L_q Estimates for Rearrangements of Trigonometric Series

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1. Main results. A. M. Garsia [1] has recently proved that every L_2 Fourier series can be rearranged so that it converges a.e. The proof rests on an estimate for finite sums $T = \sum_{\mu=1}^n \Phi_\mu$, where $\Phi_1, \dots, \Phi_n$ are orthogonal functions on a measure space. If $\sigma = (\sigma_1, \dots, \sigma_n)$ is a permutation of $\{1, \dots, n\}$, write

$$(1.1) \quad S^*(x, \sigma) = \max_{1 \leq m \leq n} \left| \sum_{\nu=1}^m \Phi_{\sigma_\nu}(x) \right|.$$

His result is that

$$(1.2) \quad \frac{1}{n!} \sum_{\sigma} \int [S^*(x, \sigma)]^2 dx \leq A \int |T|^2,$$

for all n and $\Phi_1, \dots, \Phi_n$, where the sum is over all such permutations σ , and A is an absolute constant. (From here on we will use $A_r, A(r, s, \dots)$ to denote constants, not necessarily the same at each occurrence, which depend only on the indicated parameters.)

Theorem 1 is an extension of the L_2 estimate (1.2) to L_q for even integers q . First we need a replacement for orthogonality.

(1.3) Definition. Let r be a positive integer. Let (φ_α) be a system of complex-valued functions on a measure space. We will say (φ_α) is a P -system for r if for any indices $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r$ we have

$$(1.4) \quad \int \varphi_{\alpha_1} \cdots \varphi_{\alpha_r} \bar{\varphi}_{\beta_1} \cdots \bar{\varphi}_{\beta_r} \geq 0.$$

Any orthonormal system is a P -system for 1. The systems

$$(e^{i\nu x}: \nu = 0, \pm 1, \pm 2, \dots) \quad \text{and} \quad (\cos \nu x: \nu = 0, 1, 2, \dots) \quad \text{on} \quad [0, 2\pi],$$

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and the Rademacher functions on $[0, 1]$ are P -systems for all r . However $(\sin \nu x: \nu = 1, 2, \dots)$ is not a P -system for any $r > 1$ since

$$\int_0^{2\pi} \sin^{2r-1} x \sin (2r-1)x \, dx = -\pi/4^{r-1} \quad (r = 2, 4, 6, \dots),$$

$$\int_0^{2\pi} (\sin 2x)^{2r-2} \sin x \sin (4r-5)x \, dx = -\pi/4^{r-1} \quad (r = 3, 5, 7, \dots).$$

Theorem 1. Let $\varphi_1, \dots, \varphi_n$ be a P -system for r . Let $T = \sum_{\nu=1}^n a_\nu \varphi_\nu$, $\hat{T} = \sum_{\nu=1}^n |a_\nu| \varphi_\nu$, where $a_1, \dots, a_n$ are complex numbers. Define $S^*(x, \sigma)$ by (1.1), where $\Phi_\nu = a_\nu \varphi_\nu$. Then

$$(1.5) \quad \frac{1}{n!} \sum_{\sigma} \int [S^*(x, \sigma)]^{2r} \, dx \leq A_r \int |\hat{T}|^{2r}$$

for all n and $a_1, \dots, a_n$.

This includes Garsia's result when $r = 1$ and the φ_ν are orthonormal, since $\int |T|^2 = \int |\hat{T}|^2$. Section 2 contains the proof.

Theorem 2 is an application of Theorem 1 to certain trigonometric Fourier series in L_{2r} . For a given positive integer r , the functions we consider are those

$$f \sim \sum_{-\infty}^{\infty} a_\nu e^{i\nu x}$$

such that $\sum |a_\nu| a^{i\nu x}$ is the Fourier series of a function $f \in L_{2r}$. This class of functions will be denoted by X_{2r} . We have immediately that $L_2 = X_2 \subseteq X_{2r}$ for all r . In addition,

(1.6) Theorem. a) For $r = 2, 3, \dots$ the class X_{2r} is strictly contained in L_{2r} and

$$(1.7) \quad \|f\|_{2r} \leq \|\hat{f}\|_{2r}$$

for all $f \in X_{2r}$.

b) For all r the functional $N(f) = \|\hat{f}\|_{2r}$ is a norm on X_{2r} , and X_{2r} is a Banach space under this norm.

(1.6)(a) is easy and well-known [3]. But the statements of (1.6) which apply to trigonometric polynomials are true for general P -systems (φ_ν) . Thus

$$(1.8) \quad \int \left| \sum_1^n a_\nu \varphi_\nu \right|^{2r} = \sum_{\nu_1, \dots, \nu_r, \mu_1, \dots, \mu_r=1}^n a_{\nu_1} \dots a_{\nu_r} \bar{a}_{\mu_1} \dots \bar{a}_{\mu_r} \\ \cdot \int \varphi_{\nu_1} \dots \varphi_{\nu_r} \bar{\varphi}_{\mu_1} \dots \bar{\varphi}_{\mu_r} \\ \leq \sum |a_{\nu_1} \dots a_{\nu_r} a_{\mu_1} \dots a_{\mu_r}| \int \varphi_{\nu_1} \dots \varphi_{\nu_r} \bar{\varphi}_{\mu_1} \dots \bar{\varphi}_{\mu_r} \\ = \int \left| \sum |a_\nu| \varphi_\nu \right|^{2r},$$

since all the integrals are nonnegative. By a similar expansion of

$$\int \left| \sum |a_r + b_r| \phi_r \right|^{2r},$$

it can be shown that $\|\hat{f}\|_{2r}$ satisfies the triangle inequality for finite linear combinations of the φ_r . Then in the exponential case it is not hard to show that X_{2r} is complete.

For Theorem 2 let (φ_r) be either the exponential or cosine system as follows:

$$(1.9) \quad \begin{aligned} \text{a) } & \varphi_\nu(x) = \cos(\nu - 1)x \quad (\nu = 1, 2, \dots), \\ \text{b) } & \varphi_1(x) = 1, \quad \varphi_2(x) = e^{ix}, \quad \varphi_3(x) = e^{-ix}, \quad \varphi_4(x) = e^{2ix}, \dots \end{aligned}$$

Let $0 = n_0 < n_1 < n_2 < \dots$ be a sequence of integers such that $n_{k+1}/n_k > \alpha > 1$ ($k = 1, 2, \dots$). Let $\sigma = (\sigma_1, \sigma_2, \dots)$ be a permutation of $\{1, 2, \dots\}$ which permutes the indices in each block $\{\nu : n_k + 1 \leq \nu \leq n_{k+1}\}$ ($k = 0, 1, 2, \dots$) among themselves. Given $f \sim \sum_1^\infty a_r \varphi_r$, define

$$S^*(x, \sigma) = \sup_n \left| \sum_{r=1}^n a_{\sigma_r} \varphi_{\sigma_r}(x) \right|.$$

Theorem 2. *Let $f \in X_{2r}$. Then there is a σ such that*

$$(1.10) \quad \int_0^{2\pi} [S^*(x, \sigma)]^{2r} dx \leq A(\alpha, r) \int_0^{2\pi} |\hat{f}|^{2r}.$$

This result should be compared with the Men'shov-Paley Theorem [2, vol. II, p. 189-92], which states for $q > 2$ that $S^* \in L_q$ without rearrangement provided

$$B_q^*(a) = \sum_1^\infty a_\nu^* \nu^{q-2} < \infty,$$

where $a_1^*, a_2^*, \dots$ is the nonincreasing rearrangement of $|a_1|, |a_2|, \dots$. An earlier result of Hardy and Littlewood [2, vol. II, p. 121] states that $B_q^*(a) < \infty$ implies

$$\sum a_r \varphi_r \sim f \in L_q,$$

and

$$(1.11) \quad \int |f|^q \leq A_q B_q^*(a) \quad (q \geq 2).$$

The class of such functions will be denoted by HL_q . Applying (1.11) to $\hat{f}$ instead of f we see that $HL_{2r} \subseteq X_{2r}$ for all r . Furthermore, for $r \geq 2$ the inclusion is strict. Consider

$$f(x) \sim \sum_2^\infty (k^{1/2} \log k)^{-1} \cos 2^k x.$$

This function is in L_2 but not in HL_q for any $q > 2$. However, any L_2 gap series is in L_q for all $q < \infty$ [2, vol. I, p. 215]. Thus Theorem 2 applies to a wider class of functions than the Men'shov-Paley Theorem does.

2. Proof of Theorem 2 assuming Theorem 1. Given (n_k) and σ set

$$S_0 = 0, \quad S_n = \sum_{\nu=1}^n a_\nu \varphi_\nu, \quad S_n(x, \sigma) = \sum_{\nu=1}^n a_{\sigma_\nu} \varphi_{\sigma_\nu}(x) \quad (n = 1, 2, \dots),$$

$$S_*(x) = \sup_k |S_{n_k}(x)|.$$

$$\delta_k^*(x, \sigma) = \max_{n_k < n \leq n_{k+1}} |S_n(x, \sigma) - S_{n_k}(x, \sigma)|, \quad \hat{\delta}_k = \sum_{\nu=n_k+1}^{n_{k+1}} |a_\nu| \varphi_\nu \quad (k = 0, 1, \dots).$$

Notice that $S_{n_k}(x) = S_{n_k}(x, \sigma)$. Thus for $n_k < n \leq n_{k+1}$

$$S_n(x, \sigma) = S_{n_k}(x) + [S_n(x, \sigma) - S_{n_k}(x, \sigma)],$$

which gives

$$[S^*(x, \sigma)]^{2r} \leq 2^{2r-1} \left\{ S_*(x)^{2r} + \sum_{k=0}^{\infty} [\delta_k^*(x, \sigma)]^{2r} \right\}.$$

By Theorem 1 we can choose σ so that

$$\int_0^{2\pi} [\delta_k^*(x, \sigma)]^{2r} dx \leq A_r \int_0^{2\pi} |\hat{\delta}_k|^{2r} \quad (k = 0, 1, \dots).$$

For this σ ,

$$(2.1) \quad \int_0^{2\pi} [S^*(x, \sigma)]^{2r} dx \leq 2^{2r-1} \left\{ \int_0^{2\pi} S_*^{2r} + A_r \sum_{k=0}^{\infty} \int |\hat{\delta}_k|^{2r} \right\}.$$

Littlewood-Paley theory [2, chap. XV] gives

$$\int_0^{2\pi} S_*^{2r} \leq A(\alpha, r) \int_0^{2\pi} |f|^{2r}, \quad \sum_{k=0}^{\infty} \int |\hat{\delta}_k|^{2r} \leq A(\alpha, r) \int |f|^{2r},$$

which with (2.1) proves (1.10).

Remarks. Using M. Riesz' theorem that $\|S_n\|_p \leq A_p \|f\|_p$ for $f \in L_p$, $p > 1$ [2, vol. I, p. 266], instead of Littlewood-Paley theory, we can prove a weaker form of Theorem 2 in which (n_k) depends on f and the $A(\alpha, r)$ of (1.10) becomes A_r .

We can make the σ into a probability space Ω in the manner of Garsia [1] by considering the permutations of each block $(n_k, n_{k+1}]$ to be independent. An argument like the above shows that

$$\int_{\Omega} d\sigma \int_0^{2\pi} dx [S^*(x, \sigma)]^{2r} \leq A(\alpha, r) \int_0^{2\pi} |f|^{2r}.$$

Hence $S^*(\cdot, \sigma) \in L_{2r}$ for a.e. $\sigma \in \Omega$, and (1.10) holds for all σ in a set of positive measure in Ω .

3. Proof of Theorem 1. The initial steps of this proof are essentially the same as for the case $r = 1$, treated by Garsia [1]. We apply the Bohnenblust-Spitzer Theorem [4] to the sequences $(\operatorname{Re} (a_r \varphi_r))$, $(-\operatorname{Re} (a_r \varphi_r))$, $(\operatorname{Im} (a_r \varphi_r))$, $(-\operatorname{Im} (a_r \varphi_r))$. Writing permutations in cycle form define $X_C(\pi) = 1$ if C is a cycle of the permutation π , $= 0$ otherwise. We obtain (letting $q = 2r$)

$$(3.1) \quad \frac{1}{n!} \sum_{\sigma} S^*(x, \sigma)^q \leq \frac{4^q}{n!} \sum_{\pi} \left(\sum_C X_C(\pi) \left| \sum_{r \in C} a_r \varphi_r(x) \right| \right)^q.$$

(For convenience we will often use C to denote the set of indices of the cycle C .) Integrating over x gives

$$(3.2) \quad \frac{1}{n!} \sum_{\sigma} \int S^*(x, \sigma)^q dx \\ \leq \frac{4^q}{n!} \sum_{C_1, \dots, C_q} \sum_{\pi} X_{C_1}(\pi) \cdots X_{C_q}(\pi) \int \left| \sum_{r \in C_1} a_r \varphi_r \cdots \sum_{r \in C_q} a_r \varphi_r \right|.$$

In order that

$$\sum_{\pi} X_{C_1}(\pi) \cdots X_{C_q}(\pi) \neq 0$$

for a given $(C_1, \dots, C_q)$, any two C_i, C_j must either be identical *cycles* or have disjoint sets of indices. Each such $(C_1, \dots, C_q)$ determines a partition P of $\{1, \dots, q\}$ into p disjoint classes such that $C_i = C_j$ if i and j are in the same class, $C_i \cap C_j = \emptyset$ if i and j are in different classes. Let S_k ($1 \leq k \leq p$) be the set of indices of the cycle C_i where i is the k^{th} class of P . Let l_k be the number of indices in S_k . We write

$$(3.3) \quad \frac{1}{n!} \sum_{\sigma} \int [S^*(x, \sigma)]^q dx \leq 4^q \sum_P I_P,$$

$$(3.4) \quad I_P = \frac{1}{n!} \sum_{l_1, \dots, l_p} \sum_{S_1, \dots, S_p} \sum_{C_1, \dots, C_q}^P \sum_{\pi} X_{C_1}(\pi) \cdots X_{C_q}(\pi) \\ \cdot \int \left| \sum_{r \in C_1} a_r \varphi_r \cdots \sum_{r \in C_q} a_r \varphi_r \right|,$$

where $l_k \geq 1$ ($k = 1, \dots, p$), $l_1 + \dots + l_p \leq n$, $|S_k| = l_k$, and $\sum^P$ runs over all $(C_1, \dots, C_q)$ which determine the partition P and the disjoint sets $S_1, \dots, S_p$. Performing

$$\sum_{\pi} \quad \text{and} \quad \sum_{C_1, \dots, C_q}^P$$

(The integral in (3.4) depends only on P and (S_k) .) gives

$$(3.5) \quad I_P = \sum_{l_1, \dots, l_p} \left[l_1 \cdots l_p \binom{n}{l_1, \dots, l_p, n - l_1 - \dots - l_p} \right]^{-1/q} \\ \cdot \sum_{S_1, \dots, S_p} \int \left| \sum_{r \in C_1} a_r \varphi_r \cdots \sum_{r \in C_q} a_r \varphi_r \right|.$$

Apply Hölder's inequality with indices $q, \dots, q$ to the integral, then use the concavity of the q^{th} root on $\sum_{s_1 \dots s_p}$. This yields

$$(3.6) \quad I_P \leq \sum_{l_1 \dots l_p} \frac{1}{l_1 \dots l_p} \left(l_1, \dots, l_p, n - l_1 - \dots - l_p \right)^{-1/q} \cdot \left(\sum_{s_1 \dots s_p} \prod_{i=1}^p \int \left| \sum_{\nu \in C_i} a_{\nu} \varphi_{\nu} \right|^q \right)^{1/q}.$$

Now write

$$(3.7) \quad \prod_{i=1}^p \int \left| \sum_{\nu \in C_i} a_{\nu} \varphi_{\nu} \right|^q = \int \left| \sum_{N \in C_1 \times \dots \times C_p} \Phi_N(X) dw \right|^q$$

where $N = (\nu_1, \dots, \nu_p)$, $X = (x_1, \dots, x_p)$, $dw = dx_1 \dots dx_p$, and

$$\Phi_N(X) = a_{\nu_1} \varphi_{\nu_1}(x_1) \dots a_{\nu_p} \varphi_{\nu_p}(x_p).$$

Since $q = 2r$, the right hand side of (3.7) is equal to

$$(3.8) \quad \sum_{N_1 \dots N_r, M_1 \dots M_r \in C_1 \times \dots \times C_p} \int \Phi_{N_1} \dots \Phi_{N_r} \bar{\Phi}_{M_1} \dots \bar{\Phi}_{M_r} dw.$$

Reversing the order of summation and replacing integrals by their absolute values, we have

$$(3.9) \quad \sum_{s_1 \dots s_p} \prod_{i=1}^p \int \left| \sum_{\nu \in C_i} a_{\nu} \varphi_{\nu} \right|^q \leq \sum_{\text{all } N_1 \dots N_r, M_1 \dots M_r} \zeta(N_1, \dots, N_r, M_1, \dots, M_r) \cdot \left| \int \Phi_{N_1} \dots \Phi_{N_r} \bar{\Phi}_{M_1} \dots \bar{\Phi}_{M_r} dw \right|$$

where $\zeta(N_1, \dots, N_r, M_1, \dots, M_r)$ is the number of $(s_1, \dots, s_p)$ such that $N_1, \dots, N_r, M_1, \dots, M_r$ are all in $C_1 \times \dots \times C_p$. The crucial step in the proof is to notice the fact that (φ_{ν}) being a P -system for r implies

$$(3.10) \quad \left| \int \Phi_{N_1} \dots \Phi_{N_r} \bar{\Phi}_{M_1} \dots \bar{\Phi}_{M_r} dw \right| = \int \Psi_{N_1} \dots \Psi_{N_r} \bar{\Psi}_{M_1} \dots \bar{\Psi}_{M_r} dw,$$

where

$$\Psi_N(X) = |a_{\nu_1}| \varphi_{\nu_1}(x_1) \dots |a_{\nu_p}| \varphi_{\nu_p}(x_p).$$

Then as in (3.7) we have

$$(3.11) \quad \sum_{\text{all } N_1 \dots M_r} \int \Psi_{N_1} \dots \bar{\Psi}_{M_r} dw = \left(\int \left| \sum_{\nu=1}^n |a_{\nu}| \varphi_{\nu} \right|^q \right)^q.$$

For given $(l_1, \dots, l_p)$ the maximum of ζ is just $\zeta(N, \dots, N)$, where $N = (\nu_1, \dots, \nu_p)$ is any q -tuple such that $\nu_i = \nu_j$ if and only if i and j are in the same class of P . Thus

$$\max \zeta = \left(l_1 - 1, \dots, l_p - 1, n - l_1 - \dots - l_p \right). \quad (3.12)$$

Putting (3.6), (3.9), (3.10), (3.11), and (3.12) together gives

$$(3.13) \quad I_P \leq [n(n-1) \cdots (n-p+1)]^{-1/a} \sum_{i_1, \dots, i_p} (l_1 \cdots l_p)^{1/a-1} \int |\hat{T}|^a.$$

To prove (1.5), the crude estimate

$$\sum_{i_1, \dots, i_p} (l_1 \cdots l_p)^{1/a-1} < \left(\sum_{l=1}^n l^{1/a-1} \right)^p < q^p n^{p/a}$$

is good enough. Then

$$n^p / [n(n-1) \cdots (n-p+1)] \leq p^p / p!$$

since $n \geq p$. Finally

$$(3.14) \quad I_P \leq q^p (p^p / p!)^{1/a} \int |\hat{T}|^a,$$

which together with (3.3) proves (1.5).

4. Discussion of Theorem 1. In this section we will denote the expression

$$(1/n!) \sum_{\sigma} \int [S^*(x, \sigma)]^a dx \quad \text{by} \quad J_a(T) \quad (\text{for given } \varphi_1, \dots, \varphi_n).$$

We will see that Theorem 1 is best possible in a certain restricted sense. We do not know whether

$$(4.1) \quad J_a(T) \leq A_a \int |T|^a$$

is true or not. Of course $S^*(x, \sigma) \geq |T(x)|$ for all σ , so that the reverse inequality to (4.1) holds with $A_a = 1$. This simple fact tells us

(4.2) *Let $\varphi_1, \dots, \varphi_n$ be arbitrary. Assume*

$$(4.3) \quad J_a(T) \leq N_a(|a|)$$

for all $T = \sum_{r=1}^n a_r \varphi_r$, where $N_a(|a|)$ depends only on q and on $|a| = (|a_r|)$ (as does $\int |\hat{T}|^a$). Then $N_a(|a|) \geq \int |\hat{T}|^a$, and thus (1.5) is the best inequality of type (4.3), within a factor A_a . For we have $N_a(|a|) \geq J_a(\hat{T}) \geq \int |\hat{T}|^a$.

What inequalities

$$(4.4) \quad J_a(T) \geq N_a(|a|)$$

can hold? We will later see that

$$J_{2r}(T) \geq A_r \int |\hat{T}|^{2r}$$

is false for the trigonometric system. However if all coefficients are real it is easy to see that $\|S^*(\cdot, \sigma)\|_q \geq \frac{1}{2} \|\hat{T}\|_q$ for some σ (depending on T). We can write $\hat{T} = T_1 - T_2$ where

$$T_1 = \sum_{a_r > 0} a_r \varphi_r, \quad T_2 = \sum_{a_r < 0} a_r \varphi_r.$$

We may suppose $\|T_1\|_q \geq \|T_2\|_q$. Thus

$$\|\hat{T}\|_q \leq \|T_1\|_q + \|T_2\|_q \leq 2 \|T_1\|_q \leq 2 \|S^*(\cdot, \sigma)\|_q,$$

where σ is any permutation which puts the terms of T_1 first. (There are at least $(n/2)!$ of these.) In the complex case, a basically similar argument gives $\|S^*(\cdot, \sigma)\|_{2r} \geq A_r \|\hat{T}\|_{2r}$ for some σ if (φ_r) is a P -system for r .

Assume that the φ_r are orthonormal on an interval H of length $h < \infty$, so that

$$h^{1/2-1/q} \|T\|_q \geq \|T\|_2 = \|\hat{T}\|_2 = \left(\sum_{r=1}^n |a_r|^2 \right)^{1/2}.$$

Then (4.1) with " $\geq$ " gives

$$(4.5) \quad J_q(T) \geq h^{1-q/2} \left(\sum_{r=1}^n |a_r|^2 \right)^{q/2}.$$

Analogously to (4.2) we have

(4.6) If (4.4) holds for $q = 2r$ then

$$(4.7) \quad N_{2r}(|a|) \leq A_r \int \left(\sum_{r=1}^n |a_r \varphi_r|^2 \right)^r.$$

Therefore if also $|\varphi_r| \leq M$ for all r then (4.5) is the best inequality of type (4.4), within a factor $A(r, M, h)$.

To prove this we will show that for any $b_1, \dots, b_n$ with $b_r \geq 0$ ($r = 1, \dots, n$) there is a $T = \sum_{r=1}^n a_r \varphi_r$ with $|a_r| = b_r$ ($r = 1, \dots, n$) such that

$$(4.8) \quad J_{2r}(T) \leq A_r \int \left(\sum_{r=1}^n b_r^2 |\varphi_r|^2 \right)^r.$$

In (4.8) the φ_r are quite arbitrary.

In the case of real φ_r let $R_1(w), \dots, R_n(w)$ ($0 \leq w \leq 1$) be the first n Rademacher functions. In the complex case let

$$R_1(w) = \exp [i2\pi k/(r+1)]$$

$$(k/(r+1) \leq w \leq (k+1)/(r+1); \quad k = 0, 1, 2, \dots),$$

$$R_r(w) = R_1[(r+1)^{r-1}w] \quad (r = 2, \dots, n; \quad 0 \leq w \leq 1).$$

The latter R_r are independent random variables on $[0, 1]$, contrived so that

$$(4.9) \quad \int_0^1 R_r^\alpha \bar{R}_r^\beta = \delta_{\alpha\beta} \quad (\alpha, \beta = 0, \dots, r; \quad r = 1, \dots, n).$$

This property ensures that the functions $\Psi_r(w, x) = R_r(w)\varphi_r(x)$ form a P -system for r on $[0, 1] \times H$ (no matter what the φ_r are). We can then apply Theorem 1 to $U(w, x) = \sum_{r=1}^n b_r \Psi_r(w, x)$. This gives

$$(4.10) \quad \begin{aligned} J_{2r}(U) &\leq A_r \iint \left| \sum_{r=1}^n b_r R_r(w)\varphi_r(x) \right|^{2r} dw dx, \\ \int J_{2r}(U(w, \cdot)) dw &\leq A_r \int \left(\sum_{r=1}^n b_r^2 |\varphi_r|^2 \right)^r, \end{aligned}$$

where (4.10) follows from [2] Chap. V, theorem 8.4 in the real case, and from a similar calculation in the complex case. Thus (4.8) holds with $T(x) = U(w, x)$ for an appropriate w .

If $\varphi_r(x) = \cos vx$ ($v = 0, 1, \dots$), say, then (4.8) shows that we cannot have

$$J_{2r}(T) \geq A_r \int |\hat{T}|^{2r} \quad (r > 1).$$

For there is no upper bound on $\|\hat{T}\|_q / \|T\|_2$ for a fixed $q > 2$. The same holds for any uniformly bounded complete orthonormal P -system on a bounded interval.

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