Uniformization of a Quasi-linear Hyperbolic Equation

*Part II. Solution Structure in the Large**

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Abstract. The concepts of regular variables and 'formal' solution (Meyer 1966) are used to study the limit lines of the Generalized Cauchy Problem for a typical system of homogeneous differential equations in two independent and two dependent variables. These lines are of physical interest on account of their relation to bore-formation (in oceanographical applications) and shock-formation (in gas dynamical applications); and of mathematical interest on account of their relation to the breakdown of classical existence theorems.

It is shown that a singular point in the regular plane (corresponding to a limit point) cannot be isolated and a singular curve cannot end in the interior of the domain of the solution. A generalization and new proof are given for Ludford's theorem that any nontrivial solution possesses limit points, if it is continued far enough. These are results in the large concerning domains extending beyond that on which a solution is uniquely determined.

1. Introduction. In the study of the shock-free solutions of quasi-linear hyperbolic equations with two independent variables, attention was attracted early by the connection between limit lines and the breakdown of classical existence of the solutions. Two particular questions arose from this. The first is whether such breakdown is always associated with limit points, and this is answered in the affirmative in Part I (Meyer (1966), Sections and Lemmas of which will be referred to by a prefix I.) for a particular, rather typical class of equations, under conditions eliminating singularities of the equations themselves. The second is whether isolated limit points can occur and whether limit lines can end, questions that obtruded early, since a negative answer was

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obtained for branch points and wave front points and their lines (Craggs 1948, Meyer 1949), which show a good deal of analogy with limit points and lines. In regard to the latter, however, much greater difficulties were encountered. Precise statements were precluded by the fact that the existence theorems did not cover limit points. On a less rigorous level, the local multivaluedness of the solution was partly avoided by posing the questions in regard to the corresponding 'singular' points and lines in the plane of the dependent variables. But this device proved insufficient because branch points, wave front points and singular points can coincide, and the correspondence between the planes can then assume frustrating complexity.

These difficulties were overcome in Part I by showing that, even in very general circumstances, a 'regular characteristic plane' can be introduced, in which an associated solution is singlevalued. This led to an extension of the existence theorem to the maximal domain of uniqueness, and put the discussion of shockfree solution structure on a firm basis. It emerges that singular points are related to the characteristic variables, rather than the dependent ones, and with an appropriate definition, it will be shown in Section 3 that there are indeed no isolated singular points, and singular curves can't end, in any open set on which the associated solution is defined.

The main interest, perhaps, of such results is that they identify features of the solution structure *in the large*, and remain valid, if the domain of definition is extended. This point of view has long been acknowledged as fruitful for elliptic problems. An analytic function, for instance, is known to be determined uniquely, wherever it *can* be defined, once it is defined on an infinite set with limit point. We thus think of it automatically on the whole plane or the Riemann surface. This finds typical expression in theorems such as Liouville's that an entire function, if non-trivial, cannot remain bounded everywhere. For hyperbolic problems, by contrast, the uniqueness theorems show that any given, natural data determine the solution only on a compact set, and moreover, in the absence of extended data, any continuation of the solution beyond that set must be utterly non-unique. This appears to have led to a rather general view that useful statements about hyperbolic continuation cannot be anticipated.

The primary purpose of the following is to emphasize that such a point of view is exaggerated. That branch lines, wave fronts and singular curves cannot end, must be anticipated to apply also to any continuation of a solution which respects the differential equations and weak, qualitative conditions of the Generalized Cauchy Problem. To underline this, it will be shown in Section 4 that—in almost literal analogy to Liouville's theorem—any non-trivial solution of the Generalized Cauchy Problem must possess a limit point, if the solution is continued far enough in the plane of the independent variables, regardless of how it be continued.

The theorem of Section 4 is a generalization of a result of Ludford (1952) for a class of initial-value problems. The proof, moreover, is quite different, being based on the Transport Equations of the differential equations, rather

than on the Riemann representation of the solution. This may open the way to an extension of the theorem to classes of differential equations for which the Riemann representation is more intractable or difficult to obtain.

The Ludford–Liouville theorem recalls a similar, earlier theorem of Hilbert (1901) on surfaces of negative curvature, which also concerns a hyperbolic problem. The singularities involved (Amsler 1955) are, however, quite different from those discussed here. The present results may help to stress that Hilbert's theorem is not an isolated curiosity, but that theorems on the structure in the very large of the solutions of hyperbolic problems should be anticipated as a rule, rather than as an exception.

2. GCP and regular solution. For convenience of reference, the main definitions and results of Part I that are used in Sections 3, 4 are summarized below. The Generalized Cauchy Problem (GCP) is defined as that of finding $r(x, t)$, $s(x, t)$ satisfying

$$\begin{aligned} (1) \quad & \partial r / \partial t + (u + a) \partial r / \partial x = 0, \\ (2) \quad & \partial s / \partial t + (u - a) \partial s / \partial x = 0, \\ (3) \quad & r(x^*, t^*) = r^*, \quad s(x^*, t^*) = s^*. \end{aligned}$$

Here

$$(4) \quad u = r - s, \quad a = a(r + s)$$

and r^* , s^* , x^* and t^* are any four functions of a parameter σ in the class, PCB^1 , of a single-valued functions with piecewise continuous, bounded first derivatives on $(0, 1)$ such that (with $u^* = r^* - s^*$, $a^* = a(r^* + s^*)$ and $' = d/d\sigma$)

- (i) $x'^2 + t'^2 + r'^2 + s'^2 > 0$ on $[0, 1]$,
- (ii) neither $x' - (u^* + a^*)t' \equiv r' \equiv 0$ nor $x' - (u^* - a^*)t' \equiv s' \equiv 0$ on $(0, 1)$, and
- (iii) $r^*(\sigma_1) + s^*(\sigma_2) \geq \epsilon > 0$ for arbitrary σ_1, σ_2 in $(0, 1)$.

It is assumed $a(r + s)$ and $\mu(r + s) = (2a)^{-1}(1 + da/d(r + s))$ are continuous and have positive upper and lower bounds when $(r + s)$ has such bounds. The same then holds for

$$(5) \quad \lambda = \exp \int_{\epsilon}^{r+s} \mu(\omega) d\omega.$$

The regular coordinates of the GCP are defined as

$$(6) \quad \xi(\sigma) = \int_0^\sigma [(x' - (u^* + a^*)t')^2 + r'^2]^{1/2} (a^*)^{-1} d\tau,$$

$$(7) \quad \eta(\sigma) = \int_0^\sigma [(x' - (u^* - a^*)t')^2 + s'^2]^{1/2} (a^*)^{-1} d\tau,$$

or any functions $f(\xi)$, $g(\eta)$ such that $df/d\xi$, $dg/d\eta$ exist and are respectively

either entirely positive or entirely negative on the ranges of $\xi(\sigma)$, $\eta(\sigma)$. The domain $0 < \xi < \xi(1)$, $0 < \eta < \eta(1)$ is called the range Q of the GCP, the curve $\xi = \xi(\sigma)$, $\eta = \eta(\sigma)$ is called the regular Cauchy line L , and the Cartesian products of open intervals of ξ and η are used as base of the topology of the ξ , η -plane. With $r(\xi)$, $s(\eta)$, $u(\xi, \eta)$ and $a(\xi, \eta)$ defined by

$$(8) \quad r(\xi(\sigma)) = r^*(\sigma), \quad s(\eta(\sigma)) = s^*(\sigma)$$

and (4), the problem of finding $x(\xi, \eta)$, $t(\xi, \eta)$ such that

$$(9) \quad \partial x / \partial \eta - (u + a) \partial t / \partial \eta = 0,$$

$$(10) \quad \partial x / \partial \xi - (u - a) \partial t / \partial \xi = 0,$$

$$(11) \quad x(\xi(\sigma), \eta(\sigma)) = x^*(\sigma), \quad t(\xi(\sigma), \eta(\sigma)) = t^*(\sigma),$$

is called the associated regular problem (*ARP*). It has a unique solution (symbolically referred to as Ω) of class $PCB^1(Q)$ with continuous extension to $\bar{Q}$. Similarly, $dr/d\xi = \dot{r}(\xi)$ and $ds/d\eta = \dot{s}(\eta)$ are class $PCB(Q)$. The map $(r, s) \rightarrow (x, t)$ (not generally 1 - 1) defined by Ω and (8) is called the formal solution; its relation to the classical (or 'strict') solution of the GCP is described by the theorems of Section I.3.

The *ARP* implies the 'canonical' equations

$$(12) \quad \partial U / \partial \eta = -\mu V \dot{r},$$

$$(13) \quad \partial V / \partial \xi = -\mu U \dot{s},$$

where

$$(14) \quad U(\xi, \eta) = \lambda \partial t / \partial \xi, \quad V(\xi, \eta) = \lambda \partial t / \partial \eta,$$

and the following lemmas.

Lemma I.3. $U^2 + \dot{r}^2 > 0$ and $V^2 + \dot{s}^2 > 0$ on $\bar{Q}$.

Lemma I.4. For fixed $\xi [\eta]$, $U [V] \in PCB^1$ with respect to $\eta [\xi]$ on Q .

Lemma I.5. $J_1 = \partial(x, t) / \partial(\xi, \eta)$ is of definite sign (or briefly, o.d.s.) at a point $P \in Q$, if and only if both U and V are o.d.s. at P .

Here, a quantity is called o.d.s. at P , if it does not assume zero at P and assumes only one sign on a neighborhood of P .

3. The limit line theorem. A point $P \in Q$ is called 'singular' if J_1 is not o.d.s. at P . A curve $\xi(\tau)$, $\eta(\tau)$ on $[0, 1]$ is called singular if the point with regular coordinates $\xi(\tau)$, $\eta(\tau)$ is singular for every $\tau \in [0, 1]$.

Theorem 1. Any singular point $P \in Q$ is an interior point of a singular curve which is continuous at P .

The following lemma will be proved after the theorem.

Lemma A. Let $f(x, y) \in CB[0, a]$ with respect to x for fixed $y \in (-b, b)$ and $\in PC(-b, b)$ with respect to y for fixed $x \in [0, a]$. Assume that θ_1, θ_2 can be found, for every $\theta \in (0, b)$, such that $0 < \theta_1 < \theta$, $0 < \theta_2 < \theta$ and $f(0, \theta_1)f(0, -\theta_2) < 0$. Then a function $Y(x)$ exists on $(0, a')$ with $0 < a' \leq a$ such that $Y(x) \rightarrow 0$ as $x \rightarrow 0$ and f is not o.d.s. at $(x, Y(x))$.

In view of the dual role played by U and V , and of Lemma I.5, it will suffice to prove the theorem for an arbitrary point $P \in Q$ at which U is not o.d.s. The regular variables may, without loss of generality, be chosen so that $\xi = \eta = 0$ at P , and by Lemma I.4, attention may be restricted to a neighbourhood $N(P)$ such that U and $\dot{r}$ are continuous, except possibly across the axis $\xi = 0$, and V and $\dot{s}$ are continuous, except possibly across the axis $\eta = 0$. When a distinction is necessary, the upper and righthand limits will be indicated by $+0$, and the lower and lefthand limits, by -0 . Several cases and subcases need to be distinguished.

A. If U is discontinuous and assumes both signs at P , then since both $U(+0, \eta)$ and $U(-0, \eta)$ are continuous, $\exists \gamma > 0$ such that $U(+0, \eta)U(-0, \eta) < 0$ for $|\eta| < \gamma$, and this segment of the axis $\xi = 0$ is singular.

B. Assume U is continuous at P . Then $U(0, 0) = 0$ and by Lemma I.3, $\dot{r}(\pm 0) \neq 0$. Suppose $\dot{r}$ were discontinuous, then by (I.12), the saltus of $U/\dot{r}$ would be independent of η , so that $U(0, \eta) = 0$ for all η . But at the regular Cauchy line L , this would imply $(x' - (u^* + a^*)t')/r' = 0$, by (6) to (11) and (14), and then from (6), (8), $\dot{r} = \alpha^*$, which is continuous. Thus $N(P)$ may be chosen so that U and $\dot{r}$ are continuous, and $\dot{r}$ is of definite sign, on $N(P)$.

B.1. If now $U(0, \eta) \equiv 0$ for $|\eta| < \gamma$, for some $\gamma > 0$, then this segment of the axis $\xi = 0$ is singular.

B.2. Assume that, for every $\epsilon > 0$, numbers ϵ_1, ϵ_2 in $(0, \epsilon)$ can be found such that $U(0, \epsilon_1)U(0, -\epsilon_2) \neq 0$. Then if $U(0, \eta)$ assumes both signs arbitrarily close to $\eta = 0$, the continuity of U implies the existence of $\delta(\epsilon) > 0$ such that Lemma A applies with $f = U$, $y = \eta$, $\theta = \epsilon$, $a' = \delta$ and both $x = \xi$ and $x = -\xi$. There is therefore a singular curve $\eta^*(\xi)$, defined on $(-\delta, \delta)$, such that $\eta^*(\xi) \rightarrow 0$ as $\xi \rightarrow 0$.

On the other hand, if all non-zero values of $U(0, \eta)$ in $(-\epsilon, \epsilon)$ are of the same sign, then (12) shows that ϵ'_1, ϵ'_2 can be found in $(0, \epsilon)$ such that $V(0, \epsilon'_1)V(0, -\epsilon'_2) < 0$. Therefore (i) V cannot be o.d.s. at P , and (ii) Lemma A applies with $f = V$, $y = \eta$, $\theta = \epsilon$ and both $x = \xi$ and $x = -\xi$.

B.3. There only remains the alternative that one, but not both, of the numbers ϵ_1, ϵ_2 can be found which were introduced under B.2. Since $-\xi$ and $-\eta$ are admissible regular coordinates, if ξ and η are, no generality is lost in taking $U(0, \eta) \equiv 0$ for $\eta_1 \leq \eta \leq 0$ and $U(0, \epsilon_1) > 0$. For $\eta_1 \leq \eta \leq -0$, (12) and Lemma I.3 then imply $V(0, \eta) \equiv 0$ and $\dot{s}(\eta) \neq 0$. To find a second singular curve segment ending at P , we turn our attention first to the axis $\eta = 0$.

B.3.α. If $V(\xi, -0) \equiv 0$ for $0 \leq \xi \leq \xi_1$ or for $\xi_2 \leq \xi \leq 0$, the corresponding segment of the axis $\eta = 0$ is singular.

B.3.β. Assume that δ_1 and δ_2 can be found in $(0, \delta)$, for every $\delta > 0$, such that $V(\delta_1, -0)V(-\delta_2, -0) \neq 0$, but this product cannot assume the negative sign for $\delta < \delta_0$. Then (13) implies the existence of δ'_1, δ'_2 in $(0, \delta)$ such that $U(\delta'_1, 0)U(-\delta'_2, 0) < 0$, and Lemma A applies with $f = U, y = \xi, x = \eta$ and $\theta = \delta$.

B.3.γ. There only remains the possibility that δ_1 and δ_2 can be found in $(0, \delta)$ such that $V(\delta_1, -0)V(-\delta_2, -0) < 0$. If V is continuous at P , Lemma A now applies with $f = V, y = \xi, x = \eta$ and $\theta = \delta$. If V is discontinuous, on the other hand, $N(P)$ may be chosen, by Lemma I.4, so that V is o.d.s. on the intersection of $N(P)$ with the closure of $\{\eta > 0\}$, and by (I.16), $U(0, \eta) > 0$ for $\eta > 0$ in $N(P)$. Suppose, for definiteness, that $V(0, +0) > 0$, then there are only two alternatives. First, there may be $\nu > 0$ such that $V(\xi, -0) < 0$ either for $0 < \xi < \nu$ or for $-\nu < \xi < 0$, and the corresponding segment of the axis $\eta = 0$ is then singular. Secondly, any value of ξ at which $V(\xi, -0) < 0$ (and such values exist for arbitrarily small $|\xi|$, since $V(\delta_1, -0)V(-\delta_2, -0) < 0$) is separated from zero by another value of ξ at which $V(\xi, -0) > 0$. Then by (13), for every $\delta > 0$, $U(\delta_3, 0) < 0$ for some $|\delta_3| < \delta$, and since U is continuous, also $U(\delta_3, \eta) < 0$ for $0 \leq \eta < \rho$. But $U(0, \eta) > 0$ for $\eta > 0$, and a slight simplification of the proof of Lemma A furnishes a singular curve $\xi^*(\eta)$ on $(0, \rho)$ with $\xi^* \rightarrow 0$ as $\eta \rightarrow 0$.

C. Assume, finally, that U is discontinuous and assumes zero at P . The case where V is also discontinuous, but assumes both signs at P , is disposed of by noting that it is dual to a sub-case of A, in the sense that each is obtained from the other by an interchange of $\{U, \xi\}$ and $\{V, \eta\}$. Similarly, the case where V is continuous and zero at P is dual to a sub-case of B.

Without loss of generality, we may take $U(+0, 0) = 0$, and then by Lemma I.3, $\dot{r}(+0) \neq 0$, and $N(P)$ may be chosen so that U and $\dot{r}$ are o.d.s. on the respective intersections of $N(P)$ with the closures of $\{\xi < 0\}$ and $\{\xi > 0\}$.

C.1. If V is o.d.s. at P , then $N(P)$ may also be chosen so that V is o.d.s. on $N(P)$. By (12), $\partial U / \partial \eta$ is then also o.d.s. for $\xi \geq +0$, and it follows that U changes sign discontinuously across a segment $0 < \eta < \gamma$ or $-\gamma < \eta < 0$ of the axis $\xi = 0$. In addition, the standard form of the Implicit Function Theorem (e.g. Courant 1936) assures the existence of a unique, continuous function $\eta^*(\xi)$ on $0 < \xi < \xi_1$ such that $U(\xi, \eta^*(\xi)) = 0$ and $\eta^*(0) = 0$.

C.2. The only remaining possibility is that V is also discontinuous and assumes zero at P , and again without loss of generality, $V(0, +0) = 0, \dot{s}(+0) \neq 0$, and V and $\dot{s}$ are o.d.s. on the respective intersections of $N(P)$ with $\{\eta < 0\}$ and $\{\eta > 0\}$. To fix the ideas, suppose $V > 0$ for $\eta < 0$ and $U > 0$ for $\xi < 0$ (Fig. 1); an analogous argument serves for other combinations of signs.

FIGURE 1.

C.2.α. If $\dot{s}(+0) < 0$, then for $\xi < 0$, $U(\xi, 0)\dot{s}(+0) < 0$ and by (13), $V(\xi, +0) < 0$, so that the negative ξ -axis is singular in $N(P)$. The existence of another singular curve segment containing P is immediate, if $V(\xi, +0) \equiv 0$. On the other hand, if positive values of $V(\xi, +0)$ occur for arbitrarily small, positive ξ , then Lemma A applies with $f = V$, $x = \eta$, and $y = \xi$. Finally, if negative values of $V(\xi, +0)$ occur for arbitrarily small, positive ξ , then by (13) negative values of $U(\xi, 0)$ must also occur for arbitrarily small, positive ξ , so that Lemma A applies with $f = U$, $x = \eta$ and $y = \xi$.

The dual argument disposes of the case $\dot{r}(+0) < 0$.

C.2.β. The last remaining sub-case is that where $\dot{s} > 0$ in the closure of the upper half-plane, and $\dot{r} > 0$ in the closure of the right half-plane, while still (Fig. 1) $U > 0$ in that of the left half-plane, $V > 0$ in that of the lower half-plane, and $U(+0, 0) = V(0, +0) = 0$. If now positive values of $U(+0, \eta)$ occur for arbitrarily small, positive η , then by (12), Lemma A applies with $f = V$, $y = \eta$, $x = \xi$; but if $U(+0, \eta)$ is non-positive for $0 \leq \eta < \eta_1$, then this segment of the η -axis is singular. Similarly, a segment $0 \leq \xi < \xi_1$ of the ξ -axis is singular, if $V(\xi, +0)$ is non-positive on it, but if negative values of $V(\xi, +0)$ occur for arbitrarily small positive ξ , then Lemma A applies by (13), with $f = U$, $x = \eta$ and $y = \xi$.

Proof of Lemma A. To demonstrate the existence of $a' > 0$, it suffices to consider the restriction of $f(x, y)$ to an interval $(-b', b')$ of y such that $0 < b' < b$ and b' may be chosen so that $f(x, y)$ is continuous except with respect to y at $y = 0$. Define

$$S^+(x) = \{y \mid 0 \leq y \leq b'/2 \text{ and } f(x, y) \text{ is not o.d.s. at } (x, y)\},$$

$$S^-(x) = \{y \mid -b'/2 \leq y \leq 0 \text{ and } f(x, y) \text{ is not o.d.s. at } (x, y)\}.$$

Since $f(x, y) \in C[0, a]$ with respect to x , there exists $a'(\theta) > 0$ such that $f(x, \theta_1)f(x, -\theta_2) < 0$ for $0 \leq x < a'$, and since $|f|$ is bounded, $S^+(x)$ and $S^-(x)$ cannot be both empty for any $x \in [0, a']$.

$$\begin{aligned} Y(x) &= \text{glb } S^+(x) \quad \text{if } S^+(x) \neq \emptyset \\ &= \text{lub } S^-(x) \quad \text{if } S^+(x) = \emptyset, \end{aligned}$$

therefore defines $Y(x)$ on $[0, a']$ so that $-\theta_2 < Y(x) < \theta_1$, and hence, $|Y(x)| < \theta$.

Since $f(x, y)$ is continuous with respect to y for $0 < y < b'$, the subset $s^+(x) = \{y \in S^+(x) \mid f \text{ assumes zero at } (x, y)\}$ is closed (in the standard topology of the real numbers). But $0 < y \in S^+(x)$ implies $y \in s^+(x)$, and thus if $s^+(x) \neq S^+(x) \neq \emptyset$, then $y = 0$ is an isolated element of $S^+(x)$, and $S^+(x)$ is also closed. Similarly, $S^-(x)$ is closed, and hence, $Y(x) \in (S^+(x) \cup S^-(x))$.

Remark. Intuitively, one might expect a stronger theorem, asserting that a singular point must be an interior point of a continuous singular curve. However, the problem of parametrising this curve, in the absence of information on $\partial U/\partial \xi$ and $\partial V/\partial \eta$, has so far defied the present author.

The discussion just presented shows that singular curves fall into three classes, *viz.*, those at which U , or V , or both U and V are not o.d.s., and that by-and-large, a point at which U is not o.d.s. lies on a curve at which U is not o.d.s. It is likely, however, that exceptions to this rule can be encountered. For instance, if U is continuous and vanishes at $\xi = \eta = 0$, while $V(0, +0) = 1$, $V(0, -0) = -1$, then $\dot{r}(\xi)$ is o.d.s. near $(0, 0)$ and by (12), $U(0, -\eta)U(0, \eta) > 0$ for $0 < |\eta| < \gamma$. If $\partial U/\partial \xi = \dot{U}(\xi, \eta)$ takes the limits $\dot{U}(+0, 0) = 1$, $\dot{U}(-0, 0) = -1$, then also $U(\xi, 0)U(-\xi, 0) > 0$ for $0 < |\xi| < \delta$.

4. The Ludford-Liouville Theorem. This theorem formulates a property of all possible, sufficiently extensive continuations of an arbitrary formal solution, and it is necessary to define first of all a natural way in which its hypotheses can be given a precise meaning. Consider an arbitrary GCP, choose any pair of admissible, regular coordinates ξ, η , and let Ω_0 denote the solution of the associated regular problem on the range Q_0 . Next, choose a quartet $\{x^*, \dots, s^*\}(\sigma)$ on $(\sigma_a, 0)$, and another on $(1, \sigma_b)$, subject only to the requirement that the combined data satisfy the conditions of the GCP on (σ_a, σ_b) (this requires continuity of $\{x^*, \dots, s^*\}(\sigma)$ at $\sigma = 0$ and $\sigma = 1$, and a restriction on the ranges of $r^*(\sigma), s^*(\sigma)$ to satisfy condition (iii) with some new bound ϵ). Apart from the trivial blemish that the interval of σ has not been normalized, the combined data specify a new GCP. On the understanding that regular coordinates $\xi(\sigma), \eta(\sigma)$ are chosen for it so as to agree with the earlier choice on $(0, 1)$, the new range is $Q_1 \supset Q_0$, and the solution Ω_1 of the associated regular problem satisfies $\Omega_1 \equiv \Omega_0$ on Q_0 .

The process may be repeated, and any sequence $\{\Omega_n\}$ such that $Q_{n+1} \supset Q_n$ and $\Omega_{n+1} \equiv \Omega_n$ on Q_n will be called a *continuation* of Ω_0 . It will be called *regular*

if an upper bound independent of n can be found for $|r - s|$, $(r + s)$ and $(r + s)^{-1}$ on Q_n . It will be called *indefinite*, if the least distance $d[0, \Omega_n(\partial Q_n)]$ from the x, t -origin to the image of the range boundary tends to ∞ as $n \rightarrow \infty$. (For a formal solution without limit points, the map $\Omega : Q \rightarrow R$ is onto and $1 - 1$ (Section I.3), so that $\Omega(\partial Q) = \partial\Omega(Q)$ and an indefinite continuation then continues the solution throughout the x, t -plane. It may so continue the solution even if the map Ω_n is not $1 - 1$ on Q_n for all n ; an example is given in the Appendix, and there $\partial\Omega(Q_n) \subset \Omega(\partial Q_n)$.) Finally, a regular solution Ω_0 will be called "*trivial*", if $r(\xi) \equiv \text{const.}$ and $s(\eta) \equiv \text{const.}$ on Q_0 ; this implies $u \equiv \text{const.}$ and $a \equiv \text{const.}$ so that, in the physical applications, a Newtonian observer can be found who sees the medium at rest and in equilibrium in $\Omega(Q_0)$.

Ludford-Liouville Theorem. *If the regular solution Ω_0 of a GCP has a regular, indefinite continuation without singular point, then Ω_0 is trivial.*

This is an immediate corollary of

Theorem 2. *Any regular, indefinite continuation of the regular solution of an arbitrary GCP possesses a singular line in every interval of strict monotonicity of $r(\xi)$ or $s(\eta)$.*

The proof requires, as a preliminary step, a check whether regularity and indefiniteness are compatible properties of a continuation. A sufficient confirmation is furnished in the Appendix by a proof of the following

Lemma. *If Ω_0 is free of singular points, it possesses a regular, indefinite continuation.*

Since ξ and η play a strictly symmetrical role, it suffices to consider the case of a continuation with an open interval I of ξ on which $r(\xi)$ is strictly monotone. Since $\dot{r}(\xi) \neq 0$ a.e. on I , and since U and $\dot{r}$ are piecewise continuous functions of ξ , for fixed η , there is no loss of generality in assuming the existence of $\xi_0 \in I$ such that U and $\dot{r}$ are continuous at ξ_0 and $\dot{r}(\xi_0) \neq 0$. If Γ denotes the line $\xi = \xi_0$ in the ξ, η -plane, moreover, no generality is lost in assuming $\Gamma \cap Q_0 \neq \emptyset$, and thus the existence of U and t is assured, e.g., at the point of intersection of Γ with the regular Cauchy line L_0 of Ω_0 .

Now assume that V is of definite sign on $(I \cap Q_n) = Q'_n$ for every n . The proof will be complete, in view of Theorem 1, if this is shown to imply the existence of a point on $\Gamma \cap Q_n = \Gamma_n$, for some n , at which $U(\xi_0, \eta) = 0$.

By (14), $t(\xi_0, \eta)$ is strictly monotone. Let x_{ni}, t_{ni} , $i = 1, 2$, denote the respective values of x, t at the pair of points forming $\Gamma \cap \partial Q_n$, and number them so that $t_{n2} > t_{n1}$. If $\Omega_n(\Gamma_n) = M_n$, then $k \geq j$ implies $M_j \subset M_k$, because $\Omega_k \equiv \Omega_j$ on Q_j ; M_n is therefore obtained by successive extensions of a fixed curve segment M_0 . Since the continuation is indefinite, moreover, $x_{ni}^2 + t_{ni}^2 \rightarrow \infty$ as $n \rightarrow \infty$, for $i = 1, 2$. Thus if either of t_{n1}, t_{n2} were bounded as $n \rightarrow \infty$, then

$$\text{l.u.b.}_{M_n} |dx/dt| \rightarrow \infty.$$

But $|dx/dt| = |u + a|$ is bounded independently of n , since the continuation is regular, and hence,

$$t_{n2} \rightarrow \infty, \quad t_{n1} \rightarrow -\infty, \quad \text{as } n \rightarrow \infty.$$

But since V is of definite sign, the Canonical Equation (12) may, by (14), be written in the form

$$\frac{\partial U / \partial \eta}{\partial t / \partial \eta} = -\lambda \mu \dot{r},$$

and the righthand side is bounded away from zero, since the continuation is regular. For sufficiently large n , therefore, U must assume every real value in Γ_n .

APPENDIX

To prove the lemma of Section 4, it will be sufficient to exhibit one regular solution Ω_1 defined on a range $Q_1 \supset Q_0$ such that $\Omega_1 \equiv \Omega_0$ on Q_0 , that the respective ranges of $r(\xi)$ and $s(\eta)$ are the same on Q_1 as on Q_0 , and that the distance $d[0, \Omega(\partial Q_1)]$ exceeds an arbitrary positive number.

Without loss of generality the regular coordinates may be taken to be given by (6), (7). For definiteness, assume $U < 0$, $V > 0$ on Q_0 . Then choose $\{x^*, \dots, s^*\}(\sigma)$ on (σ_a, σ_b) so as to agree with the data of Q_0 on $(0, 1)$, to be continuous at $\sigma = 0$, $\sigma = 1$, and to satisfy $t' \equiv r' \equiv s' \equiv 0$, but $x' \equiv 1$, both on $(\sigma_a, 0)$ and $(1, \sigma_b)$. [If $UV < 0$ and $V < 0$ on Q_0 , $t' \equiv 0$, $x' \equiv -1$ is the appropriate choice, and if $UV > 0$, $x' \equiv 0$, $t' = \text{const.}$ is appropriate.]

It follows immediately from (8) that $r(\xi)$ and $s(\eta)$ have respectively the same ranges on Q_0 and Q_1 . Moreover, by (6) and (7), $\xi' = \eta' = 1/a^* = \text{const.}$ on $(\sigma_a, 0)$ and also on $(1, \sigma_b)$. The regular Cauchy line L_1 of Ω_1 is therefore the union of $\bar{L}_0 = L_1 \cap \bar{Q}_0$ and two straight segments L' (on which $0 > \sigma > \sigma_a$, say) and L'' (Fig. 2). The range Q_1 , similarly, is the union of $\bar{Q}_0$ and 8 rectangles $Q'_1, \dots, Q'_8$ (which will be numbered anti-clockwise so that $L' \subset Q'_1$) and boundary segments common to pairs of the Q'_i (Fig. 2). Since $r(\xi)$ is independent of η ,

FIGURE 2.

and $s(\eta)$, of ξ , the constancy of r and s on L' and L'' implies that $r \equiv \text{const.}$ and $s \equiv \text{const.}$ on Q'_i and also on Q'_j whence either $r \equiv \text{const.}$ or $s \equiv \text{const.}$ on Q'_i for even i , and hence again $r \equiv \text{const.}$ and $s \equiv \text{const.}$ on Q'_3 and on Q'_7 . The continuation of Ω_0 is thus built up from trivial pieces and 'simple waves'.

On L' , (I.9) and (I.10) give $V = \lambda/2 = \text{const.}$ and $t^*(\sigma_a) = t^*(0)$, but $x^*(\sigma_a) = x^*(0) + \sigma_a$. For $\eta < 0$, $\dot{s} = s'/\eta' = 0$, so by (I.17), $V(\xi, \eta) \equiv \text{const.} > 0$ for $\eta < 0$. It follows from (14) that $t(\xi, \eta)$, for fixed ξ , is a strictly monotone function of η in $Q_0 \cup Q'_2$, and its range on Q'_2 is

$$t(\xi, 0) - t(\xi, \eta(\sigma_a)) = \int_{\eta(\sigma_a)}^0 \lambda^{-1} V d\eta = -k(\xi)\sigma_a$$

where $k(\xi)$ has a positive lower bound, independent of ξ , by (5). Thus $|t(\xi, \eta(\sigma_a))|$ can be made arbitrarily large by choosing $|\sigma_a|$ sufficiently large. The same holds for $x^2 + t^2$ on the boundary segment $\eta = \eta(\sigma_a)$ of Q'_1 , because it has been shown for $|x^*(\sigma_a)|$ and $|t(0, \eta(\sigma_a))|$ and because this segment has a straight image in the x, t -plane. Analogous computations show that an arbitrarily large lower bound for $x^2 + t^2$ can be obtained also on the other boundary segments of Q_1 , if $\sigma_b^{-1} + |\sigma_a|^{-1}$ is sufficiently small.

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