Some Theorems about n -Vertex Connected Graphs

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1. Introduction. In this note, G will denote an undirected graph (finite or infinite) with no loops or multiple edges. Its vertex set will be denoted by $V(G)$, or simply by V . If n is a positive integer, a connected graph G is said to be n -vertex connected if $|V(G)| \geq n + 1$ and if the removal of any set of fewer than n vertices does not disconnect G . Such graphs have been investigated by Menger [3] and characterized by H. Whitney [5] as follows:

Theorem 1. *If $|V(G)| \geq n + 1$, a necessary and sufficient condition that G be n -vertex connected is*

- (1-1) *Given $a, b \in V(G)$, there exist n arcs from a to b having only a and b in common.*

In [1] and [2], G. A. Dirac has proved that some considerably generalized formulations of (1-1) are necessary for G to be n -vertex connected. However, not all of these are sufficient. The purpose of this note is thus to generalize (1-1) as a *characterization* of n -vertex connected graphs. It is further demonstrated that the generalizations obtained are in certain respects the best possible.

2. Notation and definitions. As a general reference for the notation and vocabulary of this note, the authors recommend O. Ore [4]. If H is a subgraph of G , written $H \subset G$, then $V(H)$ denotes the vertex set of H . If $U \subset V(G)$, then $G(U)$ denotes the section subgraph of G with vertex set U .

To say that $A = A[a, b]$ is an arc shall mean that a and b are its terminal vertices. A family of arcs is said to be *openly disjoint* if the arcs in the family are pairwise disjoint except at common endpoints, if any.

If $U \subset V(G)$ and $x \in V(G)$, we say x is an *interior vertex* of $G(U)$ if all the edges incident to x lie in $G(U)$, and x is an *exterior vertex* of $G(U)$ if no edge incident to x lies in $G(U)$. Otherwise, x is an *attachment vertex* of $G(U)$.

The following equivalent formulation of an n -vertex connected graph, due to Ore [4], will be more convenient to use in the sequel.

Definition. If n is a positive integer, a graph G is n -vertex connected if for no subset $U \subset V$ does the subgraph $G(U)$ have $\mu \geq 1$ interior vertices, $\nu \geq 1$ exterior vertices, and $\lambda \leq n - 1$ attachment vertices.

The *vertex connectivity* of a graph G is the largest value of λ for which G is λ -vertex connected. It will be denoted by λ_0 .

3. An extension of Whitney's Theorem. Let $k_1, \dots, k_m$ be a sequence of positive integers, $m \geq 1$, and consider the following proposition concerning a graph G having at least $m + 1$ vertices:

$W(k_1, \dots, k_m)$: Given any set of $m + 1$ vertices $a, c_1, \dots, c_m \in V$, there exists an openly disjoint family of $\sum_{i=1}^m k_i$ arcs from a to the vertices $c_1, \dots, c_m$ such that k_i arcs join a with c_i , ($i = 1, \dots, m$).

Following from a theorem of Dirac [1], we have

Theorem 2. If G is n -vertex connected, then G satisfies $W(k_1, \dots, k_m)$ whenever

$$(3-1) \quad \sum_{i=1}^m k_i = n.$$

Let m and n be integers, $1 \leq m \leq n$. If $k_1 = n - m + 1$ and $k_2 = \dots = k_m = 1$, then (3-1) holds. The statement $W(k_1, \dots, k_m)$ in this case will be denoted by Y_m^n .

Theorem 3. Let n be a positive integer. The following $2n$ statements about the graph G (having at least $n + 1$ vertices) are equivalent:

X^n : G is n -vertex connected.

Y_m^n ($m = 1, \dots, n$).

Z_m^n ($m = 1, \dots, n - 1$): Given any set of $m + 2$ vertices $a, b, c_1, \dots, c_m \in V$, there exists an openly disjoint family of $n - m$ arcs joining a and b , each of which excludes $c_1, \dots, c_m$.

Remark. G is required to have at least $n + 1$ vertices in order to avoid the possibility of some Y_m^n or Z_m^n holding vacuously while G is not n -vertex connected. If $n = 1, 2$, this theorem is embodied in a formulation by Ore. (See [4], p. 87.)

Proof. X^n implies Y_m^n ($m = 1, \dots, n$). Use Theorem 2.

Y_1^n implies X^n . Clearly Y_1^n reduces to the condition (1-1), and the result follows from Theorem 1.

Y_m^n implies Z_{m-1}^n ($m = 2, \dots, n$). Let $m + 1$ distinct vertices $a, b, c_1, \dots, c_{m-1}$ be given. By assumption there exists an openly disjoint family of n arcs

$$\{P_i[a, b] : i = 1, \dots, n - m + 1\} \cup \{S_i[a, c_i] : i = 1, \dots, m - 1\}.$$

The arcs $P_1, \dots, P_{n-m+1}$ satisfy Z_{m-1}^n since they all exclude $c_1, \dots, c_{m-1}$.

Z_m^n implies X^n ($m = 1, \dots, n - 1$). Suppose G is not n -vertex connected. Then there exists a set $A \subset V$ such that the subgraph $G(A)$ has an interior vertex a , an exterior vertex b , and $\lambda \leq n - 1$ attachment vertices $c_1, \dots, c_\lambda$. If $\lambda \leq m$,

then, contrary to Z_m^n , no arc $P[a, b]$ can fail to contain some attachment vertex c_i . Suppose that $\lambda > m$. By Z_m^n , there exists an openly disjoint family of $n - m$ arcs $P_i[a, b]$ each of which excludes $c_1, \dots, c_m$. But since $n - m > \lambda - m$, at least two of these arcs P_i must pass through some one of the vertices $c_{m+1}, \dots, c_\lambda$, and hence the given family is not openly disjoint.

4. Concerning the converse to Theorem 2. It will be shown later that the converse to Theorem 2 does not hold in general. However, if G is a counter-example, then $|V(G)|$ is restricted by the upper bounds given in the next theorem.

Theorem 4. *Let n be a positive integer. Let G satisfy $W(k_1, \dots, k_m)$ where (3-1) holds and β_1 is the number of terms k_i equal to 1. If $\lambda_0 \leq n - 1$, then*

$$(4-1) \quad |V| \leq n - 3 + 2(m - \beta_1)$$

and

$$(4-2) \quad \begin{aligned} |V| &\leq 2n - 3, & \text{if } n \text{ is even,} \\ |V| &\leq 2n - 4, & \text{if } n \text{ is odd.} \end{aligned}$$

Proof. Let the sequence of positive integers

$$(4-3) \quad k_1, \dots, k_m$$

satisfying (3-1) be given. Since $\lambda_0 \leq n - 1$, there exists a set $A \subset V$ such that $G(A)$ has $\mu \geq 1$ interior vertices, $\nu \geq 1$ exterior vertices, and $\lambda \leq n - 1$ attachment vertices. Thus

$$(4-4) \quad |V| = \lambda + \mu + \nu.$$

If $\lambda < m$, let a be an interior vertex of $G(A)$, let the attachment vertices be $c_1, \dots, c_\lambda$, and let $c_{\lambda+1}$ be an exterior vertex. Then $W(k_1, \dots, k_m)$ is clearly violated. Hence suppose

$$(4-5) \quad \lambda \geq m.$$

For $j = 1, 2, \dots$, let β_j denote the number of terms in (4-3) which are equal to j . Thus

$$(4-6) \quad \sum_{i=1}^{\infty} \beta_i = m$$

and

$$(4-7) \quad \sum_{i=1}^{\infty} j\beta_i = n.$$

We may arrange (4-3) so that $k_1 = \dots = k_{\beta_1} = 1$. Let a be an interior vertex and let $c_1, \dots, c_{\beta_1}$ be attachment vertices. By (4-5) this is possible. Assuming $\nu \geq m - \beta_1$, we may let $c_{\beta_1+1}, \dots, c_m$ be exterior vertices. Since $W(k_1, \dots, k_m)$ holds, there exists an openly disjoint family of $n - \beta_1$ arcs from a to these exterior vertices. Hence there must exist at least $n - \beta_1$ attachment vertices

distinct from $c_1, \dots, c_{\beta_1}$, contrary to the fact that $\lambda \leq n - 1$. We conclude that $\nu \leq m - \beta_1 - 1$, and by symmetry, $\mu \leq m - \beta_1 - 1$. From (4-4) we obtain $|V| \leq (n - 1) + 2(m - \beta_1 - 1)$, whence follows (4-1).

From (4-6) and (4-7) we obtain $2(m - \beta_1) = n - \beta_1 - \sum_{i=3}^{\infty} (j - 2)\beta_i$, which combined with (4-1) gives

$$(4-8) \quad |V| \leq 2n - 3 - \beta_1 - \sum_{i=3}^{\infty} (j - 2)\beta_i.$$

Clearly, $|V| \leq 2n - 3$. Moreover, if n is odd, then by (4-7) at least one β_j for odd j must be positive, whence (4-2).

Corollary. *Let the sequence $k_1, \dots, k_m$ of positive integers be given, let β_1 be the number of terms equal to 1, and assume (3-1). If either*

$$(4-9) \quad |V| \geq 2n - 2$$

or

$$(4-10) \quad \beta_1 \geq m - 2,$$

then $W(k_1, \dots, k_m)$ is a necessary and sufficient condition that G be n -vertex connected.

Proof. $W(k_1, \dots, k_m)$ is necessary by Theorem 2. If (4-9) holds, $W(k_1, \dots, k_m)$ is sufficient by Theorem 4.

Assuming (4-10), we remark that $W(k_1, \dots, k_m)$ reduces to Y_n^n if $\beta_1 = m$ and to Y_{n-1}^n if $\beta_1 = m - 1$. Then G is n -vertex connected by Theorem 3. Hence suppose that $\beta_1 = m - 2$. Theorem 4 applies and (4-1) now reduces to $|V| \leq n + 1$. If G is not n -vertex connected, then the degree of some vertex $u \in V(G)$ must be less than n . But this is impossible if G is to satisfy $W(k_1, \dots, k_m)$ with $\sum_{i=1}^m k_i = n$.

The above corollary gives sufficient conditions on the number of vertices in G and on the sequence $k_1, \dots, k_m$ in order that $W(k_1, \dots, k_m)$ be equivalent to n -vertex connectedness. The next theorem shows that these conditions are also necessary, i.e., that Theorem 4 is the best possible result.

Theorem 5. *Let n be a positive integer. Let $k_1, \dots, k_m$ be a sequence of positive integers, β_1 of whose terms are equal to 1, such that $\sum_{i=1}^m k_i = n$ and*

$$(4-11) \quad \beta_1 \leq m - 3.$$

Then there exists a graph G on $|V| = n - 3 + 2(m - \beta_1)$ vertices which satisfies $W(k_1, \dots, k_m)$ but has vertex connectivity $\lambda_0 \leq n - 1$.

Proof. In the complete graph on $n - 3 + 2(m - \beta_1)$ vertices, decompose the vertex set into disjoint sets V_1, V_2, V_3 , where V_2 contains $n - 1$ vertices and each of V_1, V_3 contains $m - \beta_1 - 1$ vertices. Let G be the graph remaining when all edges joining points of V_1 to points of V_3 are deleted. $G - G(V_2)$

is not connected, showing that $\lambda_0 \leq n-1$. It remains to show that $\mathbf{W}(k_1, \dots, k_m)$ is satisfied.

Let $a, c_1, \dots, c_m \in V(G)$ be given. For $j = 1, 2, 3$, let r_j denote the number of vertices $c_i \in V_j$, and let $R_j = \sum_{c_i \in V_j} k_i$. Thus

$$(4-12) \quad r_1 + r_2 + r_3 = m$$

and

$$(4-13) \quad R_1 + R_2 + R_3 = n.$$

We shall now prove the existence of an openly disjoint family of n arcs from a to $c_1, \dots, c_m$ satisfying $\mathbf{W}(k_1, \dots, k_m)$. Since for any arc $A[a, b] \subset G$ there exists an arc $A'[a, b]$ of length at most 2 and such that $V(A') \subset V(A)$, the arcs in this family may be chosen to consist of single edges when possible and of precisely 2 edges otherwise.

Case 1: $a \in V_1 \cup V_3$. By symmetry we may assume $a \in V_1$. Then $r_1 + r_2$ of the required arcs may be chosen to consist of single edges, and the remaining arcs must be of length 2. These remaining arcs may be constructed provided there exist three disjoint sets of vertices, namely $R_1 - r_1$ vertices of $V_1 \cup V_2$, $R_2 - r_2$ vertices of $V_1 \cup V_2$, and R_3 vertices of V_2 , to be used as the intermediate vertices on the arcs of length 2 from a to vertices c_i lying in V_1, V_2 , and V_3 , respectively.

Since $r_3 \leq m - \beta_1 - 1$, we have by (4-12), $r_1 + r_2 \geq \beta_1 + 1$. Hence, $k_i > 1$ for at least one $c_i \in V_1 \cup V_2$, and

$$(4-14) \quad R_1 + R_2 \geq r_1 + r_2 + 1.$$

In addition to vertices c_i , V_2 contains $n - 1 - r_2$ vertices from which R_3 vertices may be selected, since by (4-13) and (4-14), $(n - 1 - r_2) - R_3 = R_1 + R_2 - r_2 - 1 \geq r_1 \geq 0$. Now in addition to a , vertices c_i , and the R_3 vertices previously selected from V_2 , $V_1 \cup V_2$ also contains $m - \beta_1 - 3 + n - r_1 - r_2 - R_3$ vertices. If from these we select $(R_1 - r_1) + (R_2 - r_2)$ vertices, by (4-11) and (4-13) there remain $m - \beta_1 - 3 \geq 0$ vertices.

Case 2: $a \in V_2$. Now $r_1 + r_2 + r_3$ of the required arcs may be chosen to consist of single edges. The construction of the remaining arcs is possible provided there exist three disjoint sets of vertices, namely $R_1 - r_1$ vertices of $V_1 \cup V_2$, $R_2 - r_2$ vertices of $V_1 \cup V_2 \cup V_3$, and $R_3 - r_3$ vertices of $V_2 \cup V_3$, to be used as the intermediate vertices on the arcs of length 2 from a to vertices c_i lying in V_1, V_2 , and V_3 , respectively.

For $j = 1, 3$, we may select up to $m_1 - \beta_1 - 1 - r_j$ vertices from V_j . We next show that if $R_j > m - \beta_1 - 1$, then the number $R_j - (m - \beta_1 - 1)$ of vertices still required does not exceed the number $n - 2 - r_2$ of vertices in V_2 other than a and those $c_i \in V_2$. For $j = 1$, it follows from (4-11) and (4-13) that $(n - 2 - r_2) - [R_1 - (m - \beta_1 - 1)] \geq R_2 - r_2 + R_3 \geq 0$. The case for $j = 3$ is analogous. Moreover, the selection for $j = 1, 3$ can be made simultaneously, since by a similar argument,

$$(4-15) \quad (n - 2 - r_2) - [R_1 + R_3 - 2(m - \beta_1 - 1)] \geq R_2 - r_2 + 2 > 0.$$

It can be verified that the left hand member of (4-15) represents the number of vertices of $V_1 \cup V_2 \cup V_3$ other than $a, c_1, \dots, c_m$ and the $(R_1 - r_1) + (R_3 - r_3)$ intermediate vertices previously selected. Since this number exceeds $R_2 - r_2$, the desired family of arcs can be constructed, and the proof is complete.

The example with the fewest vertices which illustrates Theorem 5 is a graph G with $|V| = 9$ and $\lambda_0 = 5$, constructed from the complete graph on 9 vertices by deleting the edges of a 4-cycle. G satisfies $W(2, 2, 2)$. G is by no means unique, for if any edges of a 5-cycle on the five vertices of degree 8 are also deleted, the remaining graph G' still satisfies $W(2, 2, 2)$ and is 5-vertex connected.

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