Analytic Extensions and Spectral Localization

J. G. STAMPFLI*

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An important part of the theory of spectral operators is devoted to finding conditions sufficient to insure spectrality. One necessary condition which is particularly difficult to verify is Dunford's boundedness condition ($\sigma(x) \cap \sigma(y) = \phi$ implies $\|x\| \leq K \|x + y\|$ where K is independent of x and y and depends only on the operator in question). We will give no fuller definition here but remark that this condition is intimately related to the countable additivity of a resolution of the identity.

For certain subnormal and hyponormal operators, we will show there exists a inclusion relation between the local spectrum of the operator and that of its adjoint. From this relation follows Dunford's boundedness condition as well as self adjointness of the spectral resolution if we are fortunate enough to have one available.

We will always use H to designate a separable Hilbert space. An operator T on a Hilbert space H is hyponormal if $\|Tx\| \geq \|T^*x\|$ for all $x \in H$. An operator T will be called completely hyponormal if there is no reducing subspace of T on which T is normal. Since the eigenvectors of a hyponormal operator are reducing [7], the point spectrum of a completely hyponormal operator must be empty. The symbols $\sigma_P(T)$, $\sigma_C(T)$ and $\sigma_R(T)$ will designate the point, continuous and residual spectrum of T respectively, (see [5] for definitions).

We now wish to introduce a generalization of spectrum and resolvent set due to Dunford (see [2]). Let $x \in H$, then $(T - zI)^{-1}x = R(T, x, z)$ is an analytic vector valued function for $z \in \rho(T)$. A vector valued function $f(z)$ is called an analytic extension of $R(T, x, z)$ if it is defined and analytic on an open set $D(f)$ containing $\rho(T)$ and if $(T - zI)f(z) = x$ for $z \in D(f)$.

Before proceeding further it would be well to point out that analytic extensions need not be single valued even for "reasonable" operators. Let us consider an example. If $\{e_n\}_{n=1}^\infty$ is an orthonormal basis for H set

$$Ve_n = \begin{cases} 0 & \text{for } n = 1 \\ e_{n-1} & \text{for } n \neq 1. \end{cases}$$

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We will exhibit two different analytic extensions of $R(V, e_1, z)$. First let $f_1(z) = -e_1/z$ for $z \neq 0$; then $(V - zI)f_1(z) = -1/z(V - zI)e_1 = e_1$. Since $f_1(z)$ agrees with $(V - zI)^{-1}$ on $|z| > 1$ it is clearly an analytic extension of $R(V, e_1, z)$. Second, set

$$f_2(z) = \begin{cases} -e_1/z & \text{for } |z| > 1 \\ \sum_{n=0}^{\infty} z^n e_{n+2} & \text{for } |z| < 1. \end{cases}$$

Now $f_2(z)$ certainly has the right form for $|z| > 1$. For $|z| < 1$ we have

$$\begin{aligned} (V - zI)f_2(z) &= (V - zI) \sum_{n=0}^{\infty} z^n e_{n+2} \\ &= \sum_{n=0}^{\infty} (z^n e_{n+1} - z^{n+1} e_{n+2}) = e_1. \end{aligned}$$

Thus $f_2(z)$ is also an analytic extension of $R(V, e_1, z)$. The two extensions $f_1(z)$, $f_2(z)$ do not coincide on $0 < |z| < 1$ however.

For most of the operators in which we are interested this sort of pathology can not occur (or can be ruled out by certain hypothesis).

The author would like to express his appreciation to Professor A. Lebow for suggesting the following:

Lemma 1. *Let T be completely hyponormal on H . Then for $x \in H$, $R(T, x)$ possesses only single-valued extensions.*

Proof. Assume there exists two extensions $f_1(z)$, $f_2(z)$ which differ at a common point, say z_0 . Then $(T - z_0I)[f_2(z_0) - f_1(z_0)] = x - x = 0$ where $[f_2(z_0) - f_1(z_0)] \neq 0$. Thus T has an eigenvalue contrary to our assumptions.

To obtain the conclusion of the above lemma for adjoints of hyponormal operators we must add another condition.

Lemma 2. *Let T be completely hyponormal with $\sigma_R(T) = 0$. Then for $x \in H$, $R(T^*, x)$ has only single-valued extensions.*

Proof. Again assume there exist two extensions $f_1(z)$, $f_2(z)$ which differ at a common point z_0 . Then $(T^* - z_0I)[f_2(z_0) - f_1(z_0)] = 0$ or $z_0 \in \sigma_P(T^*)$. This implies that $\bar{z}_0 \in \sigma_P(T) \cup \sigma_R(T)$. But $\sigma_P(T) \cup \sigma_R(T) = \emptyset$ which completes the proof.

Remark. The word completely could be deleted from the hypotheses of Lemmas 1 and 2 without jeopardizing the conclusion.

From now on we will be dealing with operators which have only single-valued extensions. We thus define a maximal single-valued extension $R_*(T, x, z)$ by taking the union of all extensions of $R(T, x, z)$.

Now we set $\rho(T, x) = \{z : R_*(T, x, z) \text{ is analytic at } z\}$ and $\sigma(T, x) =$ complement of $\rho(T, x)$.

The following basic facts about $\sigma(T, \cdot)$ are quoted from [2].

Proposition 1. $\sigma(T, x) = 0$ implies $x = 0$.

Proposition 2. $\sigma(T, x) =$ is a closed set contained in $\sigma(T)$.

Proposition 3. $\sigma(T, x + y) \subset \sigma(T, x) \cup \sigma(T, y)$.

Proposition 4. If P is an operator which commutes with T then

$$\sigma(T, Px) \subset \sigma(T, x).$$

Proposition 5. For $\alpha \neq 0$; $\sigma(T, \alpha x) = \sigma(T, x)$.

Lemma 3. Let T be hyponormal and $z_0 \in \sigma_c(T)$; thus $(T - z_0 I)^{-1}$ exists as a closed densely defined transformation. If $x \in \text{Domain } (T - z_0 I)^{-1}$ then $x \in \text{Domain } (T^* - \bar{z}_0 I)^{-1}$ and $\|(T^* - \bar{z}_0 I)^{-1}x\| \leq \|(T - z_0 I)^{-1}x\|$.

Proof. We may assume without loss of generality that $z_0 = 0$. Let $x \in \text{Domain } T^{-1}$, then $\|T^*T^{-1}x\| \leq \|TT^{-1}x\| = \|x\|$. Thus T^*T^{-1} may be extended to a bounded transformation on the entire space. Let $x_1, x_2 \in \text{Domain } T^{-1}$ and set $T^{-1}x_i = y_i$ for $i = 1, 2$. Then

$$\begin{aligned} L(x_2) &= (T^{*-1}x_1, x_2) = (x_1, T^{-1}x_2) \\ &= (Ty_1, T^{-1}x_2) = (y_1, T^*T^{-1}x_2), \end{aligned}$$

so $|L(x_2)| \leq \|y_1\| \cdot \|x_2\|$. Hence $L(x_2)$ is a bounded linear functional on $\text{Domain } T^{-1}$ and can be extended to all of H . By the Riesz representation theorem, there exists a vector w such that $L(x_2) = (w, x_2)$, i.e., $w = T^{*-1}x$. Now

$$\begin{aligned} |(T^{*-1}x_1, x_2)| &= |(Ty_1, T^{-1}x_2)| = |(y_1, T^*T^{-1}x_2)| \\ &\leq \|y_1\| \cdot \|x_2\| = \|T^{-1}x_1\| \cdot \|x_2\|, \end{aligned}$$

thus $\|T^{*-1}x_1\| \leq \|T^{-1}x_1\|$ which completes the proof.

Lemma 4. Let $z_0 \in \sigma_c(T)$. If $f(z)$ is an analytic extension of $R(T, x, z)$ defined at z_0 , then $x \in \text{Domain } (T - z_0 I)^{-1}$ and $f(z_0) = (T - z_0 I)^{-1}x$. Moreover, $x \in \text{Domain } (T - z_0 I)^{-n}$ and $f^{(n)}(z_0) = n! (T - z_0 I)^{-(n+1)}x$.

Proof. Let $y \in \text{Domain } (T^* - \bar{z}_0 I)^{-1}$, (which is dense in H). Then

$$\begin{aligned} L(y) &= (x, (T^* - \bar{z}_0 I)^{-1}y) = ((T - z_0 I)f(z_0), (T^* - \bar{z}_0 I)^{-1}y) \\ &= (f(z_0), y) \end{aligned}$$

clearly defines a bounded linear functional on $\text{Domain } (T^* - \bar{z}_0 I)^{-1}$ and hence on H . By the Riesz representation theorem we may conclude that there exists a $v \in H$ where $L(y) = (v, y)$ or $(T - z_0 I)^{-1}x = v$. Now $((T - z_0 I)^{-1}x - f(z_0), (T^* - \bar{z}_0 I)y) = (x - x, y) = 0$ and since vectors of the form $(T^* - \bar{z}_0 I)y$ are dense in H , this proves the first part of the lemma. The second half follows by a similar argument.

Lemma 5. If $x \in \text{Domain } (T - z_i I)^{-1}$ for $i = 1, 2$ then $x \in \text{Domain } (T - z_1 I)^{-1}(T - z_2 I)^{-1}$ and

$$[(T - z_2 I)^{-1} - (T - z_1 I)^{-1}]x = (z_1 - z_2)(T - z_2 I)^{-1}(T - z_1 I)^{-1}x.$$

Proof. This result is well known and is included for completeness.

Theorem 1. Let T be completely hyponormal with $\sigma_R(T) = \phi$. Then for $\bar{z} \in \rho(T, x)$ the vector valued function $(T^* - zI)^{-1}x$ exists and is continuous in the weak topology for fixed $x \in H$. The function is analytic for $\bar{z} \in \rho(T)$ of course.

Proof. For $z \in \rho(T, x)$ we know $x \in \text{Domain } (T - zI)^{-1}$ by Lemma 4 and hence $x \in \text{Domain } (T^* - \bar{z}I)^{-1}$ by Lemma 3. Thus $f(z) = (T^* - zI)^{-1}x$ is well defined for $\bar{z} \in \rho(T, x)$. Let $\bar{z}_0 \in \rho(T, x)$. Then $R_\epsilon(T, x, z)$ is analytic in $J = \{z : |z - \bar{z}_0| \leq \delta\}$ for some $\delta \geq 0$, and hence bounded there. Therefore $\|(T - zI)^{-1}x\| \leq M$ for $z \in J$. From Lemma 4 we know that $\|(T^* - zI)^{-1}x\| \leq M$ for $\bar{z} \in J$ and hence

$$\|[(T^* - zI)^{-1} - (T^* - z_0 I)^{-1}]x\| \leq 2M \quad \text{for } \bar{z} \in J.$$

Given $y \in H$ choose v such that $\|y - (T - \bar{z}_0 I)v\| < \epsilon$ which can be done since $\text{Range } (T - \bar{z}_0 I)$ is dense in H . Then for $\bar{z} \in J$, we have

$$\begin{aligned} |[(T^* - zI)^{-1} - (T^* - z_0 I)^{-1}]x, y| &\leq 2M \|y - (T - \bar{z}_0 I)v\| \\ &\quad + |[(T^* - zI)^{-1} - (T^* - z_0 I)^{-1}]x, (T - \bar{z}_0 I)v| \\ &\leq 2M\epsilon + |((z - z_0)(T^* - z_0 I)^{-1}(T^* - zI)^{-1}x, (T - \bar{z}_0 I)v)| \\ &\leq 2M\epsilon + |z - z_0| \|((T^* - zI)^{-1}x, v)\| \\ &\leq 2M\epsilon + |z - z_0| M \|v\| \leq 3M\epsilon, \end{aligned}$$

for $|z - z_0|$ sufficiently small, which completes the proof.

Definition. We define an *arc* to be a simple closed rectifiable curve with continuous second derivative when parametrized with respect to arc length.

Lemma 6. If T is hyponormal and $\sigma(T)$ lies on an arc, then $\sigma_R(T) = \phi$.

Proof. Assume there exists $z_0 \in \sigma_R(T)$. By translating T we may take $z_0 = 0$. Then there exists $x \in H$ with $\|x\| = 1$ and $(Th, x) = 0$ for $h \in H$. Moreover, $Tx = y$ where $(x, y) = 0$ and $y \neq 0$.

For $z \in \rho(T)$ we set $(T - zI)^{-1}x = \beta x + w$ where $(x, w) = 0$. Then $x = (T - zI)\beta x + (T - zI)w$ or $x = -z\beta x + \beta y + (T - zI)w$. Since $((T - zI)w, x) = 0$, we conclude that $\beta = -1/z$ and $(1/z)(T - zI)^{-1}y = w$. Thus

$$\|(T - zI)^{-1}x\| = \|-(1/z)x + (1/z)(T - zI)^{-1}y\| \geq |1/z| [1 + \|y\|/\|T - zI\|].$$

But

$$\|T - zI\| \leq \|T\| + |z| \leq M \quad \text{for } |z| \leq R.$$

Thus

$$|(T - zI)^{-1}x| \geq |1/z| [1 + \|y\|/M] \quad \text{for } z \in \rho(T) \quad \text{and} \quad |z| \leq R.$$

By rotating, if necessary, we may assume the tangent to $\sigma(T)$ at 0 is the x -axis. Then given $\alpha > 0$, the set $\sigma(T) \cap \{z : |z| \leq \epsilon\}$ is contained in the sector $S_\alpha = \{z : |\arg z| < \alpha\}$ for ϵ sufficiently small. Now if $z = ib$, then $\text{dist}(z, S_\alpha) = |b| \cos \alpha$ and hence $\|(T - zI)^{-1}\| \leq 1/\text{dist}(z, \sigma(T)) \leq 1/|b| \cos \alpha$ for b sufficiently small. (The first inequality above is Theorem 1 of [10].) If $z = ib$, combining what we have yields $|1/b| [1 + \|y\|/M] \leq \|(T - zI)^{-1}x| \leq |1/b| \cos \alpha$ for small b or $[1 + \|y\|/M] \leq 1/\cos \alpha$. Since α was arbitrary this is a contradiction.

Remark. The lemma above is certainly true under more general conditions on $\sigma(T)$. For instance, one may allow piecewise differentiable curves which have left and right hand derivatives at the exceptional points. In fact, by a somewhat different approach, namely cutting off pieces of the spectrum (see [1], [4], and [8], we may allow curves with self intersections which satisfy even weaker smoothness conditions.

The following theorem appeared in [10]. The proof given there contains a gap in that part of the argument which is Theorem 1 in this paper.

Theorem 2. *If T is completely hyponormal and $\sigma(T)$ lies on an arc, then $\sigma(T, x) \supseteq \sigma(T^*, x)$.*

Proof. Since $\sigma(T)$ lies on an arc, we know from Lemma 6 that $\sigma_R(T) = 0$. Thus $(T^* - zI)^{-1}x$ is analytic for $\bar{z} \in \rho(T)$ and continuous in the weak topology for $\bar{z} \in \rho(T, x)$ by Theorem 1. Hence, by a well known theorem (see [6]), $(T^* - zI)^{-1}x$ may be continued analytically across the arc, $\text{conj}[\sigma(T) \cap \rho(T, x)]$, which implies $\rho(T^*, x) \supseteq \rho(T, x)$ and the theorem follows.

Definition. We say the operator T is *quadratically hyponormal* if $(aT^2 + bT + c)$ is hyponormal for a, b, c complex.

Though this definition may seem somewhat artificial, it does include a very respectable class of operators, namely the subnormals. It goes without saying that a hyponormal operator is not necessarily quadratically hyponormal.

Theorem 3. *Let T be a completely and quadratically hyponormal on H with $\sigma_R(T) = \phi$. Then $\sigma(T, x) \supseteq \sigma(T^*, x)$.*

Proof. For $x \in H$, we know already that the function $f(z) = (T^* - zI)^{-1}x$ is continuous in the weak topology for $\bar{z} \in \rho(T, x)$. Let $\bar{z} \in \rho(T, x)$; since $R_*(T, x, z)$ is analytic for $z \in \rho(T, x)$;

$$[(T - zI)^{-1} - (T - \bar{z}_0 I)^{-1}]x/(z - \bar{z}_0) = (T - zI)^{-1}(T - \bar{z}_0 I)^{-1}x$$

is bounded for z in a suitably chosen closed neighborhood of $\bar{z}_0$. Putting this another way

$$\|(T - zI)^{-1}(T - \bar{z}_0 I)^{-1}x\| \leq M \quad \text{for} \quad |z - \bar{z}_0| \leq \delta,$$

for some $\delta > 0$. This is true for $z = z_0$ in particular and this is important since it assures us that $(T^* - z_0 I)^{-2}x$ exists. Since T is quadratically hyponormal, this implies that

$$\|(T^* - zI)^{-1}(T^* - z_0 I)^{-1}x\| \leq M \quad \text{for } |z - z_0| \leq \delta.$$

We now wish to show that $f(z)$ has a derivative at z_0 . This is equivalent to showing

$$\lim_{z \rightarrow z_0} ([(T^* - zI)^{-1} - (T^* - z_0 I)^{-1}] x / (z - z_0), y) = ((T^* - z_0 I)^{-2}x, y),$$

which in turn is equivalent to the statement

$$\lim_{z \rightarrow z_0} ((T^* - zI)^{-1}(T^* - z_0 I)^{-1}x, y) = ((T^* - z_0 I)^{-2}x, y) \quad \text{for all } y \in H.$$

Given y , choose $v \in H$ such that $\|(T - z_0 I)v - y\| < \epsilon$. Then

$$\begin{aligned} & |[(T^* - zI)^{-1} - (T^* - z_0 I)^{-1}] (T^* - z_0 I)^{-1}x, y | \\ & \leq 2M\epsilon + |[(T^* - zI)^{-1} - (T^* - z_0 I)^{-1}] (T^* - z_0 I)^{-1}x, (T - z_0 I)v | \\ & = 2M\epsilon + |[(T^* - zI)^{-1} - (T^* - z_0 I)^{-1}]x, v | \leq 3M\epsilon \end{aligned}$$

for $|z - z_0|$ sufficiently small, by Theorem 1. This shows $f'(z_0)$ exists which completes the proof.

Remark. When T is normal both $R_s(T, x, z)$ and $R_s(T^*, x, z)$ have only single valued extensions and $\sigma(T, x) = \overline{\sigma(T^*, x)}$. This means any reducing normal part of an operator behaves in the best possible manner and makes the hypothesis of complete hyponormality in the above theorem unnecessary. This condition was added as a technical convenience only.

Corollary. Let T be subnormal with $\sigma_R(T) = \phi$. Then $\sigma(T, x) \supset \overline{\sigma(T^*, x)}$.

Proof. Since T is subnormal, any polynomial in T is hyponormal which assures us that T is quadratically hyponormal.

Theorem 4. Let T be an arbitrary operator on H . If $\sigma(T, x) \supset \overline{(\sigma(T^*, x))}$ for all $x \in H$, then $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies $(x, y) = 0$.

Proof. Consider the function

$$f(z) = ((T - zI)^{-1}x, y) = \text{conj } ((T^* - \bar{z}I)^{-1}y, x).$$

Now $f(z)$ is analytic for $z \in \rho(T, x)$ and for $\bar{z} \in \rho(T^*, y)$. But $\rho(T, x) \cup \overline{\rho(T^*, x)} \supset \rho(T, x) \cup \rho(T, y) = \text{complement } [\sigma(T, x) \cap \sigma(T, y)] = \text{complex plane}$. Thus $f(z)$ is an entire function which is clearly bounded and hence a constant. Since $\lim_{z \rightarrow \infty} ((T - zI)^{-1}x, y) = 0$, we have $f(z) = 0$. But $f(z) = ((T - zI)^{-1}x, y) = -\sum_{n=0}^{\infty} (T^n x, y) z^{-(n+1)}$ for $|z| > \|T\|$ and thus the initial coefficient of $f(z)$ which is (x, y) must be zero. The basic idea for this proof is due to J. T. Schwartz.

Corollary 1. *Let T be hyponormal where $\sigma(T)$ lies on an arc. Then $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies $(x, y) = 0$.*

Corollary 2. *Let T be quadratically hyponormal with $\sigma_{\mathbb{R}}(T) = \phi$. Then $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies $(x, y) = 0$.*

Corollary 3. *Let T be subnormal with $\sigma_{\mathbb{R}}(T) = \phi$. Then $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies $(x, y) = 0$.*

Remark. An operator T is said to satisfy the boundedness condition (Dunford) if $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies $\|x\| \leq K \|x + y\|$ where K is independent of x, y and depends only on T . Now for any operator T where $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies $(x, y) = 0$ we have $\|x + y\| = [\|x\|^2 + \|y\|^2]^{1/2} \geq \|x\|$ so that K may be taken as 1. Thus the operators in Corollaries 1, 2, and 3 above all satisfy Dunford's boundedness condition.

We would now like to comment on the possible application of the foregoing material. The statement " $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies $(x, y) = 0$ " means roughly that if there are sufficient commuting projections available, they will form a countably additive resolution of the identity and these projections can in fact be taken to be self adjoint.

However, for some of the classes of operators we have considered there is in general no guarantee of any commuting projections other than 0 and I .

We mention that for any quasi-nilpotent transformation T , we have $\sigma(T, x) = \sigma(T^*, x)$ and $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies $(x, y) = 0$. (In fact, either x or y is 0 for obvious reasons.) From this, one concludes that " $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies $(x, y) = 0$ " can not yield much in the absence of other structure.

However, if T is hyponormal and $\sigma(T)$ lies on an arc, then one may generate a family of commuting projections by cutting across the spectrum. In this case T can be shown to be normal (see [10]).

In [7] the theorem which follows was observed to be true for subnormal operators and no restriction was placed on the residual spectrum. Modest as our result is, we know of no proof which avoids the rather involved machinery herein developed.

Theorem 5. *Let $\sigma(T) = A \cup B$ where A and B are closed, connected and disjoint, and let T be quadratically hyponormal on H with $\sigma_{\mathbb{R}}(T) = \phi$. Then $H = H_1 \oplus H_2$ where H is a reducing subspace of T ; $\sigma(T | H_1) = A$ and $\sigma(T | H_2) = B$.*

Proof. Choose a contour Γ containing A in its interior and B in its exterior. (This must be possible by interchanging A and B if necessary.) Then $P = (1/2\pi i) \int_{\Gamma} (zI - T)^{-1} dz$ is a projection (not necessarily self-adjoint), such that

- 1) $TPH \subset PH$ and $T(I - P)H \subset (I - P)H$,
- 2) $\sigma(T | PH) = A$,
- 3) $\sigma(T | (I - P)H) = B$.

Define $T_1 = T|_{H_1}$ and $T_2 = T|_{H_2}$ where $H_1 = PH$ and $H_2 = (I - P)H$. If we show $H_1 \perp H_2$, we are finished. But for $x_1 \in H_1$, $x_2 \in H_2$ we have $\sigma(T, x_1) \subset A$ and $\sigma(T, x_2) \subset B$ and thus $\sigma(T, x_1) \cap \sigma(T, x_2) \subset A \cap B = \phi$. From Corollary 2 of Theorem 4 we conclude $(x_1, x_2) = 0$.

Before we give a final application for spectral operators, we will need the following lemma. (A complete discussion of spectral operators can be found in [3]).

Lemma 7. *Let T be a spectral operator where $\sigma(T, x) \cap \sigma(T, y) = \phi$ implies $(x, y) = 0$. Then T has a resolution of the identity consisting of self-adjoint projections, i.e., $T = A + N$ where A is normal, N is quasi-nilpotent and A and N commute.*

Proof. Let $\{E(\cdot)\}$ be the resolution of the identity associated with the spectral operator T . We know that $\sigma(T, x) \cap \sigma(T, y) = \phi$ entails Dunford's boundedness condition with $K = 1$. Now Lemma 2 of [4] with some help from surrounding lemmas states that $\|E(\delta)\| \leq K$ for δ a Borel set. Since a projection of norm one is self-adjoint, $E(\delta)$ is self-adjoint for all Borel sets δ . But $T = \int z dE(z) + N$ where N commutes with $\{E(\cdot)\}$ and thus the conclusion follows.

It seems worthwhile to present a second direct proof of self-adjointness for the $E(\cdot)$'s. We make no claim for the originality of this proof, based as it is on ideas in the lemmas of [4] cited above.

Let δ be any closed Borel set. Then $\delta' = \text{complement } \delta = \bigcup_{n=1}^{\infty} \epsilon_n$ where ϵ_n are closed Borel sets disjoint from δ . Since $\{E(\cdot)\}$ is a spectral resolution, $\sigma(T|_{E(\gamma)H}) \subset \gamma$ for γ a closed Borel set. Thus for $x \in E(\delta)H$ and $y \in E(\epsilon_n)H$ we have $\sigma(T, x) \cap \sigma(T, y) \subset \delta \cap \epsilon_n = \phi$ and hence $(x, y) = 0$. Now $E(\delta')H = \text{closure } \bigcup_{n=1}^{\infty} E(\epsilon_n)H$ and hence $E(\delta)H$ is orthogonal to $E(\delta')H$. We observe for any operator A , that $\text{Range } A \perp \text{Range } (I - A)$ implies A is a self-adjoint projection. This completes the proof for closed Borel sets and that is sufficient.

Lemma 8. *Let T be hyponormal, A be normal and let T and A commute. Then $T + A$ is hyponormal.*

Proof. Note that

$$\begin{aligned} (T + A)^*(T + A) &= T^*T + T^*A + A^*T + A^*A \geq TT^* + T^*A \\ &\quad + A^*T + A^*A = TT^* + AT^* + TA^* + AA^* = (T + A)(T + A)^*, \end{aligned}$$

which completes the proof.

Theorem 6. *Let T be a spectral operator which is also quadratically hyponormal. Then T is normal.*

Proof. Let $w \in \sigma(T)$. Assume $E(w)H \neq \phi$ where $\{E(\cdot)\}$ is the spectral resolution of T . Now $E(w)H$ is an invariant subspace of T and $\sigma(T|_{E(w)H}) = \{w\}$. Since the restriction of a hyponormal operator to an invariant subspace is

hyponormal, $T \upharpoonright E(w)H = wI$. (Here I is the identity on $E(w)H$). Thus every vector of $E(w)H$ is an eigenvector of T and $w \in \sigma_P(T)$. Since eigenvectors of a hyponormal operator are reducing, let us assume we have removed the subspace M spanned by the eigenvectors (T is normal on M).

We now wish to show that $\sigma_\kappa(T) = \phi$. Let $w \in \sigma(T)$. We know $E(w)H = \phi$. Define $F_n = \{z : |z - w| \geq 1/n\}$ for $n = 1, 2, \dots$. Then $\sigma(T \upharpoonright E(F_n)H) \subset F_n$ and $(T - wI) \upharpoonright E(F_n)H$ is invertible on $E(F_n)H$ and thus maps $E(F_n)H$ onto $E(F_n)H$. Now

$$\text{closure} \left[\bigcup_{n=1}^{\infty} E(F_n)H \right] = E(\text{complement } w)H = H,$$

so $(T - wI)H$ is dense in H or $w \in \sigma_c(T)$.

If we combine Corollary 2 of Theorem 4 with Lemma 7, we conclude that $T = A + N$ where A is normal, N is quasi-nilpotent, and A commutes with N . Thus $N = T - A$, so by Lemma 8, N is hyponormal. But the only hyponormal, quasi-nilpotent operator is the zero operator. Hence $T = A$ and we are finished.

Corollary. *Let T be a spectral operator which is also subnormal. Then T is normal.*

It can be seen from the proof of Theorem 6 that a spectral operator with normal scalar part is hyponormal if and only if it is normal. This highlights the difficulty of connecting hyponormality with the self adjointness of spectral resolutions. For it's still an open question whether every hyponormal scalar operator is normal. This was decided affirmatively in [9] under the added hypothesis that the spectrum of the operator have zero area.

There exist unpublished proofs by both W. Saffern and the author that a subnormal scalar operator has a self-adjoint spectral resolution. The Corollary above generalizes that result.

REFERENCES

- [1] R. G. BARTLE, Spectral localization of operators in Banach spaces, *Math. Annalen*, **153** (1964) 261-269.
- [2] N. DUNFORD, Spectral theory II. Resolutions of the identity, *Pacific J. Math.*, **2** (1952) 559-614.
- [3] ———, Spectral operators, *Pacific J. Math.*, **4** (1954) 321-354.
- [4] ———, A survey of the theory of spectral operators, *Bull. Amer. Math. Soc.*, **64** (1958) 217-274.
- [5] N. DUNFORD & J. T. SCHWARTZ, *Linear Operators. Vol. I, General Theory, Vol. II, Spectral Theory in Hilbert Space*, Interscience Publishers, New York, 1962.
- [6] E. HILLE, *Analytic function theory, I*, Ginn, Boston, Mass., 1959.
- [7] A. LEBOW, On von Neumann's theory of spectral sets, *Journal Math. Anal. Appl.*, **7** (1963) 64-90.
- [8] J. T. SCHWARTZ, Subdiagonalization of operators in Hilbert space with compact imaginary part, *Comm. Pure Appl. Math.*, **15** (1962) 159-172.
- [9] J. G. STAMPFLI, Hyponormal operators, *Pacific J. Math.*, **12** (1962) 1453-1458.

- [10] ———, Hyponormal operators and spectral density, *Trans. Amer. Math. Soc.*, **117** (1965) 469–476.

The University of Michigan
and

New York University

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