Uniformization of a Quasi-linear Hyperbolic Equation

*Part I. The Formal Solution**

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Abstract. The Cauchy and characteristic boundary value problems for a typical system (Section 3) of homogeneous differential equations in two independent and two dependent variables are studied to establish a generalized solution which exists on the maximal domain of uniqueness and is identical with the classical solution wherever that exists uniquely. The new solution is parametric and removes the obstacle which the limitations of classical existence put in the way of a study of the solution structure.

1. Introduction. In the theory of quasi-linear hyperbolic differential equations, existence of the solution in the classical sense can be proved only in a restricted neighborhood of the locus carrying the initial and boundary values. By contrast, there is a much more natural region in which the solution is known to be unique—if it exists there (Courant and Hilbert, 1962). The results below contain a method for determining the extent of the region of existence. This is not, however, the main aim of this study. It is generally acknowledged that a satisfactory existence theory should be sought in terms of the concept of weak solutions of conservation laws (Lax, 1957). On the other hand, the original theorems of Lax (1957) are largely limited to a single such law and their extension has proved difficult. A closer understanding of the structure of the solutions of the differential equations is therefore sought here as a help towards a better understanding of the structure of the solutions of the conservation laws. An intimate relation between those structures is indicated by the work of Mahony (1956).

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The present investigation is thus concerned with the differential form of a simple and typical system of two conservation laws involving two independent variables. The features of the solution structure which are of most interest are the singularities that destroy existence of the classical solution. Their study requires a generalized solution concept which does not exclude them from the region of existence, and which therefore extends the classical existence theorem, preferably to the maximal region to which a uniqueness theorem can apply.

It is commonly conjectured that the main singularities of the solutions are 'limit lines', that is, envelopes of characteristics, and it will be proved below that these are indeed the only singularities curtailing the classical existence—provided the differential equations are 'non-singular' in the sense of Section 2 below. The loss of classical existence is thus caused by a loss of single-valuedness.

To overcome this and provide tools for the study of details of the solution structure, a 'formal solution' is introduced in Section 3, which involves a parametric representation of the classical solution. It will be shown that such a representation can, under very general circumstances, be chosen so as to exist on the maximal domain of uniqueness. The construction of this formal solution does not require knowledge of the classical solution; rather, it is obtained from a different differential equation (Section 3), which depends on the boundary conditions. The first task is therefore to prove two theorems (stated in Section 3), which show that the formal solution thus obtained is indeed a natural generalization of the classical solution. Results concerning the solution structure will be reported in Part II.

The formal solution is developed below for a rather sweeping generalization of the classical Cauchy and characteristic boundary problems—because no difficulty is added, but a good deal of symmetry is gained, by admitting the same types of singularities in the boundary values as are, in any case, encountered in the solutions. Otherwise, however, many obvious avenues of generalization are here left unexplored. For instance, the associated differential equation is considered only for boundary values which are piecewise continuously differentiable. 'Weak' extensions appear within easy reach. However, the real objective to be borne in mind is the extension to weak solutions of conservation laws. The formal solution is put forward, not for its own sake, but as a stepping stone.

The main limitations of the present results, then, are the restriction to solutions of the differential equations rather than of the conservation laws; the restriction to non-singular differential equations; the restriction to a pair of first-order equations, corresponding to only two conservation laws; and the restriction to two independent variables. Care has been taken in the choice of methods employed below to avoid closing the door to extensions in any of these directions.

2. A generalized Cauchy problem. Consider the equations

$$(1) \quad \partial r / \partial t + (u + a) \partial r / \partial x = 0,$$

$$(2) \quad \partial s / \partial t + (u - a) \partial s / \partial x = 0,$$

for two functions $r(x, t)$, $s(x, t)$ where

$$(3) \quad u = r - s, \quad a = a(r + s),$$

$a(r + s)$ being any function which, together with

$$(4) \quad \mu(r + s) = (2a)^{-1}(1 + da/d(r + s)),$$

is continuous and possesses positive upper and lower bounds when $r + s$ possesses such bounds.

These equations are here studied as a simple, and rather typical, example of quasi-linear equations with the properties (A) they involve two independent variables and (B) they are 'non-singular', at any rate if bounds on $|r - s|$, $r + s$ and $(r + s)^{-1}$ are prescribed (*cf.* condition (iii) below). They govern, for instance, the one-dimensional homentropic motion of a gas of very general equation of state (Meyer, 1960); the conditions on a and μ require only the existence of similar bounds for $\partial p / \partial \rho$ and $\partial^2 p / \partial (1/\rho)^2$ (at constant entropy) when the density ρ is so bounded, p being the pressure. They also apply to certain problems in hydraulics ('shallow-water theory') and oceanography ('long-wave theory').

Let x^* , t^* , r^* , s^* be any four single-valued functions of a parameter σ defined on $(0, 1)$ with piecewise continuous first derivatives x' , t' , r' , s' . (The term piecewise continuous is interpreted here and in the following in its strict sense implying boundedness.) Let

$$u^* = r^* - s^*, \quad a^* = a(r^* + s^*).$$

To find $r(x, t)$, $s(x, t)$ satisfying (1), (2) and $r(x^*, t^*) = r^*$, $s(x^*, t^*) = s^*$ on $(0, 1)$ will be called a generalized Cauchy problem for (1), (2) (or briefly, a GCP), if

- (i) $x'^2 + t'^2 + r'^2 + s'^2 > 0$ on $[0, 1]$,
- (ii) neither $x' - (u^* + a^*)t' \equiv r' \equiv 0$ nor $x' - (u^* - a^*)t' \equiv s' \equiv 0$ on $(0, 1)$, and
- (iii) $\epsilon > 0$ exists such that $r^*(\sigma_1) + s^*(\sigma_2) \geq \epsilon$ for arbitrary σ_1, σ_2 in $(0, 1)$.

It will be observed that (i) excludes only a trivial degeneracy of the parametrical representation of the boundary values, and (ii) excludes, for the sake of brevity, a well-known limiting case in which the region of determination of the solution degenerates. Condition (iii), however, is substantial and will be seen later to ensure that the solution satisfies $r + s \geq \epsilon$.

The GCP clearly includes the classical Cauchy and characteristic boundary value problems (Courant and Hilbert, 1962), for which a classical solution is known to exist in a neighborhood of the Cauchy curve C in the x, t -plane defined by $x = x^*(\sigma)$, $t = t^*(\sigma)$. The GCP, however, cannot generally possess a classical solution. Indeed, the boundary values r^* and s^* are not assumed to

FIGURE 1. Cauchy Curve of GCP.

be single-valued or differentiable, or even continuous, functions of x and t (or vice-versa) on C . The Cauchy curve, moreover, may be closed (Fig. 1)!

The generality of the GCP is motivated by the following notion. Suppose classical data are prescribed on a Cauchy curve. It is natural to desire a solution in the whole region in which it can be uniquely determined, and to desire a symmetry permitting the determination of the same solution also from its values on any other Cauchy curve in that region. Accordingly, the same degree of degeneracy is here admitted in the boundary values as is found in the solution of classical problems. The further generalization to 'weak solutions' (in the sense of boundary values with $x', t', r', s' \in L^2$, e.g.) is not here followed up.

3. Regular coordinates and formal solutions. The concept of generalized solution here put forward is based on

Lemma 1. *A pair of differentiable functions $\xi(\sigma)$, $\eta(\sigma)$, exist, the respective values of which correspond in a 1 - 1 manner to the characteristics of (1), (2) defined locally by the boundary values of the GCP.*

To see this, consider

$$(6) \quad \xi(\sigma) = \int_0^\sigma [(x' - (u^* + a^*)t')^2 + r'^2]^{1/2} (a^*)^{-1} dt,$$

$$(7) \quad \eta(\sigma) = \int_0^\sigma [(x' - (u^* - a^*)t')^2 + s'^2]^{1/2} (a^*)^{-1} dt,$$

where the positive roots are understood. These functions are differentiable and monotone. Moreover, $\xi(\sigma)$ is strictly monotone, except that $d\xi/d\sigma = 0$ for those values of σ for which

$$x' - (u^* + a^*)t' = r' = 0,$$

and similarly for $\eta(\sigma)$. It may be remarked that the lemma is established with similar ease also for equations more complicated than (1), (2), provided the

number of independent variables is two and the equations are non-singular, because only *local* correspondence at the Cauchy curve is required.

Of course, the functions $\xi(\sigma)$, $\eta(\sigma)$ are not uniquely defined by the lemma. Any function $f(\xi(\sigma))$ also satisfies it, if $df/d\xi$ exists and is either entirely positive or entirely negative on the range of $\xi(\sigma)$, and any such function $f(\xi(\sigma))$ will be considered admissible in the place of ξ for what follows. This indeterminacy is characteristic of hyperbolic problems. Actually, the functions (6), (7) will be used, for convenience, and the need to prove invariance under transformation to another admissible function pair will be avoided by restricting the inquiry to issues which are clearly invariant in that sense. For instance, if $F = F(\xi, \eta)$, the value of $\partial F/\partial \xi$ is not so invariant; but if the question is only whether $\partial F/\partial \xi$ exists, or whether $\partial F/\partial \xi = 0$, then the answer is invariant. Similarly, the sign of $\partial F/\partial \eta$ is not an invariant property, but a change of sign of $\partial F/\partial \eta$ is invariant.

The functions $\xi(\sigma)$, $\eta(\sigma)$ map $(0, 1)$ onto the rectangular domain $Q : 0 < \xi < \xi(1)$, $0 < \eta < \eta(1)$ in the ξ, η -plane. It represents precisely the set of all pairs of characteristics of (1), (2) defined locally by the data and will be referred to as the *range* of the GCP. (Intuitively, this set of pairs is the largest set on which a unique solution can be anticipated, and hence, the set on which existence is desired.)

Let the functions $r(\xi)$, $s(\eta)$, $u(\xi, \eta)$ and $a(\xi, \eta)$ be defined by

$$(8) \quad r(\xi(\sigma)) = r^*(\sigma), \quad s(\eta(\sigma)) = s^*(\sigma),$$

and (3); $|r|$ and $|s|$ will then be bounded, and $r + s$, a and μ will have positive upper and lower bounds on Q . The problem of finding $x(\xi, \eta)$ and $t(\xi, \eta)$ satisfying

$$(9) \quad \partial x/\partial \eta - (u + a) \partial t/\partial \eta = 0,$$

$$(10) \quad \partial x/\partial \xi - (u - a) \partial t/\partial \xi = 0,$$

$$(11) \quad x(\xi(\sigma), \eta(\sigma)) = x^*(\sigma), \quad t(\xi(\sigma), \eta(\sigma)) = t^*(\sigma)$$

will then be called the *regular* problem associated with the GCP.

It is *regular* in the following sense. First, (9) and (10) have continuous, distinct coefficients on $\bar{Q}$. Secondly, (11) prescribes x and t with piecewise continuous first partial derivatives on a curve L in the ξ, η -plane; *e.g.*

$$(12) \quad 2 |\partial t/\partial \xi| = [1 + r'^2(x' - (u^* + a^*)t')^{-2}]^{-1/2}$$

on L , by (9), (10), and (6). Thirdly, ξ and η are *monotone*, piecewise continuously differentiable functions of the arclength on L (Fig. 2). The regular problem is therefore a classical Cauchy or characteristic boundary value problem, or a simple combination of both. Since (9) and (10) are linear, moreover, a unique strict solution of this problem—that is, a unique solution which is single-valued and piecewise continuously differentiable—exists on the range Q (*e.g.* Courant and Hilbert, 1962; a representation is given in Section 7).

The problem is associated with the GCP as follows. The regular solution

FIGURE 2. Range Q and regular Cauchy Line L .

$x(\xi, \eta)$, $t(\xi, \eta)$, together with (8), furnishes a parametric representation of functions $r(x, t)$, $s(x, t)$ (not necessarily single-valued or differentiable on the whole of their domain) which will be called the “*formal solution*.” That it is invariant under transformation to another admissible function pair may be deduced from the homogeneity of (9), (10); it can also be read off the representation (23) below. That the formal solution is a natural generalization of the classical solution of (1), (2), is shown by the following two theorems.

Theorem 1. *The formal solution satisfies (1), (2) strictly on the image $R(x, t)$ of any characteristically convex open subset $Q' \subset Q$ on which $J_1 = \partial(x, t)/\partial(\xi, \eta)$ is of definite sign.*

A strict solution is here again understood to be a single-valued function pair $r(x, t)$, $s(x, t)$ of which the first derivatives are piecewise continuous and satisfy (1) and (2) everywhere. A quantity is called “of definite sign” at a point p , if it does not assume zero at p and there is a neighborhood of p on which the quantity assumes only one sign. Open is understood in the hyperbolic topology of the ξ, η -plane, the base of which are the Cartesian products of open intervals of ξ and η . “Characteristically convex” is defined as follows. A continuous simple curve segment consisting of a finite number of characteristic segments $\xi = \text{const.}$ and $\eta = \text{const.}$ of finite length, joined end to end (Fig. 3), will be called a ‘chain’. It will be called monotone, if ξ and η both vary monotonically as the chain is traversed from one end to the other. A characteristically convex set is defined by the property that, if a pair of points belong to it, then so does a monotone chain connecting them. (An open, bounded, convex set in the ξ, η -plane is characteristically convex, but the converse does not hold.)

For the converse theorem, note that a strict solution, if postulated without reference to data from which it is determined, cannot be expected to be interpretable as a single, formal solution obtained by the help of a single set of regular coordinates ξ, η . For definiteness, we therefore restrict attention to the solutions of classical Cauchy problems (Courant and Hilbert, 1962) with the additional

FIGURE 3. Monotone Chain.

restriction (iii) of Section 2. The situation for characteristic boundary value problems is closely analogous. A further indefiniteness arises because such a solution need not (in the absence of jump conditions adjoined to (1), (2)) be determined uniquely by the Cauchy data on the whole of the domain where it exists.

The established facts (Courant and Hilbert, 1962) may be summarized as follows. Assume that a classical Cauchy problem for (1), (2) is posed with data on a Cauchy curve C in the x, t -plane. If P is a point 'sufficiently close' to C , then a strict solution exists at P . Moreover, precisely two characteristic line segments, $C_0^\pm(P)$, of the solution can then be traced from P to C , and the solution exists on the curvilinear 'triangle' formed by these and C , and is uniquely determined at P by the data on the subsegment of C which forms the triangle-base. Furthermore, if these statements apply to P , then they apply similarly to all points lying in the closure of the triangle. The interior of the triangle is called the domain of dependence $D_0(P)$. If $P' \notin D_0(P)$, the strict solution defines precisely one pair of distinct characteristic lines through P' . The network of characteristics on $D_0(P)$ may thus be used to define a neighborhood basis at P' in which the open sets are the products of open intervals of arc length of the characteristic lines through P' . For definiteness, the arc length will be chosen as parameter also on C , and C will be understood to correspond to an open interval of this parameter. The facts quoted above then assure the existence of a unique, strict solution on a neighborhood of C which is the union of domains of dependence.

Theorem 2. *If a classical Cauchy problem possesses a strict solution, Σ , on an open set R which is the union of domains of dependence, then Σ is there a formal solution and R is the image, under the mapping defined by the associated regular solution, of a ξ, η -set on which J_1 is of definite sign.*

The same theorem holds for the classical characteristic boundary value problem, with the appropriate definition of domain of dependence. The proof differs from that of Section 6 only in regard to Lemmas 6 and 11.

The qualifications regarding characteristically convex sets and unions of domains of dependence in Theorems 1 and 2, respectively, are connected with the impairment of uniqueness of the classical solution that can be encountered in the absence of shock-jump conditions. Apart from these qualifications, the two theorems are as strong as can be desired, stating roughly that any formal solution is strict, if J_1 is of definite sign, and conversely, that any classical, strict solution is also a formal one for which J_1 is of definite sign.

A point in the ξ, η -plane at which J_1 is not of definite sign will be called a *singular point* of the associated regular solution. Its image $x(\xi, \eta)$, $t(\xi, \eta)$ is called a *limit point*. Theorem 1 shows that limit points are indeed the *only* obstacles to the existence of the strict solution of the GCP. For the range Q is characteristically convex, and if it is free of singular points, then its image $R(x, t)$ is a 'maximal domain of uniqueness' in the following sense. If the GCP is a classical Cauchy problem, then R is the union of the Cauchy curve C and two disjoint domains of dependence, each of which has the whole of C as its triangle base.

4. Canonical equations. It is convenient to prepare the proof of Theorems 1 and 2 by collecting a few properties of the regular solution. Write $dr/d\xi = \dot{r}(\xi)$ and $ds/d\eta = \dot{s}(\eta)$.

Lemma 2. $|\dot{r}|$ and $|\dot{s}|$ are bounded on Q .

Proof. By (8), these quantities may be computed directly from the boundary values of the GCP; by (6) and (7),

$$(13) \quad \begin{aligned} |\dot{r}| &= a^*[1 + r'^{-2}(x' - (u^* + a^*)l')^2]^{-1/2}, \\ |\dot{s}| &= a^*[1 + s'^{-2}(x' - (u^* - a^*)l')^2]^{-1/2}, \end{aligned}$$

and a^* is bounded, by hypothesis.

Let

$$(14) \quad U = \lambda \partial t / \partial \xi, \quad V = \lambda \partial t / \partial \eta,$$

where

$$(15) \quad \lambda = \exp \int_a^{r+s} \mu(\omega) d\omega,$$

and note that $\lambda(r + s)$, like a and μ is continuous and possesses positive upper and lower bounds, by condition (iii) of Section 2. Formally, by elimination of x from (9), (10),

$$(16) \quad \partial U / \partial \eta = -\mu V \dot{r},$$

$$(17) \quad \partial V / \partial \xi = -\mu U \dot{s}.$$

It may be shown (Meyer 1965) that (16), (17) are actually equivalent to (9), (10) in the following sense. Let $PCB^n(S)$ denote the class of functions with piecewise continuous and bounded partial derivatives of order n on the set S . Then given a solution $\{U(\xi, \eta), V(\xi, \eta)\} \in PCB(Q)$ of (16), (17) with boundary values $U^*(\sigma), V^*(\sigma)$, a solution $\{x(\xi, \eta), t(\xi, \eta)\} \in PCB^1(Q)$ of the ARP exists, with boundary conditions related to U^*, V^* consistently with (9), (10), (14), and these solutions are related by (14). And conversely, given a solution $\{x(\xi, \eta), t(\xi, \eta)\} \in PCB^1(Q)$ of the ARP, then the functions $U(\xi, \eta), V(\xi, \eta)$ given by (14) satisfy (16) and (17).

[The formal solution $r(x, t), s(x, t)$ of the GCP may be interpreted as a function pair $x(r, s), t(r, s)$ —non-degenerate, if $\partial(r, s)/\partial(\xi, \eta) = \dot{r}(\xi)\dot{s}(\eta)$ is of definite sign, since J_1 is bounded. Then $U/\dot{r} = \lambda\partial t/\partial r$, if $\dot{r} \neq 0$, and $V/\dot{s} = \lambda\partial t/\partial \eta$, if $\dot{s} \neq 0$; and if $\dot{r}\dot{s} \neq 0$,

$$(16a) \quad \frac{\partial}{\partial \eta} \left(\lambda \frac{\partial t}{\partial r} \right) = -\lambda \mu \frac{\partial t}{\partial \eta},$$

$$(17a) \quad \frac{\partial}{\partial \xi} \left(\lambda \frac{\partial t}{\partial s} \right) = -\lambda \mu \frac{\partial t}{\partial \xi},$$

which are commonly called the *Transport Equations* of (1), (2). The equivalence of (9), (10) and (16), (17), and Theorems 1, 2 imply the result—useful in applications—that the strict and formal solutions of the GCP satisfy (16a) at any point $(x, t) = \Omega(\xi, \eta)$ such that $\dot{r}(\xi) \neq 0$ (even if $\partial t/\partial s$ does not exist!) and (17a) at any point such that $\dot{s}(\eta) \neq 0$.]

Lemma 3. $U^2 + \dot{r}^2 > 0$ and $V^2 + \dot{s}^2 > 0$ on $\bar{Q}$.

Proof. By (16), for any fixed ξ , $U(\xi, \eta) = \text{const.}$ if $\dot{r}(\xi) = 0$, so it suffices to show $U^2 + \dot{r}^2 > 0$ on the regular Cauchy curve L (Fig. 2). But that follows from (14), (15), (12) and (13), since λ and a^* have positive lower bounds; and similarly for V and $\dot{s}$.

The existence theorem for the regular associated solution (Section 3) showed $\partial t/\partial \xi$ and $\partial t/\partial \eta$, and hence also U and V , to be piecewise continuous functions of ξ and η . The canonical equations (16), (17), together with Lemma 2, reveal the sharper

Lemma 4. For fixed $\xi [\eta]$, U and $\partial t/\partial \xi$ [V and $\partial t/\partial \eta$] are piecewise continuously differentiable functions of η [ξ] on Q .

Lemma 5. J_1 is of definite sign at a point $P \in Q$, if and only if both $\partial t/\partial \xi$ and $\partial t/\partial \eta$ are of definite sign at P .

Proof. By (9), (10),

$$(18) \quad J_1 = -2a(\partial t/\partial \xi) \partial t/\partial \eta,$$

and $a \geq \epsilon_1 > 0$ on $\bar{Q}$, from which the sufficiency follows. Moreover, (18) shows that the lemma can fail only at a point p where both $\partial t/\partial \xi$ and $\partial t/\partial \eta$ change

sign discontinuously so as to leave the sign of J_1 unchanged. Suppose such a point p exists. By Lemma 4, a discontinuity of $\partial t/\partial \xi$ can occur only across a line $\xi = \text{const.}$; such a line is called 'wave front' (Meyer, 1960), and by (16), the saltus of $U/\dot{r}$ is constant along it, if $\dot{r}(\xi) \neq 0$ (and that of U is constant, if $\dot{r}(\xi) = 0$). Similarly for $\partial t/\partial \eta$ and wave fronts $\eta = \text{const.}$, and hence, p must be the point of intersection of two such wave fronts and J_1 cannot be of definite sign at p .

5. Proof of Theorem 1. The formal solution is a pair of functions $r(x, t)$, $s(x, t)$ given parametrically by four single-valued functions $x(\xi, \eta)$, $t(\xi, \eta)$, $r(\xi)$ and $s(\eta)$ satisfying (8) to (11). The restriction to a set $Q' \subset Q$ on which $J_1 = \partial(x, t)/\partial(\xi, \eta)$ is of definite sign gives a pair which is locally single-valued on the image set R (that is, any point in Q' has a neighborhood such that restriction to it gives a single valued pair). Direct computation of $\partial r/\partial x$ (for fixed t) and similar derivatives gives

$$\partial r/\partial x = J_1^{-1} \dot{r} \partial t/\partial \eta,$$

$$\partial r/\partial t + (u + a) \partial r/\partial x = -J_1^{-1} \dot{r} (\partial x/\partial \eta - (u + a) \partial t/\partial \eta),$$

and similar expressions, and since $x(\xi, \eta)$ and $t(\xi, \eta)$ are piecewise continuously differentiable, the theorem follows from (9), (10) and Lemma 2, if the formal solution is also globally single-valued. Since $r(\xi)$ and $s(\eta)$ are defined as single-valued functions, this will follow from

Theorem 1a. *Let Ω denote the regular associated solution $x(\xi, \eta)$, $t(\xi, \eta)$ of a GCP, and let $Q' \subset Q$ be an open characteristically convex set on which J_1 is of definite sign. Then the restriction $\Omega|_{Q'}$ is 1 - 1.*

Let a 'coincidence pair' be defined as a pair 1, 2 of distinct points (ξ_1, η_1) and (ξ_2, η_2) for which

$$(19) \quad x(\xi_2, \eta_2) = x(\xi_1, \eta_1) \quad \text{and} \quad t(\xi_2, \eta_2) = t(\xi_1, \eta_1).$$

Theorem 1a will be established, if it is shown that the proposition

(α) Q' contains a coincidence pair
implies

(γ) Q' contains a point with a coincidence pair in every neighbourhood, because (γ) is not compatible with the local single-valuedness already established. It is convenient to show first that (α) implies

(β) every chain-contour in Q' connecting a coincidence pair 1, 2 contains another coincidence pair at a smaller distance apart than are 1 and 2.

Here 'chain-contour' denotes a simple closed curve consisting of two monotone chains connecting the same coincidence pair (Fig. 4).

To obtain (β), note first that Lemma 5 implies $0 < |b| < \infty$, if $b = (\eta_2 - \eta_1)/(\xi_2 - \xi_1)$, and since $a \geq \epsilon_1 > 0$, (α) implies $J_1 b > 0$, by (18) and (19).

FIGURE 4. Chain Contour connecting a pair of points 1, 2.

For the simplest, rectangular chain-contour connecting 1 and 2 (if it exists in Q'), a consideration of the sense of variations of t along the sides of the rectangle shows easily that violation of (β) cannot be compatible with the way in which—due to (9), (10), and the strict inequality $u - a < u + a$ following from $a \geq \epsilon_1 > 0$ —the convexity of the contour corners is either entirely preserved or entirely reversed by the transformation $(\xi, \eta) \rightarrow (x, t)$. For the general chain-contour from 1 to 2, a slight extension of the same counting argument shows (α) to imply (β) .

To obtain (γ) from (β) , begin again with the case where the simplest, rectangular chain-contour exists in Q' . Repeated application of (β) reveals a nesting sequence of such contours, connecting respectively a sequence of coincidence pairs, such that the enclosed area $\rightarrow 0$ (because otherwise (β) would apply afresh to the limit) and the diameter $\rightarrow 0$ (because of $0 < |b| < \infty$). In the general case, since 1 and 2 can be connected by a contour with distance from $\partial Q'$ bounded below by $\rho > 0$, and since an open characteristically convex set is simply connected, (β) can be used to construct a sequence of coincidence pairs all points of which have distance $\geq \rho/2$ from ∂Q . These pairs are then connected (even if not within Q') by a nesting sequence of simple, rectangular chain contours such that (γ) follows as before.

Remark. The condition of monotonicity embodied in the notion of a characteristically convex set is necessary for the theorem. The simplest type of 'singular line' $J_1 = 0$ (Meyer 1960) is similar to a parabola with characteristic axis (Fig. 5), separating a subset $J_1 > 0$ from a subset $J_1 < 0$ of the range Q . The regular solution then furnishes coincidence pairs such as 1, 2 in Fig. 5.

6. Proof of Theorem 2. With the normalized arc length as parameter σ on C , the data of the classical Cauchy problem define a GCP and hence (Section 3), regular coordinates ξ, η , a regular Cauchy line L , a range Q , and a unique,

FIGURE 5. Simple type of singular line.

regular associated solution Ω on Q . It furnishes a mapping $\Omega : (\xi, \eta) \rightarrow (x, t)$ which is single-valued and ε $PCB(Q)$; its Jacobian is J_1 . Since ξ and η are monotone functions of the arclength on L , any point $\Pi \in Q$ (but $\notin L$) is connected to L by just one characteristic segment $\Gamma^+(\Pi)$ on which $\xi = \text{const.}$, and just one such segment $\Gamma^-(\Pi)$ on which $\eta = \text{const.}$ (For definiteness, such segments Γ^+ and Γ^- will be understood in the following to correspond to open intervals of η and ξ , respectively.) The interior, $\Delta(\Pi)$, of the 'triangle' formed by $\Gamma^\pm(\Pi)$ and L is the domain of dependence of Π with respect to Ω ; it is characteristically convex, due to the monotone property of L .

Lemma 6. J_1 is of definite sign on L , and $J_1 \rightarrow 0$ on L as $\sigma \rightarrow 0$ or 1 .

(Proofs of Lemmas 6 to 9 and 11 are here omitted; they may be found in (Meyer 1965).)

Since J_1 is piecewise continuous, it follows that a neighbourhood of L exists on which J_1 is of definite sign. Let N denote the union of all such neighborhoods of L . Then if $\Delta(\Pi) \subset N$, Theorem 1 asserts that the formal solution Φ of Ω is strict on the image $\Omega(\Delta(\Pi)) = D(P)$ of Δ , where $P = \Omega(\Pi)$. By (9) and (10), moreover, Ω maps $\Gamma^\pm$ on lines, $\Omega(\Gamma^\pm) = C^\pm$, of characteristic slope with respect to Φ . Theorem 1a shows the restriction $\Omega|_{\Delta(\Pi)}$ to be $1-1$, and hence, $D(P)$ is the domain of dependence of P with respect to $\Phi|_{D(P)}$. It follows from the classical uniqueness theorem that $\Phi \equiv \Sigma$ on $D(P) \cap R$. In fact, if $R \subset D(P)$, then Φ furnishes an extension of Σ to $D(P)$ which is strict and unique, so that there is no loss of generality in assuming $D(P) \subset R$. Since Π is here any point such that $\Delta(\Pi) \subset N \subset Q$ it follows also that, if Δ is the union of all domains of dependence $\Delta(\Pi)$ such that $\Delta(\Pi) \subset N$, and if $D = \Omega(\Delta)$, then $\Phi \equiv \Sigma$ on $D \cap R$; and again without loss of generality, $D \subset R$. Theorem 2 will, therefore, be proved, if it is shown that $R \subset D$.

Lemma 7. Δ is characteristically convex.

Since $\Delta \subset N$, Theorem 1a now shows $\Omega|_{\Delta}$ to be a homeomorphism, and it only remains to prove that $p \in \bar{D} \cap R$ implies $p = \Omega(\Pi)$, $\Pi \in \Delta$.

To this end, we shall examine sequences of points $p_n \in D$, and that requires

attention to the piecewise continuity of $\partial t/\partial \xi$ and $\partial t/\partial \eta$. The actual value of J_1 at a point $\pi \in Q$ may depend on the manner of approach to π . To avoid ambiguity, a sequence of points $\pi_n \in Q$ with regular coordinates ξ_n, η_n will be said to converge monotonically,

$$\pi_n \xrightarrow{m} \pi,$$

if ξ_n and η_n are both strictly monotone functions of n for nearly all n . This ensures that $\{\pi_n\}$ approaches π from a definite, open quadrant of the ξ, η -plane referred to π as origin. Since $\partial t/\partial \xi$ and $\partial t/\partial \eta$ are piecewise continuous on $\bar{Q}$, and by Lemma 4 and (18), the values, $J_1(\pi_n)$, of J_1 at π_n must then converge to a well-defined limit, and no more than four distinct values of $\lim J_1(\pi_n)$ can be obtained from different monotone sequences

$$\pi_n \xrightarrow{m} \pi \in Q.$$

Moreover, there can be no fifth value of $\lim J_1(\pi_k)$ that is approached by the values of J_1 at the points π_k of any other sequence $\{\pi_k\}$ converging to $\pi \in Q$.

Lemma 8. *If $p \in \bar{D}$, there is a unique $\pi \in \bar{\Delta}$ such that $\Omega(\pi) = p$ and a sequence $\{\pi_n\}$ can be found such that $\pi_n \in \Delta$ and*

$$\pi_n \xrightarrow{m} \pi.$$

Lemma 9. *If $p \in \bar{D} \cap R$, $p = \Omega(\pi)$ and $C_0^+(p)$ denotes the pair of characteristic segments of Σ connecting p with the Cauchy curve C , while $\Gamma^\pm(\pi)$ denotes again the pair of characteristic segments of Ω connecting π with the regular Cauchy line L , then $\Omega(\Gamma^\pm(\pi)) = C_0^+(p)$.*

Lemma 10. *Let $p \in \bar{D} \cap R$, let $\{\pi_n\}$ be a sequence as in Lemma 8, and let $J_1(\pi_n)$ denote any one of the four values of J_1 at π_n . Then $J_1(\pi_n) \rightarrow 0$ as*

$$\pi_n \xrightarrow{m} \pi,$$

and similarly if $\pi'_n \in \Delta$ and

$$\pi'_n \xrightarrow{m} \pi' \in \Gamma^\pm(\pi),$$

then $J_1(\pi'_n) \rightarrow 0$.

It now follows that, if J_1 is continuous at π and at $\Gamma^\pm(\pi)$, then N contains a neighbourhood of π and of $\Gamma^\pm(\pi)$, and therefore, a point Π can be found such that $\pi \in \Delta(\Pi) \subset N$, whence $\pi \in \Delta$.

There only remains the case that J_1 is discontinuous at π . Lemma 10 then shows that one of the four possible values of J_1 at π has the sign which J_1 has on N . Suppose now it can be shown that $p \in \bar{D} \cap R$ implies J_1 is of definite sign at π . Lemma 9 shows $\Gamma^\pm(\pi) = C_0^\pm(p)$, and $C_0^\pm(p) \in R$, since $p \in R$. Lemma 8

shows $\pi \in \bar{\Delta}$, whence also $\Gamma^+(\pi) \subset \bar{\Delta}$ and, since Ω is a continuous map, $\Omega(\Gamma^+(\pi)) \subset \bar{D}$. It will therefore follow that J_1 is of definite sign also on $\Gamma^+(\pi)$. But $\pi \in \bar{\Delta} \subset \bar{N}$ and $\Gamma^+(\pi) \subset \bar{\Delta} \subset \bar{N}$, and since J_1 is piecewise continuous, it will also follow that $\pi \in N$ and $\Gamma^+(\pi) \subset N$, whence $\pi \in \Delta$.

That J_1 is of definite sign, on the other hand, will follow, by (13), (14), and (18), if U and V are of definite sign. In view of the symmetrical role of U and V , it therefore suffices to show that $p = \Omega(\pi) \in \bar{D} \cap R$ implies U is of definite sign at π .

This follows from Lemma 10 and from (18) and (14), if U is continuous at π . If not, we require

Lemma 11. $\Omega(\pi) \in \bar{D} \cap R$ implies $\pi \in Q$.

It assures that Lemma 4 applies at π and on $\Gamma^+(\pi)$. Thus if the regular coordinates of π are ξ_π, η_π , $U(\xi, \eta)$ has two possible limits at π , viz. $U(\xi_\pi^+, \eta_\pi)$ and $U(\xi_\pi^-, \eta_\pi)$, which are well-defined by the regular solution Ω . If both these limits of U at π can be approached on monotone sequences $\{\pi_n\} \subset \Delta$, then $\Gamma^+(\pi) \subset \Delta \subset N$ and Lemmas 10 and 5 already show U to be of definite sign at π . Thus the only case requiring further consideration is that in which only one of these limits of U can be so approached. If that limit be denoted by $U(\xi_\pi^-, \eta_\pi)$, then Lemmas 10 and 5 imply $U(\xi_\pi^-, \eta_\pi) \neq 0$, and it remains to show that the ratio $U(\xi_\pi^+, \eta_\pi)/U(\xi_\pi^-, \eta_\pi) = u(\eta_\pi) > 0$.

Let η_L denote the value of η at the point of the Cauchy line L (Fig. 2) where $\xi = \xi_\pi$; η_L is uniquely defined, because the Cauchy problem is a classical one. Without loss of generality, the arclength on the Cauchy curve C may be taken in the sense making $\eta_L \leq \eta_\pi$, and since the case $\eta_L = \eta_\pi$ is already covered by Lemmas 6 and 5, we may take $\eta_L < \eta_\pi$. Let $U(\xi_\pi^+, \eta)/U(\xi_\pi^-, \eta) = u(\eta)$. Since N is a neighbourhood of L , it must contain a subsegment $\Gamma_a: \eta_L < \eta < \eta_a \leq \eta_\pi$ of $\Gamma^+(\pi)$ on which $u(\eta) > 0$. Since $\Gamma^+(\pi) \subset \bar{\Delta}$, this implies $\Gamma_a \subset \Delta$. That $\pi \in \Delta$ therefore follows from

Lemma 12. If $p \in \bar{D} \cap R$ and $u(\eta) > 0$ for $\eta_L < \eta < \eta_a \leq \eta_\pi$, then also $u(\eta) > 0$ for $\eta_L < \eta < \eta_b$ with $\eta_b > \eta_a$.

Proof of Lemma 10. Since $\pi'_n \in \Delta$, $p'_n = \Omega(\pi'_n) \in D \subset R$, where $\Phi \equiv \Sigma$, and so $\partial r/\partial x - \partial r_0/\partial x = \partial s/\partial x - \partial s_0/\partial x = 0$ at p'_n for all n . But $p \in R$ implies $C_0^+(p) \subset R$, and it follows from Lemma 9 that, as $p'_n \rightarrow p' \in \Omega(\Gamma^+(\pi))$, $\partial r/\partial x$ and $\partial s/\partial x$ tend to limits. For the formal solution, direct computation gives

$$(22) \quad \partial r/\partial x = -\dot{r}/(2a \partial t/\partial \xi), \quad \partial s/\partial x = \dot{s}/(2a \partial t/\partial \eta),$$

where these derivatives exist, and by Lemma 3, (14) and (18) now imply $J_1(\pi'_n) \rightarrow 0$. Similarly, $J_1(\pi_n) \rightarrow 0$.

Proof of Lemma 12. By Lemma 4, $U(\xi_\pi^-, \eta)$ and $U(\xi_\pi^+, \eta)$ are continuous functions of η . By Lemmas 10 and 5, and since $\Gamma^+(\pi) \subset \bar{\Delta}$, $U(\xi_\pi^-, \eta) \neq 0$ for

$\eta_L < \eta \leq \eta_*$, so $u(\eta)$ is also continuous. It thus suffices to show that $U(\xi_\pi^+, \eta) \neq 0$ for $\eta_L < \eta < \eta_a$ implies $U(\xi_\pi^+, \eta_a) \neq 0$.

Let $\dot{r}(\xi_\pi^+)$ denote the corresponding limit of $\dot{r}(\xi)$, which exists, by Lemmas 2 and 11. If $\dot{r}(\xi_\pi^+) = 0$, then $U(\xi_\pi^+, \eta) \neq 0$ on $[0, \eta(1)]$, by Lemma 3, and only the case $\dot{r}(\xi_\pi^+) \neq 0$ needs further attention. But if $\dot{r}(\xi) \neq 0$, direct computation gives

$$U/\dot{r} = -\lambda/(2a \partial r/\partial x)$$

for the formal solution, and λ/a has a positive lower bound (Section 4). The lemma will therefore be proved, if it is shown that $\partial r/\partial x$ exists at the image $p_a = \Omega(\pi_a)$ of the point $\pi_a \in \Gamma^+(\pi)$ at which $\eta = \eta_a$.

Since $\Gamma_a \subset \Delta$, $\Omega(\Gamma_a) = C_a \subset D \subset R$ and the two 'sides' of Γ_a , corresponding to ξ_π^- and ξ_π^+ , are mapped 1-1 onto the two sides of C_a . Since $C_a \subset D$, $\partial r/\partial x - \partial r_0/\partial x \equiv 0$ on both sides of C_a , and the lemma will follow, if $\partial r_0/\partial x$ exists on both sides even at p_a . That follows, in turn, if $p_a \in R$. But since $p \in R$, also $C_0^+(p) \subset R$, and since $\pi_a \in \Gamma^+(\pi)$, Lemma 9 shows that $p_a \in C_0^+(p)$.

7. Riemann representation. Since the very differential equations of the associated problem (Section 3) depend on the boundary values of the GCP, it is of interest to note that the regular solution may be represented by the help of a Riemann function independent of those boundary values. To this end, consider first the system

$$\begin{aligned} \partial^2 R / \partial \bar{r} \partial \bar{s} &= \partial(\mu R) / \partial \bar{r} + \partial(\mu R) / \partial \bar{s}, \\ (23) \quad \partial R / \partial \bar{s} &= \mu R \quad \text{when } \bar{r} = r, \\ \partial R / \partial \bar{r} &= \mu R \quad \text{when } \bar{s} = s, \\ R(r, s; r, s) &= 1, \end{aligned}$$

for a function $R(\bar{r}, \bar{s}; r, s)$ on a domain where $\bar{r} + \bar{s}$ and $r + s$ possess positive upper and lower bounds. Let $\mu(\bar{r} + \bar{s})$ be given by (4) and assume that also $d\mu/d(\bar{r} + \bar{s})$ is bounded on the domain. Then (23) is a classical characteristic boundary value problem for a linear hyperbolic equation, and a unique solution exists. By (23) and (15)

$$R(r, \bar{s}; r, s) = \nu(r + \bar{s}), \quad R(\bar{r}, s; r, s) = \nu(\bar{r} + s),$$

where

$$\nu(\bar{r} + \bar{s}) = \lambda(\bar{r} + \bar{s})/\lambda(r + s).$$

Next, consider any GCP such that x^* , t^* , r^* and s^* all $\in PCB^2(0, 1)$ ("restricted" GCP). The regular associated solution then shares this property, and elimination of x from (9), (10) gives

$$\partial^2 t / \partial \xi \partial \eta + \mu \dot{s} \partial t / \partial \xi + \mu \dot{r} \partial t / \partial \eta = 0,$$

with which Riemann's Method (Courant and Hilbert, 1962) associates the system

$$\begin{aligned}
(24) \quad & \partial^2 H / \partial \alpha \partial \beta = \dot{s}(\beta) \partial(\mu H) / \partial \alpha + \dot{r}(\alpha) \partial(\mu H) / \partial \beta, \\
& \partial H / \partial \beta = \mu H \dot{s} \quad \text{when} \quad \alpha = \xi, \\
& \partial H / \partial \alpha = \mu H \dot{r} \quad \text{when} \quad \beta = \eta, \\
& H(\xi, \eta; \xi, \eta) = 1,
\end{aligned}$$

for a function $H(\alpha, \beta; \xi, \eta)$.

For any GCP, a pair of differentiable functions $r(\xi)$, $s(\eta)$ are given by (6), (7) and (8), and a pair $\bar{r}(\alpha)$, $\bar{s}(\beta)$ may be defined in turn by $\bar{r}(\alpha) = r(\alpha)$, $\bar{s}(\beta) = s(\beta)$. The solution of (23) then defines a function

$$H(\alpha, \beta; \xi, \eta) \equiv R(\bar{r}(\alpha), \bar{s}(\beta); r(\xi), s(\eta)),$$

which is seen to satisfy (24).

Let $\alpha(\sigma)$, $\beta(\sigma)$ be given by the integrals (6), (7) in terms of the data of the GCP, let L_0 denote the curve in the α, β -plane thus defined parametrically, let $\mu^*(\sigma) = \mu(r^* + s^*)$ and

$$\nu^*(\sigma) = \lambda(r^* + s^*) / \lambda(r(\xi) + s(\eta)).$$

Given ξ such that $\alpha(0) \leq \xi \leq \alpha(1)$, there is a unique value of σ at which $\alpha = \xi$, by (6), and similarly for η and β , so that a pair of functions $\sigma_2(\xi)$, $\sigma_1(\eta)$ may be defined by $\xi = \alpha(\sigma_2)$, $\eta = \beta(\sigma_1)$. The standard Riemann representation (Courant and Hilbert, 1962) of the solution $t(\xi, \eta)$ of the regular problem associated with the restricted GCP is then

$$\begin{aligned}
2t(\xi, \eta) &= \nu^*(\sigma_2) t^*(\sigma_2) + \nu^*(\sigma_1) t^*(\sigma_1) \\
&+ \int_{\sigma_1}^{\sigma_2} R(r^*, s^*; r(\xi), s(\eta)) \left[(\partial t / \partial \beta + 2\mu^* t^* d\bar{s} / d\beta) \frac{d\beta}{d\sigma} - \left(\frac{\partial t}{\partial \alpha} + 2\mu^* t^* \frac{d\bar{r}}{d\alpha} \right) \frac{d\alpha}{d\sigma} \right] d\sigma \\
&+ \int_{\sigma_1}^{\sigma_2} t^* \left[\frac{d\alpha}{d\sigma} \frac{\partial}{\partial \alpha} - \frac{d\beta}{d\sigma} \frac{\partial}{\partial \beta} \right] R(\bar{r}(\alpha), \bar{s}(\beta); r(\xi), s(\eta)) d\sigma,
\end{aligned}$$

where the integrands are to be evaluated on L_0 . The derivatives of t may be eliminated by the help of (9), (10) to obtain the representation

$$\begin{aligned}
(25) \quad 2t(\xi, \eta) &= \nu^*(\sigma_2) t^*(\sigma_2) + \nu^*(\sigma_1) t^*(\sigma_1) \\
&+ \int_{\sigma_1}^{\sigma_2} [a^{*-1}(x' - ut') - 2\mu^* t^* u'] R(r^*, s^*; r(\xi), s(\eta)) d\sigma \\
&+ \int_{\sigma_1}^{\sigma_2} t^* \left[r' \frac{\partial}{\partial r^*} - s' \frac{\partial}{\partial s^*} \right] R(r^*, s^*; r(\xi), s(\eta)) d\sigma
\end{aligned}$$

in terms only of the data of the GCP and the classical Riemann function of (1), (2).

The reference to ξ and η in (25) may be eliminated if the Jacobian $J_2 = (dr/d\xi)ds/d\eta$ is of definite sign, so that r and s are admissible, regular variables, and (25) is then the classical representation. In general, (25) includes the

uniformization sketched by Ludford (1952); ξ, η then serve as labels to remove any ambiguity due to multi-valuedness of $t(r, s)$, and a transformation to another admissible pair of regular coordinates does not change the correspondence between x, t and r, s . The requirement of twice differentiable data may be reduced to that of the GCP by Riemann's device of introducing a function $w(\xi, \eta)$ such that

$$\partial w / \partial \xi = (x - (u + a)t) dr / d\xi, \quad \partial w / \partial \eta = ((u - a)t - x) ds / d\eta,$$

but the requirement on $d\mu/d(r + s)$ remains.

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