (-Y) \oplus X$ [15]. Then the real inner product $\langle \cdot, \cdot \rangle_r$ of Definition 1 extends to a complex Hermitian inner product

$$\begin{aligned} \langle \cdot, \cdot \rangle_\circ &= \langle X_1 \oplus Y_1, X_2 \oplus Y_2 \rangle_\circ \\ &= \langle X_1, X_2 \rangle_r + \langle Y_1, Y_2 \rangle_r + i(\langle Y_1, X_2 \rangle_r - \langle X_1, Y_2 \rangle_r), \end{aligned}$$

and B extends similarly.

Now the argument will go as follows: By averaging in a certain way $\langle S\cdot, S\cdot \rangle$, over $\mathfrak{G}$, $\langle \cdot, \cdot \rangle_0$ will be replaced by a Hermitian inner product $\langle \cdot, \cdot \rangle_0$ with respect to which all $S \oplus S$ are unitary. On this complex Hilbert space, with conjugation $\overline{X \oplus Y} = X \oplus (-Y)$, $iB(\cdot, \cdot)$ will be a non-degenerate bounded Hermitian form equal to $\langle A\cdot, \cdot \rangle_0$ for some self-adjoint A . The positive part of A will define a positive Hermitian form, bounded away from zero, on a subspace of $\mathfrak{M}^\circ$ reducing all $S \oplus S$. This subspace will be in one-to-one correspondence $X \oplus Y \leftrightarrow X + Y$ with $\mathfrak{M}$, putting its Hermitian structure on $\mathfrak{M}$ so that the action of S is unitary and the imaginary part of the complex inner product is B .

Before averaging to obtain the invariant $\langle \cdot, \cdot \rangle_0$ it is necessary to prove that stability as we have defined it

Definition 3. $\mathfrak{G}$ is *stable* if and only if

$$\sup_{S \in \mathfrak{G}} B(SX, Y) < \infty,$$

for all X, Y .

implies a stronger definition.

Lemma 1. $\mathfrak{G}$ is *stable* if and only if $\bigcup_{\mathfrak{G}} S\mathfrak{B}$ is bounded whenever $\mathfrak{B}$ is.

Proof. To say that $\bigcup_{\mathfrak{G}} S\mathfrak{B}$ is bounded (in the B -topology) is to say that $B(\bigcup_{\mathfrak{G}} S\mathfrak{B}, Y)$ is, for any fixed Y , a bounded set of real numbers. Since any one-element set $\mathfrak{B} = \{X\}$ is certainly bounded, the “if” assertion of the lemma is obvious.

Conversely, assume that $\bigcup_{\mathfrak{G}} SX$ is bounded for any X , and let $\mathfrak{B}$ be any bounded subset of $\mathfrak{M}$. Suppose $\bigcup_{\mathfrak{G}} S\mathfrak{B}$ were not bounded. Then there would exist a Y and sequences $X_n \in \mathfrak{B}$ and $S_n \in \mathfrak{G}$ such that $B(S_n X_n, Y) = B(X_n, S_n^{-1} Y) \uparrow \infty$. But $\{X_n\}$ and $\{S_n^{-1} Y\}$ are weakly bounded, so $\|X_n\|_r \leq b_1$ and $\|S_n^{-1} Y\|_r \leq b_2$ by Banach’s Uniform Boundedness theorem. But then $|B(X_n, S_n^{-1} Y)| \leq (\text{const.}) \cdot b_1 \cdot b_2$, since B is a continuous bilinear functional on the real Hilbert space $\{\mathfrak{M}, \langle \cdot, \cdot \rangle_r\}$, contradicting $\uparrow \infty$.

Now, for a class including abelian and compact $\mathfrak{G}$, there exists an invariant inner product on $\mathfrak{M}$ topologically equivalent to $\langle \cdot, \cdot \rangle_r$, [12]. It extends to the Hermitian form $\langle \cdot, \cdot \rangle_0$ on $\mathfrak{M}^\circ$ as above. $S \oplus S$ is unitary on the complex Hilbert space $\{\mathfrak{M}^\circ, \langle \cdot, \cdot \rangle_0\}$.

Lemma 2. *There exists a bounded self-adjoint A on $\{\mathfrak{M}^\circ, \langle \cdot, \cdot \rangle_0\}$, with bounded inverse, such that $iB(\cdot, \cdot) = \langle A(\cdot), \cdot \rangle_0$.*

Proof. The bounded self-adjoint A exists because $iB(\cdot, \cdot)$ is a bounded hermitian form.

A^{-1} exists because the non-degeneracy of B prevents zero from being in the point-spectrum of A . To complete the proof we must show that A^{-1} is bounded, i.e., that its domain is $\mathfrak{M}^\circ$, i.e., that the range of A is all of $\mathfrak{M}^\circ$.

Let Z be arbitrary in $\mathfrak{M}^\circ$. Since the range of A is dense, there exist Z_n in $\mathfrak{M}^\circ$ such that $\|AZ_n - Z\|_0 \rightarrow 0$. But then $B(Z_n, \bar{Z}') = -i\langle AZ_n, Z' \rangle$ converges for

all Z' in $\mathfrak{M}^c$. Z_n is a Cauchy sequence in the B -topology on $\mathfrak{M}^c$. By the quasi-completeness there exists a Z_∞ such that $B(Z_n, \bar{Z}') \rightarrow B(Z_\infty, \bar{Z}')$ for all Z' . Therefore, $\langle AZ_\infty, \cdot \rangle_0 = iB(Z_\infty, \bar{\cdot}) = \lim iB(Z_n, \bar{\cdot}) = \lim \langle AZ_n, \cdot \rangle_0 = \langle Z, \cdot \rangle_0$. So $AZ_\infty = Z$. Z is in the range of A .

Let $A = A^+ + A^-$ be the decomposition of A into its positive and negative parts, and $\mathfrak{M}^c = \mathfrak{M}^+ + \mathfrak{M}^-$ the corresponding direct-sum decomposition of $\mathfrak{M}^c$.

For convenience in the following proof, use the temporary notation $JZ = \bar{Z}$ for the conjugation $Z \rightarrow \bar{Z}$.

Lemma 3. $J\mathfrak{M}^+ = \mathfrak{M}^-$.

Proof. $\langle AZ_1, Z_2 \rangle_0 = iB(Z_1, \bar{Z}_2)$ is pure imaginary for all real Z_1, Z_2 , and also for all pure imaginary Z_1, Z_2 .

Therefore,

$$\frac{I+J}{2} A \frac{I+J}{2} = 0,$$

since

$$\langle AZ_1, Z_2 \rangle_0 = \left\langle A \frac{I+J}{2} Z, Z_2 \right\rangle_0$$

would have a real part for some real Z_2 if

$$A \frac{I+J}{2} Z$$

did.

Similarly,

$$\frac{I-J}{2} A \frac{I-J}{2} = 0.$$

From these two equations, $JAJ + JA + AJ + A = 0$ and $JAJ - JA - AJ + A = 0$, and hence $JAJ = -A$.

Now let p_n be real polynomials, which exist by the Weierstrass approximation theorem, such that $|p_n(\lambda) - (\lambda)^{1/2}| \rightarrow 0$ uniformly on the spectrum of A^2 . Then $p_n(A^2) \rightrightarrows |A|$ (uniformly). But $Jp_n(A^2)J = p_n(JA^2J) = p_n(A^2)$. So $J|A|J = |A|$. Then

$$JA^+J = J \frac{A + |A|}{2} J = \frac{-A + |A|}{2} = -A^-.$$

Since zero is not in the spectrum of A , this implies the Lemma.

Lemma 4. The mapping $X \oplus Y \rightarrow X + Y$ is one-to-one from $\mathfrak{M}^+$ onto $\mathfrak{M}$.

Proof. If $X \oplus Y \rightarrow 0$, then $Y = -X$. But $iB(X \oplus (-X), \overline{X \oplus (-X)}) = i(1 - i^2)B(X, X) = 0$. So $X \oplus Y = 0 \oplus 0$ by the definiteness of $iB(\cdot, \cdot)$ on $\mathfrak{M}^+$, and the null space of " $\rightarrow$ " is $0 \oplus 0$.

Let X be arbitrary in $\mathfrak{M}$. Then $X \oplus 0 = (X_1 \oplus Y_1) + (X - X_1) \oplus (-Y_1)$ with $X_1 \oplus Y_1$ in $\mathfrak{M}^+$ and $(X - X_1) \oplus (-Y_1)$ in $\mathfrak{M}^-$. But then

$$\overline{(X - X_1) \oplus (-Y_1)} = (X - X_1) \oplus Y_1$$

is in $\mathfrak{M}^+$ by Lemma 3, and so $i((X - X_1) \oplus Y_1) = (-Y_1) \oplus (X - X_1)$ is in $\mathfrak{M}^+$ also. Therefore $(X_1 \oplus Y_1) + (-Y_1) \oplus (X - X_1)$ is in $\mathfrak{M}^+$. But this maps to $X_1 + Y_1 + (-Y_1) + (X - X_1) = X$. Hence " $\rightarrow$ " is onto.

Definition 4. j is the linear transformation of $\mathfrak{M}$ determined by $j(X + Y) = -Y + X$, where X and Y are uniquely determined by $X + Y$ so that $X \oplus Y$ is in $\mathfrak{M}^+$.

Lemma 5. j is symplectic.

Proof. $B(\mathfrak{M}^+, \mathfrak{M}^+) = B(\mathfrak{M}^+, \overline{\mathfrak{M}^-}) = \{0\}$, so $B(X_1, Y_2) + B(Y_1, X_2) = 0 = \text{Im } B(X_1 \oplus Y_1, X_2 \oplus Y_2)$ for all $X_i \oplus Y_i$ in $\mathfrak{M}^+$.

Therefore,

$$\begin{aligned} B(X_1 + Y_1, X_2 + Y_2) - B(-Y_1 + X_1, -Y_2 + X_2) \\ = 2B(X_1, Y_2) + 2B(Y_1, X_2) = 0 \end{aligned}$$

for all $X_i \oplus Y_i$ in $\mathfrak{M}^+$, i.e., $B(X_1 + Y_1, X_2 + Y_2) = B(j(X_1 + Y_1), j(X_2 + Y_2))$ for all $X_i + Y_i$ in $\mathfrak{M}$.

Corollary. j is skew.

Proof. $B(jX', X'') = B(j^2X', jX'') = B(-X', jX'') = B(X', -jX'')$.

Definition 5. The complex-valued bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{M}$ is given by $\langle X', X'' \rangle = B(jX', X'') + iB(X', X'')$.

Lemma 6. $\langle \cdot, \cdot \rangle$ is a positive definite, j -complex, Hermitian form.

Proof. For all X', X'', X, Y in $\mathfrak{M}$, and complex $\alpha (= a + jb$ on $\mathfrak{M}$), we have

$$(1) \quad \langle X' + X'', X \rangle = \langle X', X \rangle + \langle X'', X \rangle.$$

$$(2) \quad \langle \alpha X', X'' \rangle = \alpha \langle X', X'' \rangle.$$

(By (1) we can assume $\alpha = j$. Then

$$\langle jX', X'' \rangle = -B(X', X'') + iB(jX', X'') = i\langle X', X'' \rangle.)$$

$$(3) \quad \langle X', X'' \rangle = \overline{\langle X'', X' \rangle}.$$

$$\begin{aligned} (B(jX', X'') + iB(X', X'')) &= -B(X'', jX') - iB(X'', X') \\ &= B(jX'', X') + iB(X'', X') \end{aligned}$$

by the skewness of j .)

$$(4) \quad \langle X', X' \rangle > 0 \quad \text{if } X' \neq 0.$$

$$\begin{aligned} B(j(X + Y), X + Y) &= B(-Y + X, X + Y) \\ &= 2B(X, Y) = iB(X \oplus Y, \overline{X \oplus Y}), \end{aligned}$$

which is > 0 if $X \oplus Y$ (in $\mathfrak{M}^+$) is $\neq 0$, i.e., if $X + Y = X' \neq 0$.

Theorem. *Given a stable, amenable group of symplectics acting on a linear phase space, there exists a complex Hilbertian structure with respect to which all the symplectics are unitary and such that the imaginary part of the inner product is the Lagrange bracket.*

Proof.

$$\begin{aligned} \langle A(S \oplus S)Z_1, Z_2 \rangle_0 &= iB((S \oplus S)Z_1, \bar{Z}_2) = iB(Z_1, \overline{(S^{-1} \oplus S^{-1})Z_2}) \\ &= \langle AZ_1, (S^{-1} \oplus S^{-1})Z_2 \rangle_0 = \langle (S \oplus S)AZ_1, Z_2 \rangle_0 \quad \text{for all } Z_1, Z_2 \text{ in } \mathfrak{M}^c. \end{aligned}$$

Hence $A(S \oplus S) = (S \oplus S)A$. $S \oplus S$ is completely reduced by $\mathfrak{M}^+$. The action of $S \oplus S$ on $\mathfrak{M}^c$ can be transported by " $\rightarrow$ " onto $\mathfrak{M}$ to agree with the action of S .

$$\begin{array}{ccc} X \oplus Y & \rightarrow & X + Y \\ S \oplus S \downarrow & & S \downarrow \\ SX \oplus SY & \rightarrow & SX + SY \end{array}$$

$S \oplus S$ commutes with i on $\mathfrak{M}^+$, so it must commute with j on $\mathfrak{M}$. Therefore S is unitary on $\{\mathfrak{M}, j, \langle \cdot, \cdot \rangle\}$.

APPENDIX. UNIFORM NON-DEGENERACY

There exist B which are not uniformly non-degenerate even when the B -topology is quasi-complete:

Let $\mathfrak{X}_p$, for some $1 < p < 2$, be the reflective Banach space of all real-valued functions $x(n)$ on the integers $0, \pm 1, \pm 2, \dots$, with finite norm

$$||X||_p = \sqrt[p]{\sum_{n=-\infty}^{+\infty} |x(n)|^p}.$$

It is dual, via the bilinear form $\sum_{n=-\infty}^{+\infty} x(n)y(n)$, to the space $\mathfrak{X}_q$ with $q = p/(p-1)$. Define $\mathfrak{M}$, B and S by

$$\mathfrak{M} = \mathfrak{X}_p \oplus \mathfrak{X}_q, B(x_1 \oplus y_1, x_2 \oplus y_2) = \sum_{n=-\infty}^{+\infty} (x_1(n)y_2(n) - x_2(n)y_1(n)),$$

and $(S(x \oplus y))(n) = (x \oplus y)(n-1)$. If S were unitary with respect to some complex Hilbertian structure on $\mathfrak{M}$, then it would be spectral [16]. But the stationary sequence $\{S^n(x_0 \oplus 0)\}_{n=-\infty}^{+\infty}$, where $x_0(n) = \delta_0^n$, is not spectral [17]. Therefore S cannot be spectral [17]. So B is not uniformly non-degenerate.

The indirectness of this argument is a consequence of the indirectness of

Definition 1. A characterization is needed giving more insight into the nature of uniform non-degeneracy.

There is a hierarchy of further invariants introduced into analytical dynamics by Poincaré [18]. For any finite dimensional subspace $\mathfrak{N}$ containing the vectors $X_1, \dots, X_{\dim \mathfrak{N}}$, define

$$B_{\mathfrak{N}}(X_1, \dots, X_{\dim \mathfrak{N}}) = \sum_{\text{permutations } \pi} (\text{sgn } \pi) \cdot B(X_{\pi(1)}, X_{\pi(2)}) \cdots B(X_{\pi(\dim \mathfrak{N}-1)}, X_{\pi(\dim \mathfrak{N})}) / (\dim \mathfrak{N}/2)! 2^{\dim \mathfrak{N}/2}.$$

It measures a volume of the hyper-parallelepiped determined at one corner by the line-segments $[0, X_i]$ as edges. Extended from parallelepipeds to Borel sets, it is used to form integrals over finite dimensional subsets of $\mathfrak{M}$. When $\dim \mathfrak{M} < \infty$, $B_{\mathfrak{M}}$ is Liouville's invariant measure. Sets of Liouville measure zero could only be determined by measurements of infinite accuracy, so states of the dynamical system will be represented by probability measures absolutely continuous with respect to $B_{\mathfrak{M}}$. We wish to ensure the existence of $B_{\mathfrak{M}}$ even when $\dim \mathfrak{M} = \infty$, in the sense that there exists some generalized limit of the Poincaré measures $B_{\mathfrak{N}}$ as $\dim \mathfrak{N} \rightarrow \infty$ which can be used to form weak probability distributions [19] sufficiently non-singular to qualify as states. Loosely speaking, the condition on B is that ratios of degeneracies of Poincaré measures have a finite generalized $\liminf$. To make these statements precise, the generalized limit will first be defined.

Let $\mathfrak{F}$ be the lattice of all finite dimensional subspaces $\mathfrak{N}$ of $\mathfrak{M}$. Give $\mathfrak{F}$ the discrete topology and let $\bar{\mathfrak{F}}$ be its Stone-Čech compactification [20]. Let $\mu_{(\cdot)}$ be any assignment of probability measures $\mu_{\mathfrak{N}}$ with support in each $\mathfrak{N}$ in $\mathfrak{F}$. Then $\mu_{(\cdot)}$ maps $\mathfrak{F}$ into the set of states (normalized positive linear functionals) of the Banach algebra obtained by closing in the sup norm the algebra of all bounded tame functions on $\mathfrak{M}$ (functions of the form $f(B(X_1, \cdot), B(X_2, \cdot), \dots, B(X_n, \cdot))$ for bounded, measurable, real-valued functions f of any finite number n of real variables). By the Banach-Alaoglu theorem the set of all states is a compact Hausdorff space in the weak-* topology, and $\mu_{(\cdot)}$ is continuous on (discrete) $\mathfrak{F}$, so it can be extended uniquely to a continuous function on $\bar{\mathfrak{F}}$ by the Stone-Čech theorem.

The value of $\mu_{(\cdot)}$ at any point in $\bar{\mathfrak{F}} - \mathfrak{F}$ will be a generalized limit, but not necessarily in a direction of increasing $\mathfrak{N}$. The lattice structure of $\mathfrak{F}$ has not yet been used. Let $\mathcal{G}_{\mathfrak{N}}$ be the principal dual lattice-ideal [21] of all $\mathfrak{N}'$ in $\mathfrak{F}$ such that $\mathfrak{N}' \supset \mathfrak{N}$, and let $\bar{\mathcal{G}}_{\mathfrak{N}}$ be its closure in $\bar{\mathfrak{F}}$. $\mathcal{G}_{(\cdot)}$ is a dual isomorphism of the lattice $\mathfrak{F}$ into the lattice of all its dual ideals: $\mathcal{G}_{\mathfrak{N} \wedge \mathfrak{N}'} = \mathcal{G}_{\mathfrak{N}} \vee \mathcal{G}_{\mathfrak{N}'}$ and $\mathcal{G}_{\mathfrak{N} \vee \mathfrak{N}'} = \mathcal{G}_{\mathfrak{N}} \wedge \mathcal{G}_{\mathfrak{N}'}$. The identity map is a meet-isomorphism of the lattice of all dual ideals into the Boolean lattice $2^{\mathfrak{F}}$ of all subsets of $\mathfrak{F}$: $\mathcal{G}_{\mathfrak{N}} \wedge \mathcal{G}_{\mathfrak{N}'} = \mathcal{G}_{\mathfrak{N} \cap \mathfrak{N}'}$. And the closure in $\bar{\mathfrak{F}}$ of subsets of $\mathfrak{F}$ is Stone's representation of $2^{\mathfrak{F}}$. Therefore, $\bar{\mathcal{G}}_{(\cdot)}$ is a dual meet-homomorphism of $\mathfrak{F}$: $\bar{\mathcal{G}}_{\mathfrak{N}} \cap \bar{\mathcal{G}}_{\mathfrak{N}'} = \bar{\mathcal{G}}_{\mathfrak{N} \vee \mathfrak{N}'}$. Therefore $\bigcap_{\mathfrak{N} \in \mathfrak{F}} \bar{\mathcal{G}}_{\mathfrak{N}}$ is non-void by the finite-intersection property for compact spaces. Any element

$\mathfrak{N}_\infty$ in that intersection will give a generalized limit $\mu_{\mathfrak{N}_\infty}$ of $\mu_{\mathfrak{N}}$ in a direction of increasing $\mathfrak{N}$.

Now we want to find a condition on B permitting the construction of a function $\mu_{(\cdot)}$ such that $\mu_{\mathfrak{N}_\infty}$, for some $\mathfrak{N}_\infty$ in $\bigcap_{\mathfrak{N} \in \mathfrak{F}} \bar{\mathfrak{F}}_{\mathfrak{N}}$, will be sufficiently non-singular, *i.e.*, such that $\int B(X_1, X') B(X_2, X') d\mu_{\mathfrak{N}_\infty}(X')$ will serve as the $\langle X_1, X_2 \rangle$, of Definition 1.

If there already were a meaningful metric, the degeneracy of $B_{\mathfrak{N}}$ for $\mathfrak{N}$ in $\mathfrak{F}$ could be defined as the radius of the smallest sphere of unit $B_{\mathfrak{N}}$ -measure. Lacking such a metric, we can still use the variance $\int B(X, \cdot)^2 d\mu_{\mathfrak{N}}$ as a guage of the extent of $\mathfrak{S}_{\mathfrak{N}} \subset \mathfrak{N}$ in the $B(X, \cdot)$ -direction, where $\mu_{\mathfrak{N}}$ is the measure defined by $\mu_{\mathfrak{N}}(\mathfrak{S}') = B_{\mathfrak{N}}(\mathfrak{S} \cap \mathfrak{S}')$ for any $\mathfrak{S}' \subset \mathfrak{M}$. Let $\mu_{(\cdot)}$ be a function which assigns such measures satisfying the restriction $B_{\mathfrak{N}}(\mathfrak{S}_{\mathfrak{N}}) = 1$, to every $\mathfrak{N}$ in a subset $\mathfrak{F}_0 \subset \mathfrak{F}$ of non-degenerate ($B_{\mathfrak{N}} \neq 0$) subspaces. The extent of each $\mathfrak{S}_{\mathfrak{N}}$ in the $B(X, \cdot)$ -direction is uniform over $\mathfrak{F}_0 \subset \mathfrak{F}$ if $\int B(X, \cdot)^2 d\mu_{\mathfrak{N}} / \int B(X, \cdot)^2 d\mu_{\mathfrak{N}'}$ is bounded by a constant independent of $\mathfrak{N}$ and $\mathfrak{N}'$ in $\mathfrak{F}_0$. If such a $\mu_{(\cdot)}$ can be found satisfying the condition

$$(u) \quad \sup_{X \neq 0; \mathfrak{N}, \mathfrak{N}' \in \mathfrak{F}_0; X \in \mathfrak{N} \cap \mathfrak{N}'} \frac{\int B(X, \cdot)^2 d\mu_{\mathfrak{N}}}{\int B(X, \cdot)^2 d\mu_{\mathfrak{N}'}} = c^2 < \infty,$$

then one could say that the non-degeneracy of B itself is uniform over $\mathfrak{F}_0$; and it would be natural to say that B is uniformly non-degenerate if the lattice $\mathfrak{F}$ contains a cofinal subset $\mathfrak{F}_0$ (every $\mathfrak{N}$ in $\mathfrak{F}$ is contained in some $\mathfrak{N}_0$ in $\mathfrak{F}_0$) over which the non-degeneracy of B is uniform. We will show that this property is equivalent to Definition 1 (holding to our convention that the B -topology is quasi-complete).

If B satisfies Definition 1 then take for $\mathfrak{F}_0$ those $\mathfrak{N}_0$ in $\mathfrak{F}$ which, as given by the Theorem, are j -complex linear. For each $\mathfrak{N}_0$ pick a complex $\langle \cdot, \cdot \rangle$ -orthogonal basis $X_1, \dots, X_{\dim \mathfrak{N}_0/2}$ and let $X_{i+\dim \mathfrak{N}_0/2} = jX_i$. Let $\mathfrak{S}_{\mathfrak{N}_0}$ be the subset of all $X' = \alpha_1 X_1 + \dots + \alpha_{\dim \mathfrak{N}_0} X_{\dim \mathfrak{N}_0}$ such that $-1 \leq \alpha_i \leq 1$. Then $\{X_i\}_{i=1}^{\dim \mathfrak{N}_0}$ is both a Weyl basis ($B(X_i, X_{i'}) = \delta_{i, i' - \dim \mathfrak{N}_0/2}$ for all $i < i'$) and a $B(j(\cdot), \cdot)$ -orthonormal basis ($B(jX_i, X_{i'}) = \delta_{i, i'}$) of $\mathfrak{N}_0$. Therefore, if $X \in \mathfrak{N}_0$,

$$\begin{aligned} \int B(X, \cdot)^2 d\mu_{\mathfrak{N}_0} &= \int_{\mathfrak{S}_{\mathfrak{N}_0}} B(X, X')^2 dB_{\mathfrak{N}_0}(X') \\ &= \int_{-1}^1 d\alpha_1 \cdots \int_{-1}^1 d\alpha_{\dim \mathfrak{N}_0} B(j(-jX), \alpha_1 X_1 + \cdots + \alpha_{\dim \mathfrak{N}_0} X_{\dim \mathfrak{N}_0})^2 \\ &= \int_{-1}^1 \alpha^2 d\alpha \sum_{i=1}^{\dim \mathfrak{N}_0} B(j(-jX), X_i)^2 \\ &= \frac{2}{3} ||-jX||^2 = \frac{2}{3} ||X||^2. \end{aligned}$$

So the constant c of condition u equals one. The non-degeneracy of B is uniform over the cofinal subset $\mathfrak{F}_0$.

Conversely, assume that there exists a $\mu_{(\cdot)}$ satisfying condition $\mathfrak{U}$ for some cofinal $\mathfrak{F}_0$, and assigning probability measures arbitrarily to elements of $\mathfrak{F} - \mathfrak{F}_0$. We will show that a $\langle \cdot, \cdot \rangle_r$ for Definition 1 exists.

To each $\mathfrak{N}$ in $\mathfrak{F}$ can be associated a real, singular inner product $\langle X, Y \rangle_{\mathfrak{N}} = \int B(X, \cdot)B(Y, \cdot) d\mu_{\mathfrak{N}}$. As in the previous averaging of an inner product over $\mathfrak{G}$, a particular generalized limit of the $\langle \cdot, \cdot \rangle_{\mathfrak{N}}$'s will be chosen.

By cofinality, $\mathfrak{F}_0 \cap \mathcal{J}_{\mathfrak{N}} \neq \emptyset$ for every $\mathfrak{N}$ in $\mathfrak{F}$. Therefore $\bigcap_{\mathfrak{N} \in \mathfrak{F}} \overline{F_0 \cap \mathcal{J}_{\mathfrak{N}}} \neq \emptyset$ by the finite intersection property as before. Let $\mathfrak{N}_{\infty}$ be in $\bigcap_{\mathfrak{N} \in \mathfrak{F}} \overline{F_0 \cap \mathcal{J}_{\mathfrak{N}}}$. It will be shown that $\mu_{\mathfrak{N}_{\infty}}$ has a unique extension to the linear space spanned by unbounded tame functions of the form

$$B(X, \cdot)B(Y, \cdot); \text{ and } \langle \cdot, \cdot \rangle_{\mathfrak{N}_0} \rightarrow \langle \cdot, \cdot \rangle_{\mathfrak{N}_{\infty}} = \langle \cdot, \cdot \rangle_r$$

on the directed set $\mathfrak{F}_0$.

$\mathfrak{N}_{\infty}$, as an ultrafilter on $\mathfrak{F}$ [20] (maximal dual ideal in the Boolean lattice $2^{\mathfrak{F}}$), contains all subsets $\mathcal{J}_{\mathfrak{N}_0} \cap \mathfrak{F}_0$ for $\mathfrak{N}_0 \in \mathfrak{F}_0$. Hence it is also an untrafilter on $\mathfrak{F}_0$.

Let $X \in \mathfrak{N}_X \in \mathfrak{F}_0$. Then

$$0 < \langle X, X \rangle_{\mathfrak{N}_X} c^{-2} \leq \langle X, X \rangle_{\mathfrak{N}_0} \leq \langle X, X \rangle_{\mathfrak{N}_X} c^2 < \infty$$

for all $\mathfrak{N}_0 \supset \mathfrak{N}_X$ in $\mathfrak{F}_0$. So $\langle X, X \rangle_{(\cdot)}$ maps $\mathfrak{N}_{\infty}$, as an ultrafilter on $\mathcal{J}_{\mathfrak{N}_X} \cap \mathfrak{F}_0$, onto a filter base on the real, closed interval $[\langle X, X \rangle_{\mathfrak{N}_X} c^{-2}, \langle X, X \rangle_{\mathfrak{N}_X} c^2]$. Since the interval is compact, the filter base has at least one adherence point. Since the base is, in fact, the base of an ultrafilter, it has only one adherence point. Define $\langle X, X \rangle_{\mathfrak{N}_{\infty}}$ to be that real number. Define $\langle \cdot, \cdot \rangle_{\mathfrak{N}_{\infty}}$ by $4\langle X_1, X_2 \rangle_{\mathfrak{N}_{\infty}} = \langle X_1 + X_2, X_1 + X_2 \rangle_{\mathfrak{N}_{\infty}} - \langle X_1 - X_2, X_1 - X_2 \rangle_{\mathfrak{N}_{\infty}}$ and it is easily seen to have all the properties of a positive definite, real inner product.

To qualify as a $\langle \cdot, \cdot \rangle_r$, we need only prove that its weak topology is no weaker than the B -topology. For this it is sufficient, as in the remarks following Definition 1, to prove that each $B(X, \cdot)$ is bounded on the unit $\langle \cdot, \cdot \rangle_{\mathfrak{N}_{\infty}}$ -sphere $S(1, \mathfrak{N}_{\infty})$.

$\langle X', X' \rangle_{\mathfrak{N}_X} \leq c^2 \langle X', X' \rangle_{\mathfrak{N}_{\infty}}$ for any X' in $\mathfrak{M}$. Therefore $S(1, \mathfrak{N}_{\infty})$ is contained in the $\langle \cdot, \cdot \rangle_{\mathfrak{N}_X}$ -sphere $S(c, \mathfrak{N}_X)$ of radius c ; and $\sup_{X' \in S(1, \mathfrak{N}_{\infty})} |B(X, X')| \leq \sup_{X' \in S(c, \mathfrak{N}_X)} |B(X, X')|$. But $\langle X', X' \rangle_{\mathfrak{N}_X} = 0$ implies $B(X, X') = 0$, from the definition

$$\langle X', X' \rangle_{\mathfrak{N}_X} = \int_{\mathcal{S}_{\mathfrak{N}_X}} B(X', X'')^2 d\mathfrak{B}_{\mathfrak{N}_X}(X''),$$

since $B(X, X') \neq 0$ would imply $B(X'', X')^2 > 0$ for $X'' = \alpha X$, $\alpha \neq 0$ such that $\alpha X \in \mathcal{S}_{\mathfrak{N}_X}$. Therefore $\sup_{X' \in S(c, \mathfrak{N}_X)} |B(X, X')| = \sup_{X' \in S(c, \mathfrak{N}_X) \cap \mathfrak{N}_X} |B(X, X')|$. But $S(c, \mathfrak{N}_X) \cap \mathfrak{N}_X$ is B -compact. So $B(X, \cdot)$ is bounded on $S(1, \mathfrak{N}_{\infty})$, and the proof is complete. The two definitions of uniform non-degeneracy are equivalent.

The second definition is, perhaps, more an explanation of the name than it is a definition.

Lacking a more direct criterion, a physical argument will be given for the

assertion that all linear phase spaces should be uniformly non-degenerate. The purpose of a phase space is to support (weak [19]) probability distributions ρ representing our information about the system. Assume we have enough information to sufficiently localize its parameters that all variances $\int B(X, \cdot)^2 d\rho$ are finite. Then $\int B(X, \cdot)B(Y, \cdot) d\rho$ is an inner product on $\mathfrak{M}$. If its unit sphere were not bounded in the B -topology (such inner products can be obtained from any Hamel base), then it would contain a sequence $\{X_k\}_{k=0}^\infty$ such that

$$0 < B(X_0, X_k) \uparrow \infty \quad \text{for } k > 0.$$

Define $X^{(k)} = X_0/\sqrt{B(X_0, X_k)}$ and $Y^{(k)} = X_k/\sqrt{B(X_0, X_k)}$ for $k > 0$. Then $\{X^{(k)}, Y^{(k)}\}$ would be a pair of canonically conjugate variables, $B(X^{(k)}, Y^{(k)}) = 1$, with variances decreasing simultaneously to zero as $k \rightarrow \infty$, *i.e.*, which were known with unlimited accuracy, violating Heisenberg's Uncertainty Principle. This argument comes from outside the domain of classical mechanics, to which the paper has heretofore been carefully restricted in spite of its originally quantum field theoretic motivation [22, 11], but it does not come from outside the domain of physical fact. $\mathfrak{M}$ should be capable of supporting at least one ρ such that the $\sqrt{\int B(\cdot, X')^2 d\rho(X')}$ -norm topology is finer than the B -topology.

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