Holomorphic Functions with Gap Power Series (III)

ALFRED GRAY & S. M. SHAH*

Communicated by G. SPRINGER

1. Introduction. Let $\{r(n)\}$ be an infinite sequence of real numbers with $0 < r(1) \leq r(2) \leq \dots$ and

$$\lim_{n \rightarrow \infty} r(n) = R,$$

where $0 < R \leq \infty$. There are two natural ways that such a sequence arises in the theory of functions holomorphic for $|z| < R$: the sequence $\{r(n)\}$ may be the zeros of $f(z)$ or it may be the jump points of the central index $\nu(r)$. Thus the manner in which $\{r(n)\}$ increases determines to a great extent the behavior of the maximum modulus $M(r)$ as $r \rightarrow R$ (by Jensen's theorem in the former case and by approximation by the maximum term $\mu(r)$ in the latter).

A particularly interesting situation, which we investigate in this paper, occurs when the sequence $\{r(n)\}$ has a great many repetitions. This means that the range $\{n_k\}$ of the counting function

$$\iota(t) = \sum_{r(n) \leq t} 1$$

is a lacunary sequence or has many latent gaps. In sections 2 to 4 we introduce some definitions and prove some theorems relating the irregularity of the growth of $\iota(t)$ to the rate of increase of $\{n_k\}$. Then in section 5 we apply these results to the sequence of zeros of $f(z) - \alpha$ and in section 6 we give some applications to the central index and maximum term. Finally in section 7 we show that some recent results of Buckholtz [2] are best possible, and we give some examples.

2. Definitions. Let $\{r(n)\}$ be a sequence of the type described in section 1 with counting function $\iota(t)$ and let $\{n_k\}$ denote the range of $\iota(t)$ considered as a sequence with $n_0 = 0$. We shall allow repetitions in $\{r(n)\}$ and we shall suppose these are determined by

* The work of the first author was supported by National Science Foundation Grant GP-3990 and that of the second author by National Science Foundation Grant GP-4311. For the first two papers with this title see [4], [5].

$$(2.1) \quad r(n_k) < r(n_k + 1) = \cdots = r(n_{k+1})$$

for $k = 1, 2, 3, \dots$. Also we define a sequence $\{p(n)\}$ by

$$p(n) = \{r(1) \cdots r(n)\}^{1/n}$$

so that

$$p(n_s) = \left\{ \prod_{k=1}^s r(n_k)^{n_k - n_{k-1}} \right\}^{1/n_s}.$$

The averaging function $I(t)$ of $\iota(t)$ is defined by

$$I(t) = \int_0^t \frac{\iota(s)}{s} ds = \iota(t) \log t - \int_0^t (\log s) d\iota(s).$$

We are especially interested in the relations between $\{r(n)\}$, $\{p(n)\}$, $\iota(t)$, and $I(t)$ when $\{r(n)\}$ has a great many repetitions, that is, when $\{n_k\}$ increases rapidly. To this end we define various quantities to measure how fast $\{r(n)\}$ increases and the nature of the repetitions. We set

$$\Gamma = \limsup_{k \rightarrow \infty} \{\log r(n_{k+1}) - \log p(n_k)\}$$

$$C = \limsup_{k \rightarrow \infty} \{\log r(n_k) - \log p(n_k)\}$$

$$L = \limsup_{n \rightarrow \infty} \frac{r(n+1)}{r(n)}$$

$$\ell^* = \liminf_{k \rightarrow \infty} \frac{r(n_{k+1})}{r(n_k)}$$

$$B = \limsup_{k \rightarrow \infty} \frac{n_{k+1}}{n_k}$$

$$A = \limsup_{k \rightarrow \infty} \frac{n_{k+1} - n_k}{n_k - n_{k-1}}$$

$$S = \limsup_{k \rightarrow \infty} (n_{k+1} - n_k).$$

3. Lemmas.

Lemma 1. *We have*

$$(3.1) \quad \Gamma = \limsup_{t \rightarrow R} \frac{I(t)}{\iota(t)}$$

$$(3.2) \quad c = \liminf_{t \rightarrow R} \frac{I(t)}{\iota(t)}.$$

Proof. Let $r(n_k) \leq t < r(n_{k+1})$. Then

$$\frac{I(t)}{\iota(t)} = \log t - \log p(n_k)$$

and so

$$\log r(n_k) - \log p(n_k) \leq \frac{I(t)}{\iota(t)} \leq \log r(n_{k+1}) - \log p(n_k).$$

The lemma follows easily from this equation.

Remark. When $R < \infty$, $\Gamma = c = 0$.

Lemma 2. If γ and C are not both equal to ∞ then

$$(3.3) \quad \max(1, e^{\gamma-c}) \leq \ell^*.$$

If $C < \infty$, then

$$(3.4) \quad \begin{cases} \ell^* \leq \min(e^{\gamma-c}, e^{\Gamma-c}) \\ L \geq \max(e^{\gamma-c}, e^{\Gamma-c}). \end{cases}$$

If $c < \infty$, then

$$(3.5) \quad L \leq e^{\Gamma-c}.$$

We omit the proof. Note that when $L > 1$ either $\Gamma > c$ or $\Gamma = \infty$. In section 7 we give an example (example 5) for which $L = \ell = 1$ and $\Gamma > c$. The relations (3.3)–(3.5) hold trivially when $R < \infty$.

Lemma 3. If $1 < b \leq B < \infty$, then

$$(3.6) \quad B \leq A \leq \frac{b(B-1)}{b-1}.$$

If $1 < B < \infty$ then

$$(3.7) \quad a \geq \frac{B(b-1)}{B-1}.$$

With no restriction on b or B ,

$$(3.8) \quad \max(a, 1) \leq b \leq B \leq A.$$

Finally if $1 < b = B \leq \infty$ then $a = A$.

We omit the proof which is straightforward.

4. Properties of the counting function.

Theorem 1. Assume $R = \infty$ and let

$$\limsup_{r \rightarrow \infty} \frac{\log \iota(r)}{\log r} = \begin{cases} \rho \\ \lambda \end{cases}$$

and interpret $(\log b)/(b-1)$ as 1 when $b = 1$. Then we have

$$(4.1) \quad \rho \leq \frac{\rho(b-1)}{\log b} \leq \limsup_{r \rightarrow \infty} \frac{\iota(r)}{I(r)} \leq \frac{B-1}{\log l^*},$$

with the conventions that if either $B = \infty$ or $\ell^* = 1$, then the right extreme is equal to ∞ , and the second term is 0 when $\rho = 0$. Also

$$(4.2) \quad \frac{b-1}{b \log L} \leq \liminf_{r \rightarrow \infty} \frac{\iota(r)}{I(r)} \leq \frac{\lambda(b-1)}{b \log b} \leq \lambda$$

with the conventions that the left extreme is 0 if either $b = 1$ or $L = \infty$, and the third term is equal to ∞ when $\lambda = \infty$.

Proof. Let $r(n_k) \leq r < r(n_{k+1})$ and suppose that $1 < B < \infty$, $1 < \ell^* < \infty$. Then

$$\begin{aligned} \frac{I(r)}{\iota(r)} &= \log r - \frac{1}{n_k} \sum_{p=1}^k (n_p - n_{p-1}) \log r(n_p) \\ &= \log (r/r(n_k)) + \frac{1}{n_k} \sum_{p=1}^{k-1} n_p \log (r(n_{p+1})/r(n_p)) \\ &> (\log l^* + o(1)) \left(\sum_{p=1}^{k-1} B^{p-k} + o(1) \right) \\ &> \frac{\log l^*}{B-1} + o(1). \end{aligned}$$

This proves the right extreme of (4.1). The proof of the middle inequality is similar to that of a theorem of Buckholtz ([2]; theorem A below). Finally $(b-1)/\log b$ is an increasing function of b , and hence the left extreme of (4.1) follows.

The proof of (4.2) is similar to that of (4.1) and is omitted.

Remark. The inequalities (4.1) and (4.2) hold trivially when $r \rightarrow R$ where $R < \infty$, for in this case $\rho = \lambda = \infty$, $\Gamma = c = 0$, and $\ell^* = L = 1$.

We omit the proof of the following theorem which shows that if $\{r(n)\}$ is a sequence which tends to infinity in a regular fashion, we have $\Gamma = c$.

Theorem 2. Let ρ be any positive finite number, and let φ be a function satisfying the conditions

$$(4.3) \quad 0 < n^{1/\rho} \varphi(n) \leq (n+1)^{1/\rho} \varphi(n+1), \quad n = 1, 2, \dots,$$

$$(4.4) \quad \log \varphi(n+1) - \frac{1}{n} \log (\varphi(1) \cdots \varphi(n)) = o(1)$$

as $n \rightarrow \infty$. (In particular we may choose φ to be monotone increasing and slowly changing.) Let $\{r(n)\}$ be defined by

$$r(n) = n^{1/\rho} \varphi(n).$$

Then

$$\Gamma = c = \lim_{r \rightarrow \infty} \frac{I(r)}{\iota(r)} = \frac{1}{\rho}.$$

It is known [10] that if $R = \infty$, $0 \leq \lambda < \infty$, and $0 < \rho \leq \infty$, then

$$\limsup_{r \rightarrow \infty} \frac{I(r)}{\iota(r) \log r} \leq 1 - \frac{\lambda}{\rho}.$$

We consider here the case $\rho = 0$ and define

$$\limsup_{r \rightarrow \infty} \frac{\iota(r)}{\inf (\log r)^\Delta L(r)} = \begin{cases} D \\ d \end{cases}$$

where $0 < \Delta < \infty$, $L(r)$ is ultimately positive and non-decreasing, and D, d depend on Δ and the function L .

Theorem 3. *We have*

$$(4.5) \quad \limsup_{r \rightarrow \infty} \frac{I(r)}{\iota(r) \log r} \leq 1 - \left(\frac{d}{D}\right)^{1/\Delta} \left(\frac{b-1}{b}\right) \left(\frac{B^{1+1/\Delta}}{B^{1+1/\Delta} - 1}\right).$$

We make the conventions that if $d = \infty$ or $D = 0$, then the right hand of (4.5) is equal to 1; if $B = \infty$ we replace $B^{1+1/\Delta}(B^{1+1/\Delta} - 1)^{-1}$ by 1; and if $b = \infty$ we replace $b/(b-1)$ by 1.

If $B = 1$ then

$$(4.6) \quad \limsup_{r \rightarrow \infty} \frac{I(r)}{\iota(r) \log r} \leq 1 - \frac{\Delta}{\Delta + 1} \left(\frac{d}{D}\right)^{1/\Delta} \left(\frac{s}{S}\right)^{1+1/\Delta}$$

with the conventions that if $d = \infty$, $D = 0$, or $s = \infty$, then the right hand side of (4.6) is equal to 1.

Proof. For (4.5) we may assume that $L(r)$ is positive and non-decreasing for $r \geq n_1$. Let $r(n_k) \leq r < r(n_{k+1})$ and assume first that $B < \infty$ and $0 < d \leq D < \infty$. We have $n_k = \iota(r(n_k))$ and so

$$\log r(n_k) > \left\{ \frac{n_k}{DL(r(n_k))} \right\}^{1/\Delta} (1 + o(1)).$$

Thus

$$(4.7) \quad \begin{aligned} I(r) &= n_k \log r - \sum_{p=1}^k (n_p - n_{p-1}) \log r(n_p) \\ &\leq n_k \log r - (DL(r))^{-1/\Delta} (1 - 1/b) \sum_{p=1}^k n_p^{1+1/\Delta} (1 + o(1)). \end{aligned}$$

Now

$$\sum_{p=1}^k n_p^{1+1/\Delta} > n_k^{1+1/\Delta} \left(\frac{B^{1+1/\Delta}}{B^{1+1/\Delta} - 1} + o(1) \right),$$

and so we get (4.5). If $b < B = \infty$ and $0 < d \leq D < \infty$, we have from (4.7) that

$$I(r) \leq n_k \log r - (DL(r))^{-1/\Delta} (1 - 1/b) n_k^{1+1/\Delta} (1 + o(1)),$$

and so

$$\limsup_{r \rightarrow \infty} \frac{I(r)}{\iota(r) \log r} \leq 1 - \left(\frac{d}{D}\right)^{1/\Delta} \left(\frac{b-1}{b}\right).$$

If $b = \infty$ and $0 < d \leq D < \infty$, then

$$I(r) \leq n_k \log r - (DL(r))^{-1/\Delta} n_k^{1+1/\Delta} (1 + o(1)),$$

so that

$$\limsup_{r \rightarrow \infty} \frac{I(r)}{\iota(r) \log r} \leq 1 - \left(\frac{d}{D}\right)^{1/\Delta}.$$

We omit the proofs of (4.6) and of the remaining cases of (4.5), which are similar.

Also, we omit the proof of the following theorem.

Theorem 4. *Let*

$$L(t) = \exp \{ t^\alpha (\log t)^\beta (\log \log t)^\omega V(t) \}$$

where $V(t)$ is any logarithmico-exponential function [7] which is ultimately positive and for which

$$V'(t) = o\left(\frac{V(t)}{t(\log t)(\log \log t)}\right) \quad \text{as } t \rightarrow \infty.$$

Let

$$\limsup_{t \rightarrow \infty} \frac{\iota(t)}{L(t)} = \begin{cases} D_1 \\ d_1 \end{cases},$$

where D_1 and d_1 depend on α , β , ω , and $V(t)$. Suppose $0 < d_1 \leq D_1 < \infty$. Then we have:

(a) if $\alpha > 0$ or $\alpha = 0, \beta > 0$ or $\alpha = \beta = 0, \omega > 1$ or $\alpha = \beta = 0, \omega = 1, V(t) \rightarrow \infty$, then

$$\lim_{t \rightarrow \infty} \frac{I(t)}{\iota(t) \log t} = 0;$$

(b) if $\alpha = \beta = 0, \omega = 1$, and $V(t) \rightarrow \Omega \neq \infty$, then

$$\limsup_{t \rightarrow \infty} \frac{I(t)}{\iota(t) \log t} \leq \min \left(\frac{D_1}{d_1(\Omega + 1)}, 1 \right)$$

and

$$\liminf_{t \rightarrow \infty} \frac{I(t)}{\iota(t) \log t} \geq \frac{d_1}{D_1(\Omega + 1)};$$

(c) if $\alpha = \beta = 0$ and $0 \leq \omega < 1$, then

$$\liminf_{t \rightarrow \infty} \frac{I(t)}{\iota(t) \log t} \geq \frac{d_1}{D_1}.$$

5. Value distribution theory. Let $f(z)$ be meromorphic for $|z| < \infty$. If $|\alpha| < \infty$ we denote by $\{r(n, \alpha)\}$ the sequence of moduli of those zeros of $f - \alpha$ which are not at the origin. Similarly if $\alpha = \infty$ we denote by $\{r(n, \infty)\}$ the sequence moduli of those numbers $z \neq 0$ for which f has a pole at z . We take multiplicity into account and assume that the $r(n, \alpha)$ satisfy (2.1). If $n(r, \alpha)$ denotes, as usual, the number of times $f(z) = \alpha$ for $|z| \leq r$, and if $p(\geq 0)$ denotes the multiplicity of the number of the zeros of $f - \alpha$ (or the order of the pole if $\alpha = \infty$) at the origin, then

$$\iota(r) + p = n(r, \alpha).$$

Assume $n(r, \alpha) \rightarrow \infty$ as $r \rightarrow \infty$ and write

$$N(r, \alpha) = \int_0^r \frac{n(t, \alpha) - p}{t} dt + p \log r.$$

Then $N(r, \alpha) = I(r) + p \log r$, and we deduce from theorem 1,

Theorem 5. Let $|\alpha| \leq \infty$ and

$$\lim_{r \rightarrow \infty} \sup \frac{\log n(r, \alpha)}{\log r} = \begin{cases} \rho_1(\alpha) \\ \lambda_1(\alpha) \end{cases}.$$

Then

$$(5.1) \quad \rho_1(\alpha) \leq \frac{\rho_1(\alpha)(b-1)}{\log b} \leq \limsup_{r \rightarrow \infty} \frac{n(r, \alpha)}{N(r, \alpha)} \leq \frac{B-1}{\log l^*}$$

$$(5.2) \quad \frac{b-1}{\log L} \leq \liminf_{r \rightarrow \infty} \frac{n(r, \alpha)}{N(r, \alpha)} \leq \frac{\lambda_1(\alpha)(b-1)}{b \log b} \leq \lambda_1(\alpha).$$

We use the same conventions as in theorem 1.

The inequality (5.2) improves a known result [1, p. 17; 8]; see also [6].

We next give an application of theorem 3. Let f be an entire function of order zero and α any finite constant. It is known [1, p. 49] that the following are equivalent:

$$\log M(r) \sim N(r, \alpha)$$

$$N(r, \alpha) = O((\log r)^2)$$

$$n(r, \alpha) = O(\log r).$$

Taking $\Delta = 1$ and $L(r) \equiv 1$ in theorem 3 we have

Theorem 6. Let $f(z)$ be an entire function of zero order and assume $|\alpha| < \infty$.

Write

$$\lim_{r \rightarrow \infty} \sup \frac{n(r, \alpha)}{\log r} = \begin{cases} D \\ d \end{cases},$$

and suppose $B > 1$, $D < \infty$. Then

$$(5.3) \quad \limsup_{r \rightarrow \infty} \frac{\log M(r)}{n(r, \alpha) \log r} \leq 1 - \left(\frac{d}{D}\right) \left(\frac{b-1}{b}\right) \left(\frac{B^2}{B^2-1}\right).$$

If $B = 1$, $D < \infty$,

$$(5.4) \quad \limsup_{r \rightarrow \infty} \frac{\log M(r)}{n(r, \alpha) \log r} \leq 1 - \frac{1}{2} \left(\frac{d}{D}\right) \left(\frac{s}{S}\right)^2.$$

We use the same conventions as in theorem 3.

6. The maximum term and central index. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |a_0| = 1,$$

be holomorphic for $|z| \leq R$, $R \leq \infty$. We denote by

$$\mu(r) = \max \{|a_k| r^k : k = 0, 1, 2, \dots\}$$

the maximum term of f and $\nu(r)$ the central index of f . We shall assume that $\nu(r) \rightarrow \infty$ as $r \rightarrow R$ and that f is not a polynomial. Denote the range of $\nu(r)$ by $\{n_k\}$ and the discontinuities of $\nu(r)$ by $\{\rho(n)\}$. If we take $r(n) = \rho(n)$ then we have $\nu(r) = \iota(r)$ and $I(r) = \log \mu(r)$.

Theorem 7. If $\gamma > c$ or $\Gamma > C$,

$$\limsup_{r \rightarrow \infty} \frac{\mu(r)}{M(r)} > \liminf_{r \rightarrow \infty} \frac{\mu(r)}{M(r)}.$$

Proof. The hypotheses imply that $L > 1$ and $R = \infty$. The theorem now follows from [3].

It is not true that

$$\limsup_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r)} > \liminf_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r)}$$

implies

$$(6.1) \quad \limsup_{r \rightarrow \infty} \frac{\mu(r)}{M(r)} > \liminf_{r \rightarrow \infty} \frac{\mu(r)}{M(r)},$$

even though $L > 1$ implies (6.1). The converse is also false; see section 7(v).

The maximum term $\mu(r)$ may be extended to a (discontinuous) complex function by the formula

$$\mu(re^{i\theta}) = \mu(r)e^{i\nu(r)\theta}$$

for $r > 0$ and $0 \leq \theta < 2\pi$. Lemmas 2 and 3 together with [5] yield the following result.

Theorem 8. *Suppose*

$$f(z) = \epsilon(0) + \sum_{k=1}^{\infty} \frac{\epsilon(k)z^{n_k}}{\rho(1) \cdots \rho(n_k)}$$

has radius of convergence R ($0 < R \leq \infty$). Assume that $|\epsilon(k)| = 1$ and $\epsilon(k)$ has period h , where h is a positive integer. Also, assume

$$0 < \varphi = \lim_{k \rightarrow \infty} (n_{k+1} - n_k) \log \left\{ \frac{\rho(n_{k+1})}{\rho(n_k)} \right\} < \infty$$

and $a = A < \infty$. Let p and q be positive integers with $q < h$. For

$$\rho(n_{ph+q}) \leq |z| < \rho(n_{ph+q+1})$$

write

$$z = \rho(n_{ph+q}) \exp \left(\frac{\varphi W}{n_{ph+q+1} - n_{ph+q}} \right)$$

for $0 \leq \operatorname{Re} W \leq \beta$ (where $1/2 < \beta < 1$, β fixed). Define

$$G_q(W, 1) = \sum_{j=-\infty}^{\infty} \epsilon(j+q) \exp \left(\frac{-\varphi j}{2} (j+1-2W) \right)$$

and

$$G_q(W, x) = \sum_{j=-\infty}^{\infty} \epsilon(j+q) \cdot \exp \left\{ \frac{-\varphi}{(x-1)^2} (x^{j+1} - (j+1)x + j - (x-1)(x^j - 1)W) \right\}$$

for $1 < x < \infty$. Then

$$(i) \quad \lim_{p \rightarrow \infty} \frac{f(z)}{\mu(z)} = G_q(W, a)$$

and the limit is uniform in W .

(ii) *$f(z)$ has no asymptotic value.*

We omit the proof (see [5]).

7. Concluding remarks.

(i) We now show that the inequalities (4.1), (4.2) and the inequalities given by Buckholtz in the following theorem A are best possible.

Theorem A [2]. *Suppose*

$$\sum_{m=0}^{\infty} a_{p_m} z^{p_m}$$

is an entire function of order ρ and lower order λ whose gaps satisfy

$$\liminf_{m \rightarrow \infty} \frac{p_{m+1}}{p_m} \geq 1 + \theta > 1.$$

Then

$$(7.1) \quad \liminf_{r \rightarrow \infty} \{\mu(r)\}^{1/\nu(r)} \leq \alpha^{1/\rho},$$

$$(7.2) \quad \limsup_{r \rightarrow \infty} \{\mu(r)\}^{1/\nu(r)} \geq \beta^{1/\lambda},$$

where $\alpha = (1 + \theta)^{1/\theta}$ and $\beta = (1 + \theta)^{(1+\theta)/\theta}$.

Write

$$\lim_{r \rightarrow \infty} \sup \frac{\mu(r)}{\inf M(r)} = \begin{cases} U \\ u \end{cases}, \quad \lim_{r \rightarrow \infty} \sup \frac{\log \mu(r)}{\inf \nu(r)} = \begin{cases} \Gamma \\ c \end{cases}.$$

Example 1. Let

$$A(p, \sigma; z) = \sum_{k=0}^{\infty} p^{-\sigma k p^k} z^{p^k - 1}$$

where $\sigma > 0$ and p is an integer larger than 1. Then $A(p, \sigma; z)$ is an entire function with $\rho = \lambda = 1/\sigma$, $B = b = p$, $L = \ell^* = p^\sigma$, and $\ell = 1$. We may take $p = 1 + \theta$ and get equality in (7.1) and (7.2). Moreover, there is equality among the last three terms of (4.1) and among the first three terms of (4.2). Further $U > u$ and $\Gamma > c$.

Example 2. Let

$$(7.3) \quad g(z) = 1 + \sum_{k=1}^{\infty} \frac{z^{n_k}}{\rho(1) \cdots \rho(n_k)},$$

where $n_k = 2^{k-1} = \rho(n_k) < \rho(n_k + 1) = \cdots = \rho(n_{k+1})$, for $k = 1, 2, \dots$. Then $f(z)$ is an entire function of order $\rho = \lambda = 1$. Further $\ell^* = L = B = b = 2$ and there is equality among the last three terms of (4.1) and among the first three terms of (4.2).

Example 3. For the function e^z we have equality among the first three terms of (4.1), and among the last three terms of (4.2).

(ii) If $\theta \rightarrow \infty$, then $\alpha \rightarrow 1$, and $\beta \rightarrow \infty$ in (7.1) and (7.2). However, the left hand sides of (7.1) and (7.2) may still equal 1 and ∞ , even when θ is finite.

Example 4. There exists an entire function

$$\sum_{k=0}^{\infty} a_{n_k} z^{n_k}$$

for which $1 + \theta = b < \infty$,

$$\liminf_{r \rightarrow \infty} \mu(r)^{1/\nu(r)} = 1,$$

and

$$\limsup_{r \rightarrow \infty} \mu(r)^{1/\nu(r)} = \infty.$$

Let $f(z)$ be defined by (7.3) where we take

$$0 < \rho(1) \leq \dots, \quad \rho(n_k) < \rho(n_{k+1}) = \dots = \rho(n_{k+1}).$$

Then the range of $\nu(r, f)$ is $\{n_k\}$ and the sequence of points of discontinuity of $\nu(r, f)$ is $\{\rho(n)\}$. Hence to define these two sequences it suffices to define $\nu(r, f)$. Let $\{x_n\}$ be a sequence of positive integers such that

$$x_{n+1} = [t^3 x_n \log x_n], \quad n = 1, 2, \dots,$$

where $t > 1$. We set

$$\nu(r, f) = \begin{cases} x_n, & x_n \leq r < [tx_n], \\ [tx_n], & [tx_n] \leq r < [t^2 x_n], \\ [t^2 x_n \log x_n], & [t^2 x_n] \leq r < x_{n+1}, \end{cases}$$

$n = 1, 2, \dots$. Then $\rho = \lambda = 1$, $B = \infty$, $b = t$, $L = \infty$, $\ell^* = b$, $\Gamma = \infty$, $c = 0$. For this function there is equality between the third and fourth terms of (4.1), and between the first and second terms of (4.2).

(iii) For any entire function $f(z)$, it is known [9] that

$$(7.4) \quad \liminf_{n \rightarrow \infty} n \log \left\{ \frac{\rho(n+1)}{\rho(n)} \right\} \leq \lim_{r \rightarrow \infty} \sup \frac{\log \mu(r)}{\nu(r)} \\ \leq \limsup_{n \rightarrow \infty} n \log \left\{ \frac{\rho(n+1)}{\rho(n)} \right\}.$$

When $L = b = 1$ (e.g. $f(z) = e^z$), (7.4) gives a better upper bound for

$$\limsup_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r)}$$

than (4.2) and a better lower bound for

$$\liminf_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r)}$$

than (4.1). However, when $L \neq 1$ and $b \neq 1$, theorem 1, gives the better bounds (see example 1, above).

(iv) From (3.5) it follows that when $L > 1$, either $\Gamma > c$ or $\Gamma = \infty$. We now show that $L = 1$ does not imply $\Gamma = c$.

Example 5. Let $\{m_k\}$ be a rapidly increasing sequence of integers such that $m_{k+1} \geq m_k^2$, $k = 1, 2, \dots$, and let $r(n) = \rho(n)$ be defined as follows:

$$\rho(n) = \begin{cases} n + 1, & 1 \leq n \leq m_1, \\ \rho(m_{2k-1}) \prod_{p=m_{2k-1}+1}^n \left(1 + \frac{1}{p}\right), & m_{2k-1} < n \leq m_{2k}, \\ \rho(m_{2k}) \frac{n+1}{m_{2k}+1}, & m_{2k} < n \leq m_{2k+1}, \end{cases}$$

$k = 1, 2, \dots$. Then $L = \Gamma = C = 1$, and $\gamma = c = 0$. Further $\lambda \geq 1$,

$$c = \liminf_{n \rightarrow \infty} n \log \left\{ \frac{\rho(n+1)}{\rho(n)} \right\},$$

and

$$\Gamma = \limsup_{n \rightarrow \infty} n \log \left\{ \frac{\rho(n+1)}{\rho(n)} \right\}.$$

(v) We give examples of entire functions to show that

$$\limsup_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r)} > \liminf_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r)},$$

neither implies nor is implied by

$$\limsup_{r \rightarrow \infty} \frac{\mu(r)}{M(r)} > \liminf_{r \rightarrow \infty} \frac{\mu(r)}{M(r)},$$

(a) Let $\{m_k\}$ be a rapidly increasing sequence of integers and $\{\epsilon_n\}$ the sequence of positive numbers defined by

$$\epsilon_n = \begin{cases} 1, & 1 \leq n \leq m_1, \\ \frac{1}{n!}, & m_k < n < \frac{1}{2}m_{k+1}, \\ \frac{1}{\log n}, & \frac{1}{2}m_{k+1} \leq n \leq m_{k+1}, \quad k = 1, 2, \dots \end{cases}$$

Then set $\rho(n) = \exp(\epsilon_1 + \dots + \epsilon_n)$ and define

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{\rho(1) \dots \rho(n)}.$$

It can be verified that $f(z)$ is an entire function of order $\rho = \infty$, lower order $\lambda = 0$, and $\Gamma = \infty$, $c = 0$, $L = 1$, $U = u = 0$, and

$$\limsup_{n \rightarrow \infty} n \log \left\{ \frac{\rho(n+1)}{\rho(n)} \right\} = \begin{cases} \infty \\ 0 \end{cases}.$$

(b) For any entire function $f(z)$ for which $\log M(r) = O((\log r)^2)$ we have $U > u$ and $\Gamma = c = \infty$.

(c) Let $f(z)$ be defined by (7.3) where $n_1 = 1$ and $n_{k+1} - n_k = \max(1, [k \log k])$.

Also, define $\rho(n_k) = k^2 < \rho(n_k + 1) = \dots = \rho(n_{k+1})$ for $k = 1, 2, \dots$. Then $f(z)$ is an entire function for which

$$\rho = \lambda = 1 = U > u \quad \text{and} \quad \Gamma = c = 1 = L = B = b.$$

REFERENCES

- [1] BOAS, R. P., *Entire Functions*, Academic Press, New York, 1954.
- [2] BUCKHOLTZ, J. D., The maximum term of an entire series with gaps, *Proc. Amer. Math. Soc.*, **16** (1965) 272-276.
- [3] GRAY, A. & SHAH, S. M., A note on entire functions and a conjecture of Erdős, *Bull. Amer. Math. Soc.*, **69** (1963) 573-577.
- [4] ———, Holomorphic functions with gap power series, *Math. Zeit.*, **86** (1965) 375-394.
- [5] ———, Holomorphic functions with gap power series (II), *Jour. Math. Analysis and Applications*, **13** (1966) 370-382.
- [6] GOPALKRISHNA, J., Relations between the growths of an increasing $f(x)$ and $\int^x (f(t)/t) dt$ and applications to entire functions, *J. Ind. Math. Soc.*, **29** (1965) 49-62.
- [7] HARDY, G. H., *Orders of Infinity*, Cambridge, 1927.
- [8] SHAH, S. M., Note on a theorem of Polya, *J. Ind. Math. Soc.*, **5** (1941) 189-191.
- [9] ———, The maximum term of an entire series (V), *J. Ind. Math. Soc.*, **13** (1949) 60-64.
- [10] SHAH, S. M. & SINGH, S. K., Note on a step function, *Quat. J. Math. Oxford* (2), **9** (1958) 63-67.

University of California, Berkeley
 University of Kentucky, Lexington
Date Communicated: FEBRUARY 22, 1966