*Singularities of Real Analytic Spaces**

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Introduction. This paper contains further developments of the problem studied in [2]. Here the term real analytic space means a closed subset V of a real Euclidean space such that each point has a neighborhood in which V is the set of zeros of a collection of real analytic functions. The term is used in contrast to the term real analytic variety, which will be used with a local meaning, namely the set of zeros of a power series ideal in the neighborhood of a point. Thus the main result of this paper is a global form of the main result of [2]. The idea here is to show that if a real analytic space of dimension r has points of dimension $s < r$ which can, in a suitable sense, be analytically continued to a point where the dimension is r , then this analytic continuation can be found at p by repeated calculation of singularities at p , using only the local equations at p . This result is preliminary to a study of the sheets of V of less than maximum dimension. This will be undertaken in a later paper.

The method is to study an analytic continuation operation of singularities of V along an analytic arc in V , this operation being a generalization of the operations used in the proof of the main result of [2]. §§1–8 describe the analytic continuation operation. The remainder of the paper is devoted to the proof of the main result.

It will be noted that the main theorem is proved under a restriction, namely condition (F) in §10. Examples suggest that this condition could be dropped, but that the proof of the theorem would be much more difficult. This question will be taken up in a further paper.

1. Analytic continuation over a variety. Let V be a real or complex analytic manifold and p a point on it. Let $\mathcal{O}_p$ be the local ring of p on V and let $X = (X_1, X_2, \dots, X_n)$ be a set of indeterminates. $\mathcal{O}_p[[X]]$ will denote the ring of convergent power series in X with coefficients in $\mathcal{O}_p$. Here the convergence condition can be made more explicit as follows. Elements of $\mathcal{O}_p$ can be written as power series in some set of parameters with real or complex coefficients and so an element of $\mathcal{O}_p[[X]]$ can be written as a power series in these parameters and the X_i . The condition for convergence of such an element is then that the parameters and the X_i should all be sufficiently small.

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A second ring of power series $k_p[[X]]$ is now to be constructed with coefficients from the quotient field of $\mathfrak{O}_p$ and with the following convergence condition. For each element of $k_p[[X]]$ the parameters in $\mathfrak{O}_p$ are to be sufficiently small, but not equal to zero, while the X_i are to be bounded by quantities which may depend on these parameters. For example in the case which will be of most interest the elements of $\mathfrak{O}_p$ will be power series in a single variable t and the elements of $k_p[[X]]$ will be convergent power series (in the usual sense) in $t, Y_1, Y_2, \dots, Y_n$, where $Y_i = X_i/t^r$ for some r .

The rings $k_p[[X]]$ can now be put together to form a sheaf $K[[X]]$, to be called the sheaf of power series in the X_i with meromorphic coefficients. A section of $K[[X]]$ will be a power series in X with coefficients which are meromorphic on some open set on V . At a point p where all the coefficients are holomorphic the section will satisfy a convergence condition as described for $\mathfrak{O}_p[[X]]$, while if p is a pole for some coefficient the convergence condition is to be as described for $k_p[[X]]$.

The sheaves which are studied here, and which are to be called analytic continuations, will be sheaves over branched coverings of a given analytic space, or of some open set on it. So let Z be an open set on a real or complex analytic space and let $\tilde{Z}$ be a branched covering of Z . More precisely this is to mean that $\tilde{Z}$ is also an open set of a real or complex analytic space and that there is an analytic map $\pi : \tilde{Z} \rightarrow Z$ (not necessarily onto) which is a homeomorphism except at the points of some subspace (the branch locus). Thus in particular the dimensions of $\tilde{Z}$ and Z are equal. Then let the sheaf $K[[X]]$ be constructed as above over $\tilde{Z}$. A coherent sheaf $\mathcal{O}$ of $K[[X]]$ -modules, over $\tilde{Z}$, will be called an analytic continuation over Z . The notation $(\mathcal{O}, \tilde{Z})$ will be used for such a continuation to show both the sheaf and the branched covering. If a restriction of $\mathcal{O}$ is given then $(\mathcal{O}, \tilde{Z})$ will be called an analytic continuation of that restriction. In particular suppose that $\tilde{U}$ is an open set on $\tilde{Z}$ which is analytically homeomorphic to an open set U in Z under the map π . Then the restriction of $\mathcal{O}$ to $\tilde{U}$ can be thought of as a sheaf $\mathcal{O}_U$ over U , and $(\mathcal{O}, \tilde{Z})$ will be called an analytic continuation of $\mathcal{O}_U$. Two continuations $(\mathcal{O}, \tilde{Z})$ and $(\mathcal{O}', \tilde{Z}')$ will be called isomorphic (and for the present purpose can be identified) if the sheaves $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic and if the corresponding analytic homeomorphism between $\tilde{Z}$ and $\tilde{Z}'$ commutes with the projections maps π and π' into Z .

The following illustrates a natural way in which the above situation arises. Let V be real or complex n -space and let Z be an analytic space in V . In the real case assume that Z is coherent, that is to say, that its sheaf of ideals is coherent. Of course in the complex case this condition is automatically satisfied. Let $\mathfrak{O}$ be the sheaf of local rings in V , restricted to Z as base, and let $\mathfrak{I}$ be the sheaf of ideals of Z , also restricted to Z as base. $\mathfrak{O}_p$ and $\mathfrak{I}_p$ will denote the stalks of these sheaves at p in Z . Thus $\mathfrak{O}_p$ is the ring of real or complex convergent power series in n variables around p and $\mathfrak{I}_p$ is the ideal of Z in a neighborhood of p .

Now if f is an element of $\mathfrak{O}_p$, it is a power series in the coordinates of

a point of V , referred to p as origin, with real or complex coefficients. Write $x = (x_1, x_2, \dots, x_n)$ for these coordinates. Let q be a point near p with coordinates $(q_1, q_2, \dots, q_n)$ and write $X_i = x_i - q_i$. In other words the X_i are the coordinates of a generic point of V with the origin shifted to q . Write $f(x) = f(q + X) = \hat{f}(X)$. Thus $\hat{f}$ is a power series in the variables X with coefficients which are real or complex power series in the q_i , convergent for q sufficiently near p and for the X_i sufficiently small. In the notation introduced at the beginning of this section, $\hat{f}$ is in $k_p[[X]]$. And so the operation which maps f on $\hat{f}$ is a map $\Phi : \mathcal{O} \rightarrow K[[X]]$.

Lemma 1.1. Φ maps local sections of $\mathcal{O}$ into local sections of $K[[X]]$.

Proof. A local section of $\mathcal{O}$ can be identified with an analytic function f in x on some open set U of V , restricted to $U \cap Z$. The value of the section at a point p of Z is then the power series expansion of f around p , an element of $\mathcal{O}_p$; call this element $\hat{f}_p$.

Now construct $\hat{f}_p$ as described above. This is the expansion of f about q near p and the coefficients are power series in the coordinates of q with p as origin. That is to say these coefficients are functions of q analytic at p . To prove the lemma it is to be shown that these coefficients are the local expansions at p of meromorphic functions over $U \cap Z$. In fact they are the local expansions of functions analytic over $U \cap Z$. Namely it is easy to see that they are the local expansions of the functions whose values at a point of $U \cap Z$ are the coefficients of $\hat{f}$ at that point.

Now let $\mathcal{S}_p$ be the stalk of $\mathcal{S}$ over p and let $\mathcal{G}_p$ be the ideal in $k_p[[X]]$ generated by all $\hat{f}$ where f is in $\mathcal{S}_p$. Let $\mathcal{O}$ be the union of $\mathcal{G}_p$ for all p in Z .

Lemma 1.2. $\mathcal{O}$ is a sheaf of $K[[X]]$ -modules.

Proof. Since $\mathcal{O}$ is a subset of $K[[X]]$, this amounts to showing that $\mathcal{O}$ is an open set of $K[[X]]$ (cf. [1], §7).

Let $g \in \mathcal{O}$, and suppose that g is in the stalk $\mathcal{G}_p$. Then $g = \sum a_i \hat{f}_i$ where $a_i \in k_p[[X]]$ and $\hat{f}_i \in \mathcal{S}_p$. It must be shown that there is a section of $K[[X]]$ passing through g and contained in $\mathcal{O}$. Clearly it is sufficient to show that this is true for each of the $\hat{f}_i$. But there is a section of $\mathcal{O}$ through $\hat{f}_i$ and contained in $\mathcal{S}$, since $\mathcal{S}$ is a sheaf of $\mathcal{O}$ -ideals. The last lemma implies that this section is mapped by Φ into a section of $K[[X]]$, which will certainly pass through $\hat{f}_i$ and be contained in $\mathcal{O}$. This completes the proof.

Lemma 1.3. $\mathcal{O}$ is a coherent sheaf of $K[[X]]$ -modules.

Proof. Since $\mathcal{O}$ is a subsheaf of $K[[X]]$ it is sufficient to show that it is of finite type (cf. [1], §12). Since $\mathcal{S}$ is coherent, automatically in the complex case and by hypothesis in the real case, there is, for each p in V , a neighborhood U of p and a finite collection of local sections of $\mathcal{S}$ over U generating each $\mathcal{S}_q$ for q in U . These local sections can be identified with functions $f_1, f_2, \dots, f_r$.

analytic over U . Then by definition the corresponding f_1 generate $\mathcal{G}_q$ at each point q of U , and so $\mathcal{O}$ is of finite type.

Thus in the terminology introduced here $(\mathcal{O}, Z)$ is an analytic continuation over Z . In the present case $\tilde{Z} = Z$.

There are obvious variations of this construction; for example, restriction of $\mathcal{O}$ to obtain a continuation over a subvariety of Z . An example involving an actual branched covering (that is with $\tilde{Z} \neq Z$) will be introduced presently.

2. Analytic continuations along a segment. From now onward a very special case of analytic continuations will be considered. Namely, taking the ground field as the field of real numbers, V will be taken as the real line and Z (in the notation of §1) as a closed interval on C . The branched covering space will be a second closed interval $\tilde{C}$, and it will be assumed that the covering map $\pi : \tilde{C} \rightarrow C$ satisfies the following finiteness conditions. In the first place there are to be finitely many branch points, and in the second, if s_0 is a branch point on $\tilde{C}$ and $t_0 = \pi(s_0)$ then parameters can be chosen so that π is given locally by $t - t_0 = (s - s_0)^r$ for some integer r . Also the continuations to be considered will be assumed to satisfy a finiteness condition. Namely, in the notation of §1, at all but a finite number of points of $\tilde{C}$, the stalk $\mathcal{G}_p$ is to be generated by an ideal in $\mathcal{O}_p[[X]]$. This can be stated by saying that the continuations are to have only finitely many poles on $\tilde{C}$. If some emphasis is needed, continuations satisfying the finiteness conditions just stated will be said to be of finite type. Since all continuations to be studied here have this property, the terminology will not generally be used. To avoid troublesome end-effects, it will be assumed that the map π and all sheaves are defined on an open segment containing $\tilde{C}$, and that the finiteness conditions extend to the open segment.

The behavior of π at a branch point should be noted. Choose parameters s and t on $\tilde{C}$ and C , respectively, so that $s = t = 0$ at the point in question, and so that the map π is given by $t = \pi(s) = s^r$. If the integer r is odd, since only real values are considered π is 1-1 locally. If r is even, π is locally 2-1, the points s and $-s$ having the same image. In this case π , locally, maps an interval around $s = 0$ on $\tilde{C}$ onto a half interval on C ending at $t = 0$.

Each element of a stalk $\mathcal{G}_p$ of a continuation $(\mathcal{O}, \tilde{C})$ is a power series in the indeterminates X with coefficients meromorphic in a parameter s on $\tilde{C}$ around p . The convergence condition is that s should be sufficiently small (but not equal to 0 if p is a pole) and that the X should be bounded by quantities which may depend on s . If the map π is given locally at p by $t = \pi(s) = s^r$ then f can be written as a power series in X with coefficients which are fractional power series in t , in fact, power series in $t^{1/r}$. If p is not a branch point r will of course be 1. In the case where r is even note that there will only be stalks of $\mathcal{O}$ over points for which $t \geq 0$ locally around p .

Definition 2.1. In the situation just described, the coefficients of all the elements of $\mathcal{G}_p$ being meromorphic in $t^{1/r}$, suppose that r is the smallest integer

for which this true. Then the branch point p will be called *effective for the continuation* $(\mathcal{O}, \tilde{C})$. Otherwise it will be called *trivial*.

Definition 2.2. If $(\mathcal{O}, \tilde{C})$ is a continuation over C for which all the branch points are effective, then the continuation $(\mathcal{O}, \tilde{C})$ itself will be called *effective*.

3. Initial and final elements. Let $(\mathcal{O}, \tilde{C})$ be a continuation over the segment C as described above. Since there is only a finite number of branch points for the map $\pi : \tilde{C} \rightarrow C$, there are intervals $\tilde{U}_1$ and $\tilde{U}_2$ at the ends of $\tilde{C}$ containing no branch points. Thus π maps $\tilde{U}_1$ and $\tilde{U}_2$ by analytic homeomorphisms on intervals U_1 and U_2 on C , which means that the restrictions of $\mathcal{O}$ to $\tilde{U}_1$ and $\tilde{U}_2$ can be identified with sheaves $\mathcal{O}_1$ and $\mathcal{O}_2$ over U_1 and U_2 respectively. In this context it is convenient to think of $\tilde{C}$ as being directed from the initial subinterval $\tilde{U}_1$ to the final subinterval $\tilde{U}_2$, and this terminology will be transferred to the sheaves $\mathcal{O}_1$ and $\mathcal{O}_2$.

Definition 3.1. $(\mathcal{O}_1, U_1)$ will be called the *initial element* of $(\mathcal{O}, \tilde{C})$ and $(\mathcal{O}_2, U_2)$ its *final element*. $\tilde{C}$ will be called the *path of the continuation*.

Clearly the whole continuation $(\mathcal{O}, \tilde{C})$ is fully determined by its initial element and the path. In fact this motivates the terminology used both here and in Definition 3.2. For one can think of the continuation as being an operation performed by transferring the initial element $\mathcal{O}_1$ along $\tilde{C}$ till it is transformed into its final element. It will turn out that the final element so obtained depends only on the effective branch points passed on the way. Thus the effective continuations have a sort of minimal property. This is made explicit in the following lemma.

Lemma 3.1. Let $(\mathcal{O}, \tilde{C})$ be an effective continuation with initial element $(\mathcal{O}_1, U_1)$ and let $(\mathcal{O}', \tilde{C}')$ be any other continuation over C with the same initial element $(\mathcal{O}_1, U_1)$. Then there is an analytic map $\phi : \tilde{C} \rightarrow \tilde{C}'$ making the diagram

$$\begin{array}{ccc} \tilde{C}' & \xrightarrow{\quad} & \tilde{C} \\ \pi' \searrow & & \swarrow \pi \\ & C & \end{array}$$

commute, π and π' being the appropriate branched covering maps, and such that $\mathcal{O}' = \phi^{-1}\mathcal{O}$. In addition if $(\mathcal{O}', \tilde{C}')$ is also effective, ϕ is an analytic homeomorphism.

The last sentence here means that an effective continuation is uniquely determined by its initial element.

Proof. Since $(\mathcal{O}, \tilde{C})$ and $(\mathcal{O}', \tilde{C}')$ have a common initial element, a map ϕ with the required properties can be defined on a neighborhood at the beginning of $\tilde{C}'$, namely on $\pi'^{-1}(U_1)$, and can be defined there as $\phi = \pi^{-1}\pi'$. Let U be the longest interval containing $\pi'^{-1}(U_1)$ on which an analytic map ϕ into $\tilde{C}$

can be defined such that $\pi' = \pi\phi$ on U and $\mathcal{O}' = \phi^{-1}\mathcal{O}$ (the sheaves being restricted to U and $\phi(U)$ respectively). If U is not the whole of $\tilde{C}'$, let p' be its upper bound and let p be the limit of $\phi(q)$ as q tends to p' . Let s' , s and t be parameters on $\tilde{C}'$, $\tilde{C}$ and C around p' , p and $\pi'(p') = \pi(p)$ chosen so that the maps π' and π are given locally by $\pi'(s') = s'^{r'}$ and $\pi(s) = s^r$, respectively. Assume that points of U near p' correspond to positive values of s' . The equation $\mathcal{O}' = \phi^{-1}\mathcal{O}$ holds over U ; in particular it holds for small positive values of s and s' , the map ϕ being 1-1 for such values, and being given by $s'^{r'} = s^r$. This means that $\mathcal{O}'$ and $\mathcal{O}$ have the same stalks over points s' and s for small positive values of these parameters satisfying $s'^{r'} = s^r$. Then if $\mathcal{G}_s$ is such a stalk, it consists of power series whose coefficients are meromorphic in both $t^{1/r}$ and $t^{1/r'}$ (cf. §2) for small t . And, since $(\mathcal{O}, \tilde{C})$ is assumed effective, r is minimal with this property. It follows that $r' = kr$ for some integer k . And so the equation of ϕ locally is $s = s'^k$ ($s > 0$, $s' > 0$). However this equation can now be used to extend ϕ to small negative values of s' and for the extension the relation $\pi' = \pi\phi$ and $\mathcal{O}' = \phi^{-1}\mathcal{O}$ will continue to hold. This contradicts the maximality of U , which must therefore be the whole of $\tilde{C}'$ as was to be shown.

The last part of the lemma follows from the same kind of argument, letting U now be the maximum interval containing $\pi'^{-1}(U_1)$ on which ϕ is an analytic homeomorphism. Then since $(\mathcal{O}, \tilde{C}')$ and $(\mathcal{O}, \tilde{C})$ are both assumed effective, the integers r and r' (in the notation of the above argument) must be equal. It follows that ϕ extends as an analytic homeomorphism, again contradicting the maximality of U , unless U is the whole of $\tilde{C}'$. This completes the proof.

Note that if, in the notation of this lemma, there is a ϕ such that $\pi' = \pi\phi$ it is unique. For any two such maps agree on a neighborhood of one end of C' , and, being analytic, must therefore coincide.

The following is more or less an alternative way of stating the first part of the above lemma.

Lemma 3.2. *If $(\mathcal{O}, \tilde{C})$ is an effective continuation with initial element $(\mathcal{O}_1, U_1)$, then $(\mathcal{O}_1, U_1)$ has a continuation along a path $\tilde{C}'$ if and only if there is an analytic map $\phi : \tilde{C}' \rightarrow \tilde{C}$ such that $\pi' = \pi\phi$.*

Proof. If $(\mathcal{O}_1, U_1)$ has a continuation $(\mathcal{O}', \tilde{C}')$ the last lemma gives the existence of ϕ . On the other hand if ϕ is given, then $(\phi^{-1}\mathcal{O}, \tilde{C}')$ is a continuation of $(\mathcal{O}_1, U_1)$.

The case that will be of interest later is that in which the initial and final elements of the continuation project onto the same interval U on C . Thus in the above notation the initial and final elements are $(\mathcal{O}_1, U_1)$ and $(\mathcal{O}_2, U_2)$ with $U_1 = U_2 = U$. Usually U will be an interval at one end of C , but this is not necessarily so.

Definition 3.2. If the initial and final elements are over the same neighborhood U of C the continuation $(\mathcal{O}, \tilde{C})$ will be called *closed* and $\tilde{C}$ will be called a *closed continuation path*. $\mathcal{O}_2$, determined by $\mathcal{O}_1$ and $\tilde{C}$, will be called the

transform of $\mathcal{O}_1$ along $\tilde{C}$ and will be written as $T\mathcal{O}_1$. T will be called the *transformation corresponding to continuation along $\tilde{C}$* .

Lemma 3.3. *Let T be the transformation associated with the closed continuation path $\tilde{C}$ and suppose the initial element $(\mathcal{O}_1, U)$ has a continuation along $\tilde{C}$. Then if this continuation is effective, $T\mathcal{O}_1 \neq \mathcal{O}_1$.*

Proof. Suppose $T\mathcal{O}_1 = \mathcal{O}_1$. That is to say the continuation $(\mathcal{O}, \tilde{C})$ with initial element $(\mathcal{O}_1, U)$ has also final element $(\mathcal{O}_1, U)$. Now the continuation $(\mathcal{O}, \tilde{C})$ can be thought of either as a continuation along $\tilde{C}$ or along $\tilde{C}$ in reverse, in each case being effective and in each case with the same initial element. Lemma 3.1 then implies that there is an analytic homeomorphism ϕ of $\tilde{C}$ onto itself reversing direction and satisfying the conditions $\pi\phi = \pi$ and $\mathcal{O} = \phi^{-1}\mathcal{O}$. Since ϕ is direction reversing it has a fixed point p . Take parameters s and t around p and $\pi(p)$ on $\tilde{C}$ and C such that π is given locally by the equation $t = \pi(s) = s^r$. Since $\pi\phi = \pi$, π cannot be 1-1 around p and so must be 2-1. And so ϕ is given locally by $\phi(s) = -s$, and r is even. The coefficients of elements of $\mathcal{G}_p$ are meromorphic in $s = t^{1/r}$ around p (cf. §2) and the condition $\mathcal{O} = \phi^{-1}\mathcal{O}$ implies that these functions are invariant under the sign change, substituting $-s$ for s . This means that they are power series in $t^{2/r}$. But this contradicts the fact that the continuation $(\mathcal{O}, \tilde{C})$ is effective, and so the initial assumption that $T\mathcal{O}_1 = \mathcal{O}_1$ is false, as was to be shown.

Lemma 3.4. *Continue with the notation of the last lemma and let the initial element $(\mathcal{O}_1, U)$ also have a closed continuation $(\mathcal{O}', \tilde{C}')$ and let T' be the transformation associated with $\tilde{C}'$. Then $T'\mathcal{O}_1 = T\mathcal{O}_1$ if the map ϕ of Lemma 3.2 is onto, and otherwise $T'\mathcal{O}_1 = \mathcal{O}_1$.*

Proof. If ϕ is onto it induces an analytic homeomorphism between open intervals at the ends of $\tilde{C}$ and similar intervals at the ends of $\tilde{C}'$, and the relation $\mathcal{O}' = \phi^{-1}\mathcal{O}$ implies that $(\mathcal{O}', \tilde{C}')$ has the same final element as $(\mathcal{O}, \tilde{C})$, as required. On the other hand if ϕ is into, this same reasoning implies that $T'\mathcal{O}_1 = \mathcal{O}_1$.

The last lemma leads at once to a more general comparison of continuations along different paths.

Lemma 3.5. *Let $\tilde{C}_1$ and $\tilde{C}_2$ be two closed paths for continuation over C , with branched covering maps π_1 and π_2 and corresponding transformations T_1 and T_2 . Then if there is an analytic map $\psi : \tilde{C}_1 \rightarrow \tilde{C}_2$ such that the diagram*

$$(3.1) \quad \begin{array}{ccc} \tilde{C}_1 & \xrightarrow{\psi} & \tilde{C}_2 \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & C & \end{array}$$

is commutative and ψ is onto, $T_1 = T_2$ on any initial element with continuations

along both $\tilde{C}_1$ and $\tilde{C}_2$. On the other hand if ψ is into $\tilde{C}_2$, then T_1 acts trivially on any initial element which has a continuation along $\tilde{C}_2$.

Proof. Suppose that there is a ψ making (3.1) commutative, and let $(\mathcal{O}, \tilde{C})$ be an effective closed continuation with an initial element which can be continued along $\tilde{C}_2$. Then there is an analytic map $\phi_2 : \tilde{C}_2 \rightarrow \tilde{C}$ as in Lemma 3.2, such that $\pi_2 = \pi\phi_2$. Thus the combined diagram

$$(3.2) \quad \begin{array}{ccccc} \tilde{C}_1 & \xrightarrow{\psi} & \tilde{C}_2 & \xrightarrow{\phi_2} & \tilde{C} \\ & \searrow \pi_1 & \downarrow \pi_2 & \swarrow \pi & \\ & & C & & \end{array}$$

is commutative. Hence the initial element $(\mathcal{O}_1, U_1)$ of $(\mathcal{O}, \tilde{C})$ can be continued along $\tilde{C}_1$ (Lemma 3.2). Write $\phi_1 = \phi_2\psi$. Then ϕ_1 is as in Lemma 3.2, applied to $\tilde{C}_1$ (the remark on uniqueness, following Lemma 3.1, is used here) and so the continuations of $(\mathcal{O}_1, U_1)$ along $\tilde{C}_1$ and $\tilde{C}_2$ are given by $(\phi_1^{-1}\mathcal{O}, \tilde{C}_1)$ and $(\phi_2^{-1}\mathcal{O}, \tilde{C}_2)$. Here $\phi_1^{-1}\mathcal{O} = \psi^{-1}(\phi_2^{-1}\mathcal{O})$ and so, arguing as in Lemma 3.4, if ψ is onto $T_1\mathcal{O}_1 = T_2\mathcal{O}_1$ but if ψ is into $T_1\mathcal{O}_1 = \mathcal{O}_1$.

The following is an immediate corollary.

Lemma 3.6. *Let $\tilde{C}_1$ and $\tilde{C}_2$ be two paths for closed continuation over C with projection maps π_1 and π_2 and corresponding transformations T_1 and T_2 . Suppose that there is a continuation path $\tilde{C}$, with projection map π into C , and that there are analytic maps ϕ_1 and ϕ_2 making the following diagram commutative:*

$$(3.3) \quad \begin{array}{ccccc} \tilde{C}_1 & \xleftarrow{\phi_1} & \tilde{C} & \xrightarrow{\phi_2} & \tilde{C}_2 \\ & \searrow \pi_1 & \downarrow \pi & \swarrow \pi_2 & \\ & & C & & \end{array}$$

Then if ϕ_1 and ϕ_2 are both onto, $T_1 = T_2$ on any initial element with continuations along both $\tilde{C}_1$ and $\tilde{C}_2$. On the other hand if ϕ_1 is onto and ϕ_2 is into then T_2 acts trivially on any initial element with a continuation along both $\tilde{C}_1$ and $\tilde{C}_2$.

Proof. This is simply a double application of Lemma 3.5.

4. Ordering of paths. In what follows the equality, or otherwise, of the transformations associated with continuations will be the most significant basis for their comparison, and so the last lemma prompts the following definition.

Definition 4.1. Let $\tilde{C}_1, \tilde{C}_2, \tilde{C}$ be closed paths for continuations over C , with projection maps π_1, π_2, π into C and let ϕ_1 and ϕ_2 be analytic maps such that

$$(4.1) \quad \begin{array}{ccccc} \tilde{C}_1 & \xleftarrow{\phi_1} & \tilde{C} & \xrightarrow{\phi_2} & \tilde{C}_2 \\ & \searrow \pi_1 & \downarrow \pi & \swarrow \pi_2 & \\ & & C & & \end{array}$$

is commutative. If $\tilde{C}$ and the ϕ_i can be chosen so that the ϕ_i are both onto, $\tilde{C}_1$ and $\tilde{C}_2$ will be called *equivalent paths*. The notation for this will be $\tilde{C}_1 \sim \tilde{C}_2$. If $\tilde{C}$ and the ϕ_i can be chosen so that ϕ_1 is onto and ϕ_2 is into, $\tilde{C}_1$ will be called *greater than $\tilde{C}_2$* , in symbols $\tilde{C}_1 > \tilde{C}_2$.

Thus in terms of this definition, Lemma 3.6 says that continuations along equivalent paths with the same initial elements also have the same final elements. On the other hand if $\tilde{C}_1 > \tilde{C}_2$ then for any initial element $(\mathcal{P}, U)$ with a continuation along both paths, $T_2\mathcal{P} = \mathcal{P}$.

In order to study the relations just defined it is convenient to represent the continuation paths geometrically. If $\tilde{C}$ is a path for a closed continuation, π the corresponding projection map into C , $\tilde{C}$ is analytically homeomorphic to, and so may be identified with, an arc of the curve in the (x, y) -plane with equation $x = \pi(y)$. The assumption (cf. §2) of finiteness of type implies that this arc is met in a finite and in fact bounded number of points by any line parallel to the y -axis. Also, near the end points there are no branch points for π and so the equation $x = \pi(y)$ can be inverted locally near the end points giving local equations $y = f(x)$ and $y = g(x)$ for the initial and final arcs of $\tilde{C}$.

Now let $\tilde{C}_1$ and $\tilde{C}_2$ be two continuation paths, identified as above with the curves $x = \pi_1(y)$ and $x = \pi_2(y)$. Let the local equations of the initial and final arcs of $\tilde{C}_1$ be $y = f_1(x)$ and $y = g_1(x)$ and let those of $\tilde{C}_2$ be $y = f_2(x)$ and $y = g_2(x)$. Consider the curve Γ given in 3-space by the equations

$$x = \pi_1(y), \quad x = \pi_2(z).$$

Γ is a union of a finite number of loops and arcs, contained in a bounded set of 3-space. In fact there can only be two arcs, since there are just four end points for Γ , namely the four neighborhoods having the four sets of local equations:

$$(4.2) \quad \begin{array}{cccc} y = f_1(x) \Big\{ & y = f_1(x) \Big\{ & y = g_1(x) \Big\{ & y = g_1(x) \Big\{ \\ z = f_2(x) \Big\} & z = g_2(x) \Big\} & z = f_2(x) \Big\} & z = g_2(x) \Big\} \end{array}$$

The two arcs in Γ must be formed by joining these end neighborhoods in pairs.

Let Γ_0 be the arc of Γ starting off with the local equations $y = f_1(x), z = f_2(x)$. Γ_0 is the analytic image of a line segment $\tilde{C}_0$ and so defines a branched covering $\pi_0: \tilde{C}_0 \rightarrow C$, obtained by mapping $s \in \tilde{C}_0$ on the point $(x(s), y(s), z(s))$ of parameter s on Γ_0 , and then projecting on the point $x(s)$ on C . It is easy to check that $\tilde{C}_0$ satisfies the conditions for being of finite type.

Note that here $(x(s), y(s))$ is a point on the curve in the (x, y) -plane repre-

sending $\tilde{C}_1$. That is to say $x(s) = \pi_1(y(s))$. Thus if the map $\phi_1 : \tilde{C}_0 \rightarrow \tilde{C}_1$ is defined by setting $\phi_1(s) = y(s)$, then $\pi_1\phi_1 = \pi_0$. Similarly if $\phi_2(s) = z(s)$ then $\pi_2\phi_2 = \pi_0$. Thus there is a commutative diagram

$$(4.3) \quad \begin{array}{ccccc} & & \tilde{C}_1 & \xleftarrow{\phi_1} & \tilde{C}_0 & \xrightarrow{\phi_2} & \tilde{C}_2 \\ & & \searrow \pi_1 & & \downarrow \pi_0 & & \swarrow \pi_2 \\ & & & & C & & \end{array}$$

The maps ϕ_1 and ϕ_2 are clearly analytic.

Lemma 4.1. *At least one of the maps ϕ_1, ϕ_2 is onto. $\tilde{C}_1 \sim \tilde{C}_2$ if and only if both the ϕ_i are onto. $\tilde{C}_1 > \tilde{C}_2$ if and only if ϕ_1 is onto and ϕ_2 is into.*

Proof. The arc Γ_0 starts off with the arc having local equations $y = f_1(x)$, $z = f_2(x)$ and must join this to one of the other ends of the curve Γ . For example if Γ_0 ends with the arc having local equations $y = f_1(x)$, $z = g_2(x)$ then the projection of a point (x, y, z) of Γ_0 on the point (x, z) of the curve $x = \pi_2(z)$ maps the two ends of the former on the two ends of the latter, and so ϕ_2 is onto. Similarly in the other two cases listed in (4.2) ϕ_1 is onto and ϕ_2 is onto, respectively.

Certainly if both ϕ_1 and ϕ_2 are onto then $\tilde{C}_1 \sim \tilde{C}_2$, by Definition 4.1. Suppose conversely that $\tilde{C}_1 \sim \tilde{C}_2$. Then there is a commutative diagram

$$(4.4) \quad \begin{array}{ccccc} & & \tilde{C}_1 & \xleftarrow{\psi_1} & \tilde{C} & \xrightarrow{\psi_2} & \tilde{C}_2 \\ & & \searrow \pi_1 & & \downarrow \pi & & \swarrow \pi_2 \\ & & & & C & & \end{array}$$

with the maps ψ_1 and ψ_2 both onto. Then for $s \in \tilde{C}$, the point

$$(\pi_1\psi_1(s), \psi_1(s), \psi_2(s)) = (\pi_2\psi_2(s), \psi_1(s), \psi_2(s))$$

is in Γ and in fact it can be assumed that the initial arc of $\tilde{C}$ is mapped on the initial arc of Γ_0 . And so an analytic map $\phi : \tilde{C} \rightarrow \tilde{C}_0$ is defined such that the diagram

$$(4.5) \quad \begin{array}{ccccc} & & \tilde{C}_1 & \xleftarrow{\psi_1} & \tilde{C} & \xrightarrow{\psi_2} & \tilde{C}_2 \\ & & \searrow \phi_1 & & \downarrow \phi & & \swarrow \phi_2 \\ & & & & \tilde{C}_0 & & \\ & & \searrow \pi_1 & & \downarrow \pi_0 & & \swarrow \pi_2 \\ & & & & C & & \end{array}$$

is commutative, along with the additional relation $\pi_0\phi = \pi$. Now remembering

that onto-ness here can be detected by examining how the ends are mapped, it follows that, since ψ_1 and ψ_2 are onto, so is ϕ . Then if ϕ_1 were into, the onto property of ψ_1 would be contradicted. Hence ϕ_1 is onto and similarly also ϕ_2 , as was to be shown.

Finally if ϕ_1 is onto and ϕ_2 is into then $\tilde{C}_1 > \tilde{C}_2$ by Definition 4.1. Conversely if $\tilde{C}_1 > \tilde{C}_2$ then a commutative diagram (4.4) can be set up as before, but now with ψ_1 onto and ψ_2 into. Arguing as before ϕ_1 is then also onto. But now, since ψ_2 is into the second end of Γ_0 must have local equations $y = g_1(x)$, $z = f_2(x)$ and so ϕ_2 is into as required.

The next three lemmas are immediate consequences of the last one.

Lemma 4.2. *Given $\tilde{C}_1$ and $\tilde{C}_2$, exactly one of the relations $\tilde{C}_1 > \tilde{C}_2$, $\tilde{C}_1 \sim \tilde{C}_2$ or $\tilde{C}_2 > \tilde{C}_1$ holds.*

Proof. Using the notation of the last lemma, either ϕ_1 is onto and ϕ_2 into, or ϕ_1, ϕ_2 are both onto, or ϕ_2 is onto and ϕ_1 is into, and by the last lemma these mutually exclusive cases are exactly equivalent to the three cases in the statement of this lemma.

Lemma 4.3. *Let $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ be three paths for closed continuations of finite type and suppose that $\tilde{C}_1 \succeq \tilde{C}_2$ and $\tilde{C}_2 \succeq \tilde{C}_3$. Then $\tilde{C}_1 \succeq \tilde{C}_3$.*

Proof. By hypothesis, using Definition 4.1, there are paths $\tilde{K}_1, \tilde{K}_2$ over C and a commutative diagram

$$(4.6) \quad \begin{array}{ccccccc} \tilde{C}_1 & \xleftarrow{\tau_1} & \tilde{K}_1 & \xrightarrow{\tau_2} & \tilde{C}_2 & \xleftarrow{\tau_3} & \tilde{K}_2 & \xrightarrow{\tau_4} & \tilde{C}_3 \\ & \searrow \pi_1 & & \searrow \kappa_1 & \downarrow \pi_2 & & \searrow \kappa_2 & \searrow \pi_3 & \\ & & & & C & & & & \end{array}$$

all the maps being analytic, and ψ_1 and ψ_3 being onto. On the other hand, by the construction preceding Lemma 4.1 there is a path $\tilde{K}_0$ and analytic maps forming the commutative diagram

$$(4.7) \quad \begin{array}{ccccc} \tilde{K}_1 & \xleftarrow{\phi} & \tilde{K}_0 & \xrightarrow{\phi} & \tilde{K}_2 \\ & \searrow \kappa_1 & \downarrow \kappa_0 & \searrow \kappa_2 & \\ & & C & & \end{array}$$

And either ϕ_1 or ϕ_2 is onto by Lemma 4.1. Suppose that ϕ_1 is into and ϕ_2 is onto. Then there is a commutative diagram

$$(4.8) \quad \begin{array}{ccccc} \tilde{C}_1 & \xleftarrow{\tau_1 \phi} & \tilde{K}_0 & \xrightarrow{\tau_2 \phi} & \tilde{C}_2 \\ & \searrow \pi_1 & \downarrow \kappa_0 & \searrow \pi_2 & \\ & & C & & \end{array}$$

with $\psi_1 \phi_1$ into and $\psi_3 \phi_2$ onto. But this would mean that $\tilde{C}_2 > \tilde{C}_1$, contrary

to the hypothesis that $\tilde{C}_1 \geq \tilde{C}_2$, since both relations cannot hold, by Lemma 4.2. Hence ϕ_1 is onto. But this means that there is a commutative diagram

$$(4.9) \quad \begin{array}{ccccc} \tilde{C}_1 & \xleftarrow{\psi_1 \phi_1} & \tilde{K}_0 & \xrightarrow{\psi_2 \phi_2} & \tilde{C}_3 \\ & \searrow \pi_1 & \downarrow \kappa_0 & \swarrow \pi_3 & \\ & & C & & \end{array}$$

with $\psi_1 \phi_1$ onto. Hence $\tilde{C}_1 \geq \tilde{C}_3$ (Definition 4.1).

Lemma 4.4. *Continuing with the notation of the last lemma, $\tilde{C}_1 \sim \tilde{C}_3$ if and only if both $\tilde{C}_1 \sim \tilde{C}_2$ and $\tilde{C}_2 \sim \tilde{C}_3$.*

Proof. Suppose that $\tilde{C}_1 \sim \tilde{C}_2$ and $\tilde{C}_2 \sim \tilde{C}_3$. Then the ψ_i are all onto. To show that $\tilde{C}_1 \sim \tilde{C}_3$ it must be shown that ϕ_2 is onto, for then this will make both $\psi_1 \phi_1$ and $\psi_4 \phi_2$ onto in diagram (4.9). The diagrams (4.6) and (4.7) put together give a commutative diagram

$$(4.10) \quad \begin{array}{ccccc} \tilde{K}_1 & \xleftarrow{\phi_1} & \tilde{K}_0 & \xrightarrow{\phi_2} & \tilde{K}_2 \\ \downarrow \psi_1 & \searrow \kappa_1 & \downarrow \kappa_0 & \swarrow \kappa_2 & \downarrow \psi_2 \\ \tilde{C}_1 & & C & & \tilde{C}_3 \\ & \searrow \psi_3 & & \swarrow \psi_4 & \\ & & \tilde{C}_2 & & \end{array}$$

ϕ_1 , ψ_2 and so $\psi_2 \phi_1$ are onto, from which it follows that $\psi_3 \phi_2$ is onto. But this means, by examining how the ends of $\tilde{K}_0$ are mapped, that ϕ_2 is onto, as was to be shown.

Next suppose that $\tilde{C}_2 > \tilde{C}_3$. Then ψ_4 will be into and so $\psi_4 \phi_2$ is into and so $\tilde{C}_1 > \tilde{C}_3$.

Finally suppose that $\tilde{C}_1 > \tilde{C}_2$. Then ψ_2 is into and so $\psi_2 \phi_1$ is into. Using diagram (4.10) it follows that $\psi_3 \phi_2$ is into and so ϕ_2 is into. Hence $\psi_4 \phi_2$ is into and this means, using diagram (4.9), that $\tilde{C}_1 > \tilde{C}_3$. This completes the proof of the lemma.

It is practically trivial that the relation $\sim$ is symmetric and reflexive and so the last two lemmas show that $\sim$ is an equivalence relation and that $>$ is an order relation on the equivalence classes.

The definition of the relation $\sim$ says that two equivalent paths are the analytic images of some other path, with a compatibility condition on the projections into C . It will now be shown that this is also true for any number of equivalent paths.

Lemma 4.5. *If $\tilde{C}_1 \sim \tilde{C}_2 \sim \dots \sim \tilde{C}_r$, there is a path $\tilde{C}_0$ and analytic maps $\phi_i : \tilde{C}_0 \rightarrow \tilde{C}_i$ which are onto and such that $\pi_0 = \pi_i \phi_i$, where the π denote the appropriate projection maps into C .*

Proof. The proof is by induction. As pointed out, the result is true for $r = 2$. Suppose that there is a path C'_0 and maps $\phi'_i : \tilde{C}'_0 \rightarrow \tilde{C}'_i$ ($i = 1, 2, \dots, r - 1$), all onto and such that $\pi'_0 = \pi_i \phi'_i$. Then $\tilde{C}'_0 \sim \tilde{C}'_{r-1}$ and so (by Lemma 4.4) $\tilde{C}'_0 \sim \tilde{C}'_r$. Hence there is a commutative diagram

$$(4.11) \quad \begin{array}{ccccc} \tilde{C}'_0 & \xleftarrow{\phi} & \tilde{C}'_r & \xrightarrow{\phi_r} & \tilde{C}'_r \\ & \searrow \pi'_0 & \downarrow \pi_r & \swarrow \pi_r & \\ & & C & & \end{array}$$

with the maps ϕ and ϕ_r both onto. Then the required result is obtained for r paths by taking $\phi_i = \phi'_i \phi$ ($i = 1, 2, \dots, r - 1$) along with ϕ_r .

5. Symmetrization. The aim of this section is to show that a path for a closed continuation of finite type is equivalent to one which is, in a sense, symmetric. This will imply among other things that the corresponding transformation is involutory.

Definition 5.1. A closed continuation path $\tilde{C}$ with the projection map π into C will be called *symmetric* if there is an analytic homeomorphism ϕ of $\tilde{C}$ onto itself, reversing direction and satisfying $\pi\phi = \pi$.

Note that in this case, if $\tilde{C}'$ denotes the path $\tilde{C}$ taken in reverse, then $\tilde{C}' \sim \tilde{C}$.

Let T be the transformation associated with the symmetric path $\tilde{C}$. Then the reversed path $\tilde{C}'$ is associated with the transformation T^{-1} . Then Lemmas 3.2, 3.4 imply that if $(\mathcal{P}, U)$ is an initial element with a continuation along $\tilde{C}$, it also has a continuation along $\tilde{C}'$ and $T^{-1}\mathcal{P} = T\mathcal{P}$. That is to say, the following lemma holds.

Lemma 5.1. If T is the transformation corresponding to a symmetric path then T^2 is the identity.

The following lemma aims at showing that this result also holds for the transformation associated with any closed continuation path.

Lemma 5.2. Let $\tilde{C}$ be a path for a closed continuation over C . Then there is a symmetric path $\tilde{C}_0$ for continuation over C and an analytic map ϕ of $\tilde{C}_0$ onto $\tilde{C}$ such that

$$(5.1) \quad \begin{array}{ccc} \tilde{C}_0 & \xrightarrow{\phi} & \tilde{C} \\ & \searrow \pi_0 & \swarrow \pi \\ & & C \end{array}$$

is commutative.

Proof. Let $\tilde{C}'$ be the path $\tilde{C}$ taken in reverse. Then in any case there is a path $\tilde{C}_0$ with maps ϕ_1, ϕ_2 constructed as in §4 and forming a commutative diagram

$$(5.2) \quad \begin{array}{ccccc} & & \tilde{C}_0 & & \\ \tilde{C} & \xleftarrow{\phi_1} & & \xrightarrow{\phi_2} & \tilde{C}' \\ & \searrow \pi & \downarrow \pi_0 & \swarrow \pi' & \\ & & C & & \end{array}$$

If the initial and final arcs of $\tilde{C}$ are given by the local equations $y = f_1(x)$ and $y = f_2(x)$ respectively (using the geometric set-up of §4) then these are the final and initial arcs of $\tilde{C}'$, respectively. For s on $\tilde{C}_0$, $\phi_2(s) - \phi_1(s)$ is an analytic function on $\tilde{C}_0$ given locally at the beginning of $\tilde{C}_0$ by $f_2(x) - f_1(x)$. If either ϕ_2 or ϕ_1 is into (the other being, of course onto, by Lemma 4.1) this function will be zero in a neighborhood of the end of $\tilde{C}_0$, and so will be identically zero, which is impossible. Hence ϕ_1 and ϕ_2 are both onto.

Thus the initial arc of $\tilde{C}_0$ is $y = f_1(x)$, $z = f_2(x)$ and its final arc is $y = f_2(x)$, $z = f_1(x)$. Then, still using the geometric set-up and notation of §4, the map carrying (x, y, z) on Γ_0 into (x, z, y) induces a map ϕ of $\tilde{C}_0$ into $\tilde{C}'_0$ which is certainly analytic and makes the following diagram commute:

$$(5.3) \quad \begin{array}{ccccc} & & \tilde{C}_0 & & \\ & \swarrow \phi_1 & \downarrow \phi & \searrow \phi_2 & \\ \tilde{C} & & & & \tilde{C}' \\ & \nwarrow \phi_2 & \downarrow \phi & \swarrow \phi_1 & \\ & & \tilde{C}'_0 & & \end{array}$$

Since ϕ_1 and ϕ_2 are onto it follows that ϕ is onto. It is clear also that $\tilde{C}_0$ and ϕ make the diagram (5.1) commute and so the lemma is proved.

Now, continuing with the notation of the last lemma, Lemmas 3.2, 3.5 imply that any initial element with a continuation along $\tilde{C}$ also has a continuation along $\tilde{C}_0$, and the corresponding transformations are the same. Thus the following is an immediate consequence of Lemmas 5.1, 5.2.

Lemma 5.3. *Let $\tilde{C}$ be a path for closed continuation over C , and let T be the corresponding transformation. Then T^2 is the identity.*

The next step is to obtain a more general form of Lemma 5.2, showing how to replace a whole collection of paths by symmetric paths, at the same time getting a simple relation between the corresponding transformations.

Theorem 1. *Let a collection of paths for closed continuation of finite type over C be given, a typical member being $\tilde{C}_i$, with projection map π_i into C . Then there is a collection of symmetric paths $\tilde{K}_i$, the projection maps being $\kappa_i : \tilde{K}_i \rightarrow C$, such that*

- (1) *For each $i > j$ there are analytic maps $\phi_{i,j}$, $\phi'_{i,j}$ making the diagrams*

$$(5.4) \quad \begin{array}{ccc} \tilde{K}_j & \xrightarrow{\phi_j} & \tilde{K}_i \\ \kappa_j \searrow & & \swarrow \kappa_i \\ & C & \end{array} \quad \begin{array}{ccc} \tilde{K}_j & \xrightarrow{\phi'_j} & \tilde{K}'_i \\ \kappa_j \searrow & & \swarrow \kappa_i \\ & C & \end{array}$$

commute, where $\tilde{K}'_i$ denotes K_i taken in reverse and all the ϕ_{ij} and ϕ'_{ij} are into.

(2) For each $\tilde{K}_i$ there is an analytic map ψ_i of $\tilde{K}_i$ onto one of the $\tilde{C}_i$ making the diagram

$$(5.5) \quad \begin{array}{ccc} \tilde{K}_i & \xrightarrow{\sim} & \tilde{C}_j \\ \kappa_i \searrow & & \swarrow \pi_j \\ & C & \end{array}$$

commute, and all the $\tilde{C}_i$ are obtained as the images of such maps.

Proof. If a number of the $\tilde{C}_i$ are equivalent under the relation $\sim$, Lemma 4.5 implies that there is a $\tilde{C}_i^*$ mapped onto each by an analytic map compatible with the projection maps into C , and it is clear that, to prove the theorem, it is sufficient to prove it for the $\tilde{C}_i^*$ instead of for the $\tilde{C}_i$, the essential simplification being that no two of the $\tilde{C}_i^*$ are equivalent. So assume that this arrangement has already been made, and drop the stars. Thus the theorem is now to be proved under the additional assumption that no two of the $\tilde{C}_i$ are related under $\sim$. By Lemma 4.3 the relation $>$ is a total ordering on the $\tilde{C}_i$. So let these paths be arranged in the order $\tilde{C}_1 > \tilde{C}_2 > \cdots > \tilde{C}_r$.

The theorem will now be proved by showing that there is a collection of symmetric paths $\tilde{K}_i$ ($i = 1, 2, \dots, r$) satisfying condition (1) in the statement of the theorem and also the following:

(2)' For each i there is an analytic map ψ_i of $\tilde{K}_i$ onto $\tilde{C}_i$ making

$$(5.6) \quad \begin{array}{ccc} \tilde{K}_i & \xrightarrow{\quad} & \tilde{C}_i \\ \kappa_i \searrow & & \swarrow \pi_i \\ & C & \end{array}$$

commute.

This will be proved by induction. Lemma 5.2 shows that it is true for $r = 1$. Suppose now that paths $\tilde{K}_i$ and maps ψ_i onto the corresponding $\tilde{C}_i$ have been found satisfying (1) and (2)' for $i = 2, 3, \dots, r$. Consider the paths $\tilde{C}_1$ and $\tilde{K}_2$. $\tilde{K}_2 \sim \tilde{C}_2$ and so $\tilde{C}_1 > \tilde{K}_2$. Also $\tilde{K}_2$ is symmetric and so $\tilde{K}'_2 \sim \tilde{K}_2$, which in turn implies that $\tilde{C}_1 > \tilde{K}'_2$. Hence there are paths L_1 and $\tilde{M}_1$ and commutative diagrams of analytic maps

$$(5.7) \quad \begin{array}{ccccc} \tilde{C}_1 & \xleftarrow{\theta_1} & \tilde{L} & \xrightarrow{\theta_2} & \tilde{K}_1 \\ & \searrow \pi_1 & \downarrow \lambda & \swarrow \kappa_2 & \\ & & C & & \end{array} \quad \begin{array}{ccccc} \tilde{C}_1 & \xleftarrow{\theta'_1} & \tilde{M} & \xrightarrow{\theta'_2} & \tilde{K}_1 \\ & \searrow \pi_1 & \downarrow \mu & \swarrow \kappa_2 & \\ & & C & & \end{array}$$

where θ_1 and θ'_1 are onto and θ_2 and θ'_2 are onto. Since θ_1 and θ'_1 are onto, $\tilde{L} \sim \tilde{C}_1$ and $\tilde{M} \sim \tilde{C}_1$ and so, by Lemma 4.4, $\tilde{L} \sim \tilde{M}$. Now apply the definition of the relation $\sim$ along with Lemma 5.2 to get a symmetric path $\tilde{K}_1$ and a commutative diagram of analytic maps

$$(5.8) \quad \begin{array}{ccccc} \tilde{L} & \xleftarrow{\chi} & \tilde{K}_1 & \xrightarrow{\chi'} & M \\ & \searrow \lambda & \downarrow \kappa_1 & \swarrow \mu & \\ & & C & & \end{array}$$

with χ and χ' both onto. Then $\tilde{K}_1$, along with the $\tilde{K}_i$ already given for $i \geq 2$, satisfies the required conditions. Namely ψ_1 can be taken to be $\theta_1\chi$, ϕ_{21} to be $\theta_2\chi$, ϕ_{i1} to be $\phi_{i2}\theta_2\chi$ for $i > 2$, ϕ'_{21} to be $\theta'_2\chi'$ and ϕ_{i1} to be $\phi'_{i2}\theta'_2\chi'$ for $i > 2$.

This theorem will be applied in particular to the case where the $\tilde{C}_i$ are given as paths of effective continuations, and in this context it is convenient to reformulate it as follows in terms of the corresponding transformations.

Going back to the statement of the theorem, let $\tilde{C}_{i1}, \tilde{C}_{i2}, \dots, \tilde{C}_{i, n(i)}$ be those $\tilde{C}_h$ equivalent to $\tilde{K}_i$ (that is to say, admitting an onto map ψ_i as in (2)). If $(\mathcal{P}_{i1}, U)$ is an initial element having an effective continuation along $\tilde{C}_{i1}$, it can be continued along $\tilde{K}_i$ and the corresponding transformation T_i is non-trivial on it. On the other hand $\tilde{K}_i$ has analytic maps, compatible with the projections into C , into each $\tilde{C}_{hi}$ and $\tilde{C}'_{hi}$ for $h > i$, which means (cf. Lemma 3.5) that $\mathcal{P}_{hi}$ and $T_h\mathcal{P}_{hi}$ have continuations along $\tilde{K}_i$, and are both left invariant by T_i .

Summing all this up gives the following version of Theorem 1.

Theorem 2. *Given a collection of initial elements of closed effective continuations of finite type over C , they may be indexed as $(\mathcal{P}_{ij}, U)$ with $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n(i)$ for each such i , such that, for fixed i , all the $\mathcal{P}_{ij}$ have continuations along a symmetric path $\tilde{K}_i$ with corresponding transformation T_i operating non-trivially on these $\mathcal{P}_{ij}$, while for $h > i$, $\mathcal{P}_{hi}$ and $T_h\mathcal{P}_{hi}$ have continuations along $\tilde{K}_i$ and $T_i\mathcal{P}_{hi} = \mathcal{P}_{hi}$, $T_iT_h\mathcal{P}_{hi} = T_h\mathcal{P}_{hi}$.*

There is a complement to this theorem involving the case of non-closed paths. Let $\tilde{C}_1, \tilde{C}_2$ be two paths for continuations of finite type over C with projections π_1 and π_2 , respectively. Assume that $\tilde{C}_1$ is closed while π_2 maps $\tilde{C}_2$ onto C . Then, arguing as for the proof of Lemma 4.1 there is a path $\tilde{C}_0$ with projection π_0 into C making the diagram

$$(5.9) \quad \begin{array}{ccccc} & & \tilde{C}_0 & & \\ \phi_1 \swarrow & & & \searrow \phi_2 & \\ \tilde{C}_1 & & \tilde{C}_0 & & \tilde{C}_2 \\ & \searrow \pi_1 & \downarrow \pi_0 & \swarrow \pi_2 & \\ & & C & & \end{array}$$

commutative, with analytic maps ϕ_1 onto $\tilde{C}_1$ and ϕ_2 into $\tilde{C}_2$.

In terms of transformations this implies the following lemma.

Lemma 5.4. *Let $\tilde{C}_1$ and $\tilde{C}_2$ be as described above, let T_1 be the transformation corresponding to $\tilde{C}_1$ and let $(\mathcal{O}, U)$ be an initial element with continuations along both $\tilde{C}_1$ and $\tilde{C}_2$. Then $T_1\mathcal{O} = \mathcal{O}$.*

6. Inclusion relations and ideal decomposition. Let $(\mathcal{O}, \tilde{C})$ and $(\mathcal{Q}, \tilde{C})$ be two continuations over $\tilde{C}$ with stalks $\mathfrak{p}_p$ and $\mathfrak{q}_p$, respectively, over p . Suppose that $\mathfrak{p}_p \supset \mathfrak{q}_p$ for all p in some open set on $\tilde{C}$. Actually since $\mathcal{O}$ and $\mathcal{Q}$ are coherent subsheaves of $K[[X]]$ over $\tilde{C}$ it is enough to assume this inclusion relation at one point, from which it then extends at once to a neighborhood. The hypothesis then means that the set on which the sheaf $(\mathcal{O} + \mathcal{Q})/\mathcal{O}$ is zero is open and non-empty. However this a coherent sheaf of $K[[X]]$ -modules and so the set on which it is zero is closed, from which it follows that it is zero over all of $\tilde{C}$. That is to say $\mathcal{O} \supset \mathcal{Q}$.

A special case of this will be applied below, namely where $\mathcal{Q}$ is a sheaf of principal ideals. In fact suppose a power series f in the indeterminates X is given with coefficients meromorphic on $\tilde{C}$, with only a finite number of poles, such that f induces an element f_p of $k_p[[X]]$ at all points of $\tilde{C}$ and an element of $\mathcal{O}_p[[X]]$ at all but a finite number of points. f_p generates the principal ideal (f_p) in $k_p[[X]]$ and the (f_p) form a coherent sheaf which can be taken as $\mathcal{Q}$ above. Thus if $f_p \notin \mathfrak{p}_p$ for all p in some open set, then $f_p \notin \mathfrak{p}_p$ for all p in $\tilde{C}$; as noted above it is sufficient here to assume that $f_p \notin \mathfrak{p}_p$ for one point p .

The general inclusion result obtained above will be used later in the case where $(\mathcal{O}, \tilde{C})$ and $(\mathcal{Q}, \tilde{C})$ are continuations of initial elements $(\mathcal{O}_1, U)$ and $(\mathcal{Q}_1, U)$ and it is given that $\mathcal{O}_1 \supset \mathcal{Q}_1$. The above then implies that $\mathcal{O} \supset \mathcal{Q}$.

Another topic related to the above is the decomposition of stalks of a continuation $(\mathcal{O}, \tilde{C})$ into primary components. Again it will be shown that this is an operation which extends automatically from a single stalk to the whole of $\mathcal{O}$.

Fix attention first on a stalk $\mathfrak{p}_p$ and assume that it is prime. Thus, taking a parameter t on $\tilde{C}$, with value 0 at p , $\mathfrak{p}_p$ is a prime ideal in the ring $R((t))[[X]]$, with convergence conditions for the elements as described in §1. The argument of Lemma 6.5 of [2] shows that any decomposition of $\mathfrak{p}_p$ must be in the ring of power series in X over $C((t^r))$, where C denotes the complex numbers and r is a rational number. Assume that there is no such decomposition for $\mathfrak{p}_p$.

Consider now the possible decomposition of $\mathfrak{p}_q$ for some q near p . Here $\mathfrak{p}_q$ will be generated by a set of generators of $\mathfrak{p}_p$, with coefficients written as power series around q instead of around p . It can be assumed that the regularization process of Lemma 6.5 of [2] has already been carried out on $\mathfrak{p}_p$, so that it appears

as generated by a prime ideal $\mathfrak{p}_p$ in $R((t))[[X_1, X_2, \dots, X_k]][X_{k+1}, \dots, X_n]$, this ideal being of dimension zero over $R((t, X_1, X_2, \dots, X_k))$. Thus $\mathfrak{p}_q$ will be generated by an ideal $\mathfrak{p}_q$ in $R((t'))[[X_1, X_2, \dots, X_k]][X_{k+1}, \dots, X_n]$, where $R((t'))$ denotes functions meromorphic on $\tilde{C}$ around q . Any prime component $\mathfrak{q}_q$ of $\mathfrak{p}_q$ will be generated by a prime component $\mathfrak{q}_q$ of $\mathfrak{p}_p$, and the coefficients of the polynomials in $X_{k+1}, X_{k+2}, \dots, X_n$ generating $\mathfrak{q}_q$ will be algebraic over the coefficients of polynomials generating $\mathfrak{p}_p$ (this is the argument of Lemma 6.5 of [2] again). They can thus be continued analytically back to p . That is to say $\mathfrak{q}_q$ is generated by an ideal $\mathfrak{q}_p$ in $C((t^r))[[X_1, X_2, \dots, X_n]]$ for some rational r . Here $\mathfrak{q}_p$ certainly contains $\mathfrak{p}_p$, but these ideals have the same dimension, and so, since $\mathfrak{p}_p$ has no decomposition over $C((t^r))$ it follows that $\mathfrak{q}_p = \mathfrak{p}_p$. This means that $\mathfrak{p}_q$ is equal to the prime ideal $\mathfrak{q}_q$, and so the following has been proved.

Lemma 6.1. *If $\mathfrak{p}_p$ is prime and remains so under any extension of the ground field to $C((t^r))$ for rational r , then $\mathfrak{p}_q$ is prime for q near p .*

A similar result also holds for primary ideals.

Lemma 6.2. *If $\mathfrak{p}_p$ is a primary ideal with prime $\mathfrak{q}_p$, the latter remaining prime under any ground field extension to $C((t^r))$ for any rational r , then $\mathfrak{p}_q$ is also primary for q near p .*

Proof. There is an integer ρ such that $\mathfrak{q}_p^\rho \subset \mathfrak{p}_p \subset \mathfrak{q}_p$. Also the last lemma implies that $\mathfrak{q}_p$ induces a prime ideal $\mathfrak{q}_q$ over q for q near p . The above inclusion thus induces an inclusion $\mathfrak{q}_q^\rho \subset \mathfrak{p}_q \subset \mathfrak{q}_q$ with $\mathfrak{q}_q$ prime, and this means that $\mathfrak{p}_q$ is primary, with the prime $\mathfrak{q}_q$.

The above will now be extended to primary decompositions.

Lemma 6.3. *Let the stalk $\mathfrak{p}_p$ of $\mathcal{O}$ have the primary decomposition*

$$\mathfrak{p}_p = \mathfrak{q}_p^{(1)} \cap \mathfrak{q}_p^{(2)} \cap \dots \cap \mathfrak{q}_p^{(m)},$$

where it is assumed that the primes of the $\mathfrak{q}_p^{(i)}$ remain prime under any extension of ground field to $C((t^r))$ for rational r . Then for q near p the given decomposition induces a primary decomposition

$$\mathfrak{p}_q = \mathfrak{q}_q^{(1)} \cap \mathfrak{q}_q^{(2)} \cap \dots \cap \mathfrak{q}_q^{(m)}.$$

Proof. In any case it is clear that the given intersection relation implies the required one, the $\mathfrak{q}_q^{(i)}$ being induced by the corresponding $\mathfrak{q}_p^{(i)}$. And the last lemma shows that the $\mathfrak{q}_q^{(i)}$ are in fact primary.

It is actually the following slightly weaker form of this result that will be applied later.

Lemma 6.4. *Assume that the minimal primes of the stalk $\mathfrak{p}_p$ of $\mathcal{O}$ remain prime under any extension of the ground field $R((t))$ to $C((t^r))$ for rational r . Then they induce the minimal primes of $\mathfrak{p}_q$ for q near p .*

Proof. Let $\mathfrak{p}_p = \mathfrak{q}_p^{(1)} \cap \mathfrak{q}_p^{(2)} \cap \cdots \cap \mathfrak{q}_p^{(m)}$ be a primary decomposition where the first h ideals correspond to minimal primes. Then $\mathfrak{p}_q = \mathfrak{q}_q^{(1)} \cap \mathfrak{q}_q^{(2)} \cap \cdots \cap \mathfrak{q}_q^{(m)}$, for q near p , where $\mathfrak{q}_q^{(i)}$ is induced by $\mathfrak{q}_p^{(i)}$. Here the hypothesis, along with Lemma 6.2, implies that $\mathfrak{q}_q^{(1)}, \mathfrak{q}_q^{(2)}, \dots, \mathfrak{q}_q^{(h)}$ are primary. The other $\mathfrak{q}_q^{(i)}$ may not be primary, but in any case, since inclusion relations at p induce corresponding relations at q , all their prime components must be embedded. Conversely, using the remarks at the beginning of this section, any inclusion relation between the primes of $\mathfrak{q}_q^{(1)}, \dots, \mathfrak{q}_q^{(h)}$ would imply an inclusion relation between the minimal primes of $\mathfrak{p}_p$, which is impossible. It follows that the primes of $\mathfrak{q}_q^{(1)}, \mathfrak{q}_q^{(2)}, \dots, \mathfrak{q}_q^{(h)}$ are the minimal primes of $\mathfrak{p}_q$, as required.

The following is an immediate consequence of Lemma 6.4.

Lemma 6.5. *If $(\mathcal{O}, \tilde{C})$ and the $(\mathcal{O}_i, \tilde{C})$ are such that the stalks of the $\mathcal{O}_i$ are the minimal primes of the stalk of $\mathcal{O}$ at one point of $\tilde{C}$, then this is so for all points of $\tilde{C}$.*

The argument preceding Lemma 6.1 yields some further information. Take a continuation $(\mathcal{O}, \tilde{C})$ with stalks not necessarily prime, and decompose these stalks in a neighborhood of p on $\tilde{C}$. First a primary decomposition of $\mathfrak{p}_p$ can be carried out over $C((t))$, taking parameter t on $\tilde{C}$ with $t = 0$ at p . This induces a decomposition of the $\mathfrak{p}_q$ for q near p (but not necessarily into primaries) and the component ideals so obtained form coherent sheaves over a neighborhood of p . Reasoning as for Lemma 6.1, any further decomposition of the stalks of $\mathcal{O}$ around p will be into ideals whose generators are power series in X with coefficients analytic over a neighborhood of p , with the exception of some branch points. And by taking a sufficiently small neighborhood of p it can be assumed that p is the only such branch point. This means that over an open interval U in a neighborhood of p (U not necessarily containing p) the primary components of stalks of $\mathcal{O}$ form coherent sheaves. A similar statement can be made about minimal primes.

7. An operation on sheaves. The above results are to be applied to continuations whose initial elements are obtained from analytic varieties more or less as in the example in §1. First the algebraic operations will be examined.

Let C be an analytic arc in real n -space. C is the analytic image of a line segment C_0 , and, while one should strictly speaking talk of continuations over C_0 , it will be convenient to transfer attention to the arc C by speaking of continuations over C . Let $\mathcal{O}$ be the sheaf of local rings in the ambient n -space, restricted to C as base. Thus the stalk $\mathcal{O}_p$ of $\mathcal{O}$ over p is the ring of real analytic functions in n variables around p . On the other hand one can also construct the sheaf $K[[X]]$ over C , where $X_1, X_2, \dots, X_n$ are n indeterminates (cf. §1). Here the stalk of $K[[X]]$ at p is the ring of power series in the X with coefficients meromorphic in t , the parameter on C , at p , with convergence conditions as

described in §1. This is precisely the situation of §1 with Z replaced by C , the ground field here being the real numbers.

A map Φ of $\mathcal{O}$ into $K[[X]]$ is to be defined now, as in §1. Let f be in the stalk $\mathcal{O}_p$ of $\mathcal{O}$. Thus f is a real power series in $x - x(p)$, where $x(p)$ denotes the coordinates of p and vector notation is being used. f is convergent in some neighborhood of p . Write

$$f(x) = f(x - x(t) + x(t)) = \check{f}(t, x - x(t)),$$

where $x(t)$ is the point of parameter t on C . Then $\check{f}(t, X)$ is in the stalk $k_p[[X]]$ of $K[[X]]$. The map Φ is defined by setting $\check{f} = \Phi(f)$.

It was shown in Lemma 1.1 that Φ maps local sections of $\mathcal{O}$ into local sections of $K[[X]]$.

Now let $\mathfrak{A}$ be any coherent sheaf of $\mathcal{O}$ -ideals over C with stalk $\mathfrak{A}_p$ over p . Let $\check{\mathfrak{A}}_p = \mathfrak{p}_p$ be the ideal in $k_p[[X]]$ generated by all $\check{f}$ where f is in $\mathfrak{A}_p$. Let $\check{\mathfrak{A}} = \mathcal{O}$ be the union of all the $\check{\mathfrak{A}}_p$ for p on C . The case described in §1 was the special one in which $\mathfrak{A}$ was the sheaf of ideals of Z (equal to C here). The argument of Lemmas 1.2, 1.3 shows that $\mathcal{O}$ is a coherent sheaf of $K[[X]]$ -ideals over C .

It is possible to define a partial inverse to the above operation. Namely let $\mathcal{O}$ be a coherent sheaf of $K[[X]]$ -ideals over C and let p be a point at which the stalk $\mathfrak{p}_p$ of $\mathcal{O}$ is generated by an ideal in the ring of power series in $X_1, X_2, \dots, X_n$ with coefficients analytic in t at p . Let $f(t, X)$ be an element of $\mathfrak{p}_p$ analytic in t at p and write $\check{f}(x) = f(t(p), x - x(p))$, where $t(p)$ is the value of t at p and $x(p)$ are the coordinates of p in the ambient n -space. Write $\check{\mathfrak{A}}_p = \check{\mathfrak{p}}_p$ for the ideal in the ring of real analytic functions at p generated by all $\check{f}$ with f in $\mathfrak{p}_p$, analytic in t at p . Let $\check{\mathcal{O}} = \check{\mathfrak{A}}$ be the union of all the $\check{\mathfrak{p}}_p$. It is not necessarily true that $\check{\mathfrak{A}}$ is a sheaf. But if $\mathfrak{A}$ is given as a sheaf of $\mathcal{O}$ -ideals over C , it is easy to see that $(\check{\mathfrak{A}})^\sim = \mathfrak{A}$.

The following is another way of looking at the above operations. The sheaf $\mathcal{O}$ and its subsheaf $\mathfrak{A}$ are given over C in n -space R^n . Embedding R^n in R^{n+1} , the projection of the latter on the former induces inverse images of $\mathcal{O}$ and $\mathfrak{A}$ over $C \times R$, and by restriction these induce sheaves $\mathcal{O}'$ and $\mathfrak{A}'$ over the subset C' of $C \times R$ defined by $x_{n+1} = t$, where t is the parameter on C . Now define new coordinates in R^{n+1} in a neighborhood of C' by the equations

$$(7.1) \quad \begin{aligned} X_i &= x_i - x_i(x_{n+1}) & (i = 1, 2, \dots, n) \\ X_{n+1} &= x_{n+1}. \end{aligned}$$

Here the $x_i(x_{n+1})$ are the coordinates in R^n of the point on C with parameter value x_{n+1} . This transformation is analytic and invertible in a neighborhood of C' . Then the stalk of $\mathcal{O}'$ over p on C' becomes the ring of convergent power series, with real coefficients, in the variables $X_1, X_2, \dots, X_n, X_{n+1} - X_{n+1}(p)$, while the stalk of $\mathfrak{A}'$ becomes the ideal in this ring generated by the image of $\mathfrak{A}$ under the coordinate change (7.1). $\hat{\mathcal{O}}$ and $\hat{\mathfrak{A}}$ are then obtained by extending the ring $\mathcal{O}'_p$ in each case to the ring of power series in $X_1, X_2, \dots, X_n$,

$X_{n+1} - X_{n+1}(p)$, now allowing negative powers of the last variable. This arrangement makes it clear that the coherent sheaf $\mathfrak{A}$ leads to a coherent sheaf $\mathfrak{A}'$. Conversely the operation going from $\mathfrak{O}$ to $\mathfrak{O}'$ can be described by making the transformation (7.1) in reverse and then fixing the value of the parameter on C . However, as pointed out, this operation is not defined at all points of C in any case $\mathfrak{O}'$ need not be a sheaf.

It will now be checked that the operation just described, carrying $\mathfrak{A}$ into $\mathfrak{A}'$, transforms sheaves of prime ideals into sheaves of primes. Suppose first that the given sheaf $\mathfrak{A}$ over C consists of prime ideals. That is to say, the stalk $\mathfrak{A}_p$, for each p on C , is a prime ideal in the ring of real analytic functions at p in R . Using the second description above for the construction of $\mathfrak{A}'$, it is immediately clear that the stalks of $\mathfrak{A}'$ over C' are all prime, and remain so after the change of variables (7.1). Here a stalk $\mathfrak{A}'_q$ is a prime ideal in the ring of convergent power series with real coefficients in $X_1, X_2, \dots, X_n$ and $t = X_{n+1} - X_{n+1}(p)$, where t is taken as parameter on C (or C') around p . It could happen that $\mathfrak{A}'_q$ would cease to be prime when the ring is extended to allow negative powers of t . Suppose that f and g are power series in X with coefficients meromorphic in t at p such that fg is in $\mathfrak{A}'_p$. If $f_1, f_2, \dots, f_r$ are generators of $\mathfrak{A}_p$, also generating all $\mathfrak{A}_q$ for q near p (a choice which can be made since $\mathfrak{A}$ is coherent) then fg can be written as $\sum a_i f_i$, with the a_i power series in the X_i with coefficients meromorphic in t at p . And here it can be assumed that a neighborhood of p is taken on C in which at most p itself is a pole for the coefficients of the a_i . Thus $fg = \sum a_i f_i$ in a neighborhood of some q near p with all the a_i analytic at q . This however says that fg is in $\mathfrak{A}'_q$, which is prime, and so one of the factors, say f , is in $\mathfrak{A}'_q$. This means that f is a section of $\mathfrak{A}$ over a neighborhood of q , and so (cf. beginning of §6) it is a section of $\mathfrak{A}$ over the whole selected neighborhood of p . In particular it induces an element of $\mathfrak{A}_p$, and so $\mathfrak{A}_p$ is prime, as was to be shown.

Conversely, suppose that, starting with $\mathfrak{A}$, it turns out that all the stalks of $\mathfrak{A}$ are prime. Let f, g be analytic functions in x around p on C and suppose that fg is in $\mathfrak{A}_p$. Then fg is in $\mathfrak{A}'_p$ and so one factor, say f , is in $\mathfrak{A}'_p$. It then follows that f is in $\mathfrak{A}_p$, and so $\mathfrak{A}_p$ is prime.

Summing up, the following has thus been proved:

Lemma 7.1. *Let $\mathfrak{A}$ be a coherent sheaf of $\mathfrak{O}$ -ideals over C and let $\mathfrak{A}'$ be constructed as described above. Then the stalks of $\mathfrak{A}'$ are prime if and only if those of $\mathfrak{A}$ are.*

The correspondence between the minimal primes of stalks of $\mathfrak{A}$ and those of $\mathfrak{A}'$ (when these stalks are not themselves prime) will now be examined.

To begin with, assume that the minimal primes of stalks of $\mathfrak{A}$ form coherent sheaves of $\mathfrak{O}$ -ideals over C and that the minimal primes of stalks of $\mathfrak{A}'$ form coherent sheaves of $K[[X]]$ -ideals over C . It is easy to see, by going from $\mathfrak{A}$ to $\mathfrak{A}'$ via the intermediate sheaf of rings $\mathfrak{O}'$, that, if $\mathfrak{A}^{(1)}, \mathfrak{A}^{(2)}, \dots, \mathfrak{A}^{(k)}$ are the sheaves of minimal primes of stalks of $\mathfrak{A}$, then the $\mathfrak{A}'^{(i)}$ are the sheaves

of minimal primes of the stalks of $\mathfrak{A}$. This can now be combined with the remark at the end of §6 to give the following result.

Lemma 7.2. *Let $\mathcal{O} = \mathfrak{A}$ over C as above and assume that the minimal primes of the stalks of $\mathfrak{A}$ form coherent sheaves of $\mathcal{O}$ -ideals $\mathfrak{A}^{(1)}, \mathfrak{A}^{(2)}, \dots, \mathfrak{A}^{(k)}$ over C . Then for any p in C there is an open interval U in a neighborhood of p such that the minimal primes of the stalks of $\mathfrak{A}$ form the sheaves $\mathfrak{A}^{(i)}$ over U .*

Proof. It is only necessary to take U (cf. the end of §6) such that the minimal primes of stalks of $\mathcal{O}$ over U form coherent $K[[X]]$ -sheaves and then to use the remark preceding this lemma, restricted to U .

The next step will be to see how one can recognize a sheaf $\mathcal{O}$ of $K[[X]]$ -ideals over C as being of the form $\mathfrak{A}$, where $\mathfrak{A}$ is a sheaf of $\mathcal{O}$ -ideals. If $\mathcal{O} = \mathfrak{A}$ and if $f_1, f_2, \dots, f_r$ are generating sections of $\mathfrak{A}$ over a neighborhood of p , then $\mathcal{O}$ has local generating sections $f_i(t, X)$ over a neighborhood of p , where

$$f_i(t, x - x(t)) = f_i(x).$$

Clearly the function $f_i(t, x - x(t))$ is actually independent of t , and so its derivative with respect to t is zero. If now f is any local section of $\mathcal{O}$ in a neighborhood of p , then $f(t, X) = \sum a_i f_i$, where the a are power series in X with coefficients meromorphic in t . Then it is clear that

$$\frac{d}{dt} f(t, x - x(t)) = \sum \frac{d}{dt} a_i(t, x - x(t)) f_i(t, x - x(t)).$$

That is to say it is of the form $g(t, x - x(t))$, where g is also a local section of $\mathcal{O}$ over a neighborhood of p . It will turn out that this is more or less a necessary and sufficient condition for $\mathcal{O}$ to be of the form $\mathfrak{A}$. The reservation is that a finite number of points on C may have to be omitted. Before proving this result, it is convenient at this stage to put things into a more general setting, where, while the $\mathcal{O}$ -sheaves are to be over C , the $K[[X]]$ -sheaves are to be over $\tilde{C}$, which is projected into C by an analytic map π . The operations described earlier in this section will now be applied over certain open subsets of C and $\tilde{C}$.

So let C be an analytic arc in n -space as above and let $\tilde{C}$ be a line segment mapped into C by an analytic map π , with only a finite number of branch points. $K[[X]]$ will mean the sheaf of power series in X with coefficients meromorphic on $\tilde{C}$, while $\mathcal{O}$ will be the sheaf of local rings in the n -space containing C , with base restricted to C . If U is an open set on $\tilde{C}$ which is mapped homeomorphically into C by π , so that it can be identified with $\pi(U)$ on C , and if $\mathfrak{A}$ is a sheaf of $\mathcal{O}$ -ideals over $\pi(U)$, then $\mathfrak{A}$ is to be constructed as a sheaf of $K[[X]]$ -ideals over U . Conversely if $(\mathcal{O}, \tilde{C})$ is a continuation over $\tilde{C}$, its restriction to U is a sheaf over U and the set $\mathcal{O}$ is then constructed as a set of ideals over $\pi(U)$.

Now suppose that $(\mathcal{O}, \tilde{C})$ is a continuation over C such that its restriction to an open set U is of the form $\mathfrak{A}$ where $\mathfrak{A}$ is a sheaf of $\mathcal{O}$ -ideals over $\pi(U)$. Then, as remarked above, $\mathcal{O}$ has the following property.

For each p in U , if f is a local section of $\mathcal{O}$ over a neighborhood of p ,
 (D) $df(t, x - x(\pi(t)))/dt$ is of the form $g(t, x - x(\pi(t)))$ where g is also a local section of $\mathcal{O}$ over a neighborhood of p .

The idea now is to show first that this property transmits itself from U to the whole of $\tilde{C}$, and second to deduce from this that $\mathcal{O}$ is of the form $\hat{\mathcal{A}}$ over any open set not containing branch points of π or a finite number of additional exceptional points. This will be done in the following two lemmas.

Lemma 7.3. *If the property (D) holds at all points of an open set U , then it holds for all p in $\tilde{C}$.*

Proof. Let U be the maximal open set in $\tilde{C}$ over which the property (D) holds and let p_0 be a point in $\tilde{U}$ but not in U . Such a point can always be found unless U is already all of $\tilde{C}$. Thus for any open interval N around p_0 on $\tilde{C}$ the points of $N \cap U$ have the property (D). Choose N so that $\mathcal{O}$ has a finite set of generating sections over N ; that is to say sections generating each $\mathfrak{p}_p$ for p in N . Let f be such a generating section of $\mathcal{O}$ over N . Then for p in $N \cap U$, f is a section over a neighborhood of p and so, by hypothesis, $df(t, x - x(t))/dt$ is a section of $\mathcal{O}$ over a neighborhood of p . That is to say $df(t, x - x(t))/dt$ is a section of $\mathcal{O}$ over all of $N \cap U$. But then (cf. beginning of §6) it is a section of $\mathcal{O}$ all over N . If now $\Phi(t, X)$ is any section of $\mathcal{O}$ over a neighborhood of some p in N , then $\Phi = \sum a_i f_i$, where the f_i are generating sections of $\mathcal{O}$ over N and the a_i are in $k_p[[X]]$. It follows at once from what has been proved for the generating sections that $d\Phi(t, x - x(t))/dt$ is a section of $\mathcal{O}$ over a neighborhood of p . Thus the property (D) is extended to $U \cup N$, contradicting the maximality of U , unless U is already all of $\tilde{C}$. This completes the proof of the lemma.

Lemma 7.4. *Suppose that $(\mathcal{O}, C)$ is a continuation with the property (D) along $\tilde{C}$. Then, for each p in $\tilde{C}$ with a finite number of exceptions, there is a neighborhood U of p mapped homeomorphically into C by π and a sheaf $\hat{\mathcal{A}}$ of $\mathcal{O}$ -ideals over $\pi(U)$ such that the restriction of $\mathcal{O}$ to U is $\hat{\mathcal{A}}$.*

Proof. For any p in $\tilde{C}$ let $f_1, f_2, \dots, f_r$ be generating sections of $\mathcal{O}$ over a neighborhood of p . The property (D) implies that, if t is the parameter on C around p

$$(7.2) \quad \frac{d}{dt} f_i(t, x - x(\pi(t))) = \sum a_{ij} f_j(t, x - x(\pi(t))),$$

$i = 1, 2, \dots, r$, where the a_{ij} are power series in $x - x(\pi(t))$ with coefficients meromorphic in t . The neighborhood of p can be chosen so that the f_i and a_{ij} are analytic in t at all points except possibly p . $\tilde{C}$ can be covered by a finite number of such neighborhoods. Thus for all except a finite number of points, each p has a neighborhood U over which $\mathcal{O}$ has generating sections satisfying differential equations of the form (7.2) with the a_{ij} analytic in t . In addition

assume that the branch points of π are taken among the exceptional points, so that π maps U homeomorphically into C .

To solve the equations (7.2), consider first the auxiliary matrix differential equations $dZ/dt = AZ$, where A is the matrix with entries $a_{ij}(t, x - x(\pi(t)))$, and the equations are to be solved for the square matrix Z . There is a unique solution Z whose initial value, for $t = 0$ at p , is the unit matrix, and this solution is analytic in t for t in U , and analytic in x for a sufficiently small value of $x - x(\pi(t))$. Clearly Z is non-singular in a neighborhood of p and has an inverse also analytic in t and x for t and the $x - x(\pi(t))$ sufficiently small. Then if $Z = Z(t, x - x(\pi(t)))$, write, in matrix notation, $f(t, X) = Z(t, X)g(t, X)$. This defines $g = (g_1, g_2, \dots, g_r)$ as a set of generating sections of $\mathcal{O}$ over a neighborhood of p . Also

$$\frac{d}{dt} f(t, x - x(\pi(t))) = \frac{d}{dt} (Zg) = AZg + Z \frac{d}{dt} g = Af + Z \frac{dg}{dt}.$$

Hence $dg_i(t, x - x(\pi(t)))/dt = 0$ for each i . That is to say $\mathcal{O}$ has generating sections g_i over a neighborhood of p such that the $g_i(t, x - x(\pi(t)))$ are independent of t . This means that the $\mathfrak{p}_q$ for q near p form the stalks of a coherent sheaf $\mathfrak{A}$ such that $\mathcal{O} = \mathfrak{A}$.

In fact, by piecing together neighborhoods for which this result holds, it is clear that a sheaf $\mathfrak{A}$ such that $\mathcal{O} = \mathfrak{A}$ is obtained over the image $\pi(U)$ of each component U of C after the removal of the exceptional points appearing in the proof.

8. Geometric interpretation. The results of the last section will now be put in a geometric setting. As in §7, let C be a real analytic arc in real n -space. Let W be a real analytic variety given in a neighborhood of p on C and containing an arc U of C . Let $\mathfrak{A}$ be the sheaf of $\mathcal{O}$ -ideals of W restricted to U as base and suppose that $\mathfrak{A}$ is coherent. Note that this condition is weaker than saying that W is coherent, since only points along U are mentioned. It is natural to say in this case that W is *coherent along* U .

Let the sheaf $\mathcal{O}_0 = \mathfrak{A}$ of $K[[X]]$ -ideals over U be constructed as in §7, and suppose that $(\mathcal{O}_0, U)$ can be taken as initial element of a continuation $(\mathcal{O}, \tilde{C})$ of finite type along some path $\tilde{C}$ with projection π into C . Now according to Lemma 7.4, if q is any point of $\tilde{C}$, with a finite number of exceptions, q has a neighborhood U_1 projecting homeomorphically into C under π , such that the restriction $\mathcal{O}_1$ of $\mathcal{O}$ to U_1 is of the form $\mathfrak{A}_1$ where $\mathfrak{A}_1$ is a coherent sheaf of $\mathcal{O}$ -ideals over $\pi(U_1)$. The set of zeros of $\mathfrak{A}_1$ in the complexification of n -space in a neighborhood of $\pi(q)$ is a complex analytic variety W_1 at $\pi(q)$.

Definition 8.1. W_1 will be called *the continuation of W along $\tilde{C}$ to q* . If W_1 is a real analytic variety (that is, the complexification of its set of real points) at each point q of $\tilde{C}$ where it is defined, W will be said to have *a real continuation along $\tilde{C}$* .

Continuing with the same notation, suppose in addition that W is *irreducible along* U , meaning by this that the local variety induced by it at each point of U is irreducible. Then the stalks of $\mathfrak{A}$ are all prime and so, by Lemma 7.1, the stalks of $\mathcal{O}_0 = \mathfrak{A}$ are prime. But then Lemma 6.3 implies that all the stalks of $\mathcal{O}$ are prime. And so, applying the same lemma in reverse, the stalks of $\mathfrak{A}_1$ over a neighborhood of q are prime. Then if W_1 is real, the stalks of $\mathfrak{A}_1$ will be the ideals of W_1 at points along C near q . In other words W_1 is coherent along C in a neighborhood of q , and in fact it satisfies the same condition around q as the original W satisfies around p .

The inclusion result of §6 can be put into geometric terms as follows.

Lemma 8.1. *Let W and W' be two real analytic varieties coherent along a neighborhood U of p on C and suppose they have continuations W_1 and W'_1 along $\tilde{C}$ to a neighborhood of q on $\tilde{C}$. Then if $W \subset W'$ it will follow that $W_1 \subset W'_1$.*

This is true without any further restriction on W and W' . The following is a variant in which the continuation of a local variety W' is replaced by a real analytic space V containing C . Here however no coherence condition is imposed on V and to compensate the condition of reality will have to be imposed on the continuation of W .

Lemma 8.2. *Let V be a real analytic space containing the analytic arc C and let W be an irreducible real analytic variety containing the initial interval U of C and contained in V . Suppose that the analytic continuation of W along $\tilde{C}$ is real wherever it is defined. Then it is contained in V wherever it is defined.*

Proof. Let $(\mathcal{O}, \tilde{C})$ be the continuation whose initial element is $\mathfrak{A}$, where $\mathfrak{A}$ is the sheaf of ideals of W over the initial interval of C . Let N be the maximal open interval on $\tilde{C}$ containing the initial interval and such that, if an analytic continuation W_1 of W is defined at q in N , then $W_1 \subset V$. Certainly N is not empty, and it is now to be shown to be the whole of $\tilde{C}$. If not, let p_0 be the upper bound of N on $\tilde{C}$. Let U_0 be a neighborhood of $\pi(p_0)$ in the ambient n -space such that there are local generators $f_1, f_2, \dots, f_r$ in U_0 for the ideal of V at $\pi(p_0)$, and let N_0 be a neighborhood of p_0 on $\tilde{C}$ such that $\pi(N_0) \subset U_0$. Now at any q of $N \cap N_0$ where the analytic continuation of W along $\tilde{C}$ is defined (and at most a finite set of q are excluded from this) $W_1 \subset V$, by hypothesis, and so each f_i is zero on W_1 . That is to say, each f_i is in the ideal of W_1 at $\pi(q)$. However, in the case of an irreducible W with real continuation along $\tilde{C}$, the sheaf of ideals of W_1 over a neighborhood of $\pi(q)$ is a coherent sheaf of primes $\mathfrak{A}_1$ such that $\mathfrak{A}_1$ is the restriction of $\mathcal{O}$ to a neighborhood of q , (cf. the remarks preceding this lemma). And so if $f_i(x) = \hat{f}_i(t, x - x(\pi(t)))$ this means that each $\hat{f}_i(t, X)$ is a section of $\mathcal{O}$ over a neighborhood of q and so (beginning of §6) over all of N_0 . But then, arguing in the opposite direction, this implies that the analytic continuation of W to points of N_0 not in N is contained in V , contrary to the maximality of N . N is therefore the whole of $\tilde{C}$ as required.

It will be convenient to express some more of the terminology of §3 in geometric

terms. Namely suppose that in Definition 8.1 the continuation $(\mathcal{O}, \tilde{C})$ is closed with associated transformation T . Here the initial and final intervals both project on the same interval of C . Assume that an analytic continuation W_1 is defined at points in the final interval of $\tilde{C}$. Then W_1 will be called the *transform of W associated with the closed path $\tilde{C}$* , and the relation will be written as $W_1 = TW$.

9. Analytic continuation of singularities. Let C be an analytic arc contained in a real analytic space V . The object of this section is to discuss continuations of singularities of V over C . First the singular bunch of V at a point p will be defined.

Definition 9.1. Let V^* be the complexification of V in a neighborhood of p in the complexification of the ambient n -space. $(\mathfrak{S}V)_p$, the *singular bunch of V at p* , is the smallest bunch of complex analytic varieties at p containing the components of V at p and closed under the operations of intersection and the operation S of computation of singularities by the Jacobian criterion.

More explicitly, if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are the ideals of components W_1, W_2 of $(\mathfrak{S}V)_p$, then the intersection $W_1 \cap W_2$ of these is the bunch of varieties defined by the minimal primes of $(\mathfrak{p}_1, \mathfrak{p}_2)$. And if $\mathfrak{p}$ is the ideal of a component W of $(\mathfrak{S}V)_p$, with generators $f_1, f_2, \dots, f_r$, say, and if $\delta\mathfrak{p}$ denotes the ideal generated by the minors of appropriate order of the Jacobian of the f_i , then SW is the bunch of varieties defined by the minimal primes of $(\mathfrak{p}, \delta\mathfrak{p})$.

All this is done in the ring of power series at p with complex coefficients. However it is clear that the ideals of the components of V^* itself all have real generators. And so all the ideals appearing in the computation of $(\mathfrak{S}V)_p$ will either have real generators, or will be the decomposition into complex conjugate pairs of ideals with real generators. It is convenient in the present context to recombine all such complex conjugate pairs of components to get varieties which are irreducible in the real sense but reducible in the complex sense and whose ideals have real generators. When this adjustment is made, the components of $(\mathfrak{S}V)_p$, either singly or in complex conjugate pairs, will correspond to a family of ideals in the ring of real power series around p with the properties:

- (1) The ideals of components of V at p belong to the family;
- (2) If $\mathfrak{p}_1, \mathfrak{p}_2$ belong to the family, so do the minimal primes of $(\mathfrak{p}_1, \mathfrak{p}_2)$;
- (3) If $\mathfrak{p}$ belongs to the family so do the minimal primes of $(\mathfrak{p}, \delta\mathfrak{p})$.

Now there are two coherence properties involved here. In the first place Lemma 5.6 of [2] implies that in any neighborhood on C of a point p of C , there is an open interval U on C along which each real component of $(\mathfrak{S}V)_p (p \in U)$ is coherent. In fact for any two points p and q of U , the real components of $(\mathfrak{S}V)_p$ induce those of $(\mathfrak{S}V)_q$. In the second place, Lemmas 7.1, 7.2 above show that U can be chosen so that if $\mathfrak{A}$ is the sheaf of ideals of a component of $(\mathfrak{S}V)_p (p \in U)$ then $\mathfrak{A}$ is a coherent sheaf of $K[[X]]$ -ideals over U .

It can be assumed that the initial neighborhood U of C has the properties just described.

Consider now continuations of components of $(\mathfrak{S}V)_p (p \in U)$ along a path $\tilde{C}$ over C . Suppose the real components W, W' of $(\mathfrak{S}V)_p$ and all components of $W \cap W'$ have continuations along $\tilde{C}$ to q . Write W_1 and W'_1 for the continuations of W and W' . Apply first Lemma 7.2 over U , then Lemma 6.5 and finally Lemma 7.2 the other way round over a neighborhood of q , and the following result is obtained.

Lemma 9.1. *The continuations to q of the components of $W \cap W'$ are the components of $W_1 \cap W'_1$.*

Similarly the following is obtained.

Lemma 9.2. *If the real component W of $(\mathfrak{S}V)_p$ and all the components of SW have continuations along $\tilde{C}$ to q , and if W_1 is the continuation of W , then the continuations of the components of SW are the components of SW_1 .*

Finally the following remark on the reality of continuations will be useful.

Lemma 9.3. *Let W and W' have continuations along $\tilde{C}$ to give varieties W_1 and W'_1 . Suppose that W_1 is real and that W'_1 is not, and that $W \subset W'$. Then W is contained in some component of SW' which has a continuation along $\tilde{C}$ containing W_1 .*

Proof. Since $W \subset W'$, also $W_1 \subset W'_1$ (Lemma 8.1). Then since W_1 is real and W'_1 is not, W_1 is contained in some component of SW'_1 ([2], Lemma 3.2) and by Lemma 9.2 this is a continuation of some component of SW' which in turn must contain W (using Lemma 8.1 in the opposite direction along $\tilde{C}$).

10. Statement of the main theorem. As in the last section let V be a real analytic space and C an analytic arc contained in it. Assume that the initial interval U of C is such that the real components of $(\mathfrak{S}V)_p$ for p in U all satisfy the coherence conditions described in §9 namely each is coherent along U and induces a coherent sheaf of $K[[X]]$ -ideals over U . Thus the real components of $(\mathfrak{S}V)_p, p \in U$, all have continuations along paths over C . The following condition will be assumed to hold. This is a restriction on V and C ; examples can be given to show that it is not always satisfied.

(F) *Each real component of the singular bunch of V at points of the initial interval of C has an effective continuation of finite type along some path $\tilde{C}$ over C which is either closed or projects onto C .*

Thus the final interval of $\tilde{C}$ lies over either the initial or the final interval of C . The idea of this condition is to preclude the possibility that an analytic continuation of some real component of $(\mathfrak{S}V)_p, p \in U$, should lead to some sort of natural boundary or essential singularity.

Next, it will be supposed that a situation holds which is a generalization of the hypothesis for the main theorem of [2]. Namely, suppose that the local dimension of V at points of the initial arc of C is r , while its local dimension

at points of the final arc is $s < r$. Also suppose that there is an s -dimensional component Z of V at q in the final arc U_1 of C , coherent along U_1 and having a real continuation along C itself in reverse to a point p of the initial arc. That is to say the sheaf of $K[[X]]$ -ideals associated with this continuation is actually a continuation $(\mathcal{O}, C)$ over C . If Z' is the continuation of Z to a point of the initial interval of C , note that this means that Z' has a real continuation along any path $\tilde{C}$ over C , closed or not. This remark will be used frequently below without further comment.

There are variations and slight generalizations of the assumption just made, but in this form it will show the principle of the argument to be given in its most straightforward form. Clearly the variety Z appearing here has a property generalizing that of the variety Z in [2].

The main result to be proved can now be stated.

Theorem 3. *If Z' is the continuation along C in reverse of the variety Z described above, then Z' is a component of $(\mathfrak{S}V)_p$, p in the initial arc U of C .*

The proof will follow essentially the pattern of that of the main theorem of [2]. The idea is to consider the set of real components of $(\mathfrak{S}V)_p$ for p in the initial interval of C , which contain Z' , and to show that the minimum member of this set is Z' itself.

11. First steps in the proof of Theorem 3. Theorem 3 will be proved by indirect argument, assuming it false and then obtaining a contradiction. Let W be the minimum real component of the bunch $(\mathfrak{S}V)_p$ which contains Z' . On the assumption that Theorem 3 is false, $W \neq Z'$, and so, in particular $\dim W > \dim Z = \dim Z'$. Suppose that W has a continuation along a path $\tilde{C}$ which projects onto C . The continuation cannot be real all along $\tilde{C}$. For, if it were, it would give a real subvariety of V at a point q of the final interval of C (cf. Lemma 8.2) with dimension greater than $\dim V$ at q . Now Z' continues along $\tilde{C}$ to give Z at a point of the final interval and so, by Lemma 9.3, Z' is contained in some component of SW , contradicting the minimal property assumed for W . Hence by the hypothesis (F), W must have an effective continuation along a closed path.

Now let $\mathfrak{B}$ be the bunch of those real components of $(\mathfrak{S}V)_p$ which have effective continuations along closed paths over C . Let $\mathfrak{C}$ be the corresponding set of closed paths and $\mathfrak{T}$ the set of corresponding transformations. The members of $\mathfrak{C}$ will be denoted by $\tilde{C}_i$, the corresponding transformations by T_i . Applying Theorem 2, $\mathfrak{C}$ is an ordered set $\tilde{C}_1 > \tilde{C}_2 > \dots$, and correspondingly the bunch $\mathfrak{B}$ can be subdivided into sub-bunches $\mathfrak{B}_i$ such that T_i operates non-trivially on the components of the i -th bunch $\mathfrak{B}_i$ while, if W is a component of the h -th bunch $\mathfrak{B}_h$, $h > i$, then T_i operates trivially on W and $T_h W$. If W is in $\mathfrak{B}_h$ and $h < i$, T_i is not defined on W . It is convenient to order the T_i in the same way as the corresponding $\tilde{C}_i$. Thus T_i is the last transformation in $\mathfrak{T}$ which operates on the i -th sub-bunch $\mathfrak{B}_i$ of $\mathfrak{B}$. In particular note that the first member

T_1 of $\mathfrak{T}$ operates on all the components of $\mathfrak{X}$. It will be convenient to denote T_1 by T and the corresponding path $\tilde{C}_1$ by $\tilde{C}$.

The argument given at the beginning of this section shows that, if W is the minimum real component of $(\mathfrak{S}V)_p$ containing Z' , then W is a component of $\mathfrak{X}$. Suppose in fact that W is in $\mathfrak{X}_m$ and consider the continuation of W along $\tilde{C}_m$. This continuation must be real all along $\tilde{C}_m$. For Z' has a real continuation along $\tilde{C}_m$ and so Lemma 9.3 implies that if the continuation of W were not real along $\tilde{C}_m$, Z' would be contained in some component of SW , and this would contradict the minimality of W . On the other hand, $T_m W$ cannot belong to $(\mathfrak{S}V)_p$. For $T_m W \supset Z'$ (Z' being invariant under T_m), and so, if $T_m W$ were a component of $(\mathfrak{S}V)_p$, Z' would be contained in a component of $W \cap T_m W$ which would be a component of $(\mathfrak{S}V)_p$ properly contained in W , contradicting the minimality of W .

Thus W is a real component of $(\mathfrak{S}V)_p$ with the property that it has a real continuation along C_m , while $T_m W$ is not a component of $(\mathfrak{S}V)_p$. The idea now is to start with a component of $(\mathfrak{S}V)_p$ having just this property and to build up inductively an ascending chain of components with the same property. When the maximum dimension is reached, a contradiction will evidently be obtained.

So let W_m be a component of the bunch $\mathfrak{X}_m$ and suppose that $T_m W_m$ is real but is not a component of $(\mathfrak{S}V)_p$. Since $T_m W_m$ is real it is contained in V (Lemma 8.2) and so is contained in some real component W_i of $(\mathfrak{S}V)_p$ with $\dim W_i > \dim W_m$. Take W_i minimal with this property.

The following cases are possible.

(1) $TW_i = W_i$ (using here the convention that $T_1 = T$). Note that this means that W_i is not in the bunch $\mathfrak{X}_1$. In particular it also means that TW_i is real.

(2) $TW_i \neq W_i$; $TW_m = W_m$. Note that since $T_m W_m \neq W_m$, this means that $T \neq T_m$, while the inequality $TW_i \neq W_i$ means that W_i is in $\mathfrak{X}_1$. Here T precedes T_m in $\mathfrak{T}$ and so T operates trivially on both W_m and $T_m W_m$. The inclusion $T_m W_m \subset W_i$ thus implies that $TT_m W_m = T_m W_m \subset TW_i$. Now if TW_i were not real this would mean, since $T_m W_m$ is real, that $T_m W_m$ would be contained in some component of STW_i . Apply T again, and Lemma 9.2 along with the fact that T^2 is the identity, and it would follow that $T_m W_m$ would be contained in some component of SW_i . But this would contradict the minimal property of W_i , and so in this case TW_i is real.

(3) $TW_i \neq W_i$; $TW_m \neq W_m$. In this case both W_i and W_m are in $\mathfrak{X}_1$. The inclusion $TW_m \subset W_i$ along with the fact that T^2 is the identity implies that $W_m \subset TW_i$. Here W_m is real and so TW_i must be real. For otherwise W_m would be contained in a component of STW_i , and so, arguing as in case (2) a contradiction would be obtained to the minimality of W_i .

Thus, taking the three cases together, the following lemma has been proved.

Lemma 11.1. *Let W_m be a real component of $(\mathfrak{S}V)_p$ such that $T_m W_m$ is real*

but is not a component of $(\mathfrak{S}V)_p$. Then there is a real component W_i of $(\mathfrak{S}V)_p$ such that $T_m W_m \subset W_i$ and TW_i is real.

Now take W_i maximal with respect to the properties stated in the above lemma. It will be shown that in this case TW_i is a component of $(\mathfrak{S}V)_p$.

Suppose that this is false; namely suppose that TW_i is not a component of $(\mathfrak{S}V)_p$. In particular $TW_i \neq W_i$, which means that W_i is in $\mathfrak{B}_1$. Thus W_i is a real component of $(\mathfrak{S}V)_p$ with TW_i real but not contained in $(\mathfrak{S}V)_p$, and so Lemma 11.1 can be applied to it, replacing W_m there by W_i . It follows that there is a real component W_j of $(\mathfrak{S}V)_p$ such that $TW_i \subset W_j$ and TW_j is real. Consider the following two cases.

(a) Suppose that TW_i is a component of $(\mathfrak{S}V)_p$. Then $T_m W_m \subset W_i$ and $TW_i \subset W_j$, which is the same as $W_i \subset TW_j$, together imply $T_m W_m \subset TW_j$. Here TW_j is a real component of $(\mathfrak{S}V)_p$ and $T(TW_j) = W_j$ is real. This contradicts the maximality of W_i .

(b) In this case suppose that TW_j is not a component of $(\mathfrak{S}V)_p$. Apply Lemma 11.1 to W_j (which must be in $\mathfrak{B}_1$ since TW_j is certainly not W_j). Then there is a real component W_k of $(\mathfrak{S}V)_p$ such that $TW_j \subset W_k$ and TW_k is real. Thus $T_m W_m \subset W_i \subset TW_i \subset W_k$. Again this contradicts the maximality of W_i . Hence both cases (a) and (b) lead to contradictions and so it follows that the initial assumption, that TW_i is not a component of $(\mathfrak{S}V)_p$ is false, as was to be shown.

Summing up this argument, the following result has thus been obtained.

Lemma 11.2. *If W_m is a real component of the sub-bunch $\mathfrak{B}_m$ of $(\mathfrak{S}V)_p$ and if $T_m W_m$ is real but is not a component of $(\mathfrak{S}V)_p$ then there is a real component W_i of $(\mathfrak{S}V)_p$ such that $T_m W_m \subset W_i$ and TW_i is a real component of $(\mathfrak{S}V)_p$.*

Presently W_i is to be taken minimal with respect to the properties stated in this lemma, but first the following auxiliary lemma will be proved.

Lemma 11.3. *Let W_m be a component of $\mathfrak{B}_m$ such that $T_m W_m$ is real but is not a component of $(\mathfrak{S}V)_p$ and suppose that W_i is a component of $(\mathfrak{S}V)_p$ such that $T_m W_m \subset W_i$ and TW_i is a component of $(\mathfrak{S}V)_p$. Then if W_i is taken of lowest possible dimension with these properties, both W_i and TW_i will be real.*

Proof. Suppose W_i is not real. Here the ideal of W_i is real and may remain prime over the complex field or may decompose into complex conjugate primes. In any case $T_m W_m$ is real and contained in W_i and so in some component of SW_i . Here W_i is a component of $(\mathfrak{S}V)_p$. On the other hand since W_i is a component of SW_i , TW_i is a component of STW_i (Lemma 9.2) and so TW_i is also a component of $(\mathfrak{S}V)_p$. But $\dim W_i < \dim W_i$, contradicting the minimality assumption of this lemma. Hence W_i is real as was to be shown.

For the second part of the proof take two cases separately. The first case is that T precedes T_m in the ordering of $\mathfrak{T}$. Then T operates trivially on $T_m W_m$ and so the inclusion $T_m W_m \subset W_i$ implies $TT_m W_m = T_m W_m \subset TW_i$. The

above argument for the first part of the lemma then implies that TW_i is real. The second case is that $T = T_m$. Then the inclusion $T_m W_m \subset W_i$ implies that $W_m \subset T_m W_i = TW_i$. If TW_i were not real, W_m (being real) would be contained in a component of STW_i and so, as above, $T_m W_m = TW_m$ would be contained in a component of SW_i which, along with its transform by T , would be a component of $(\mathfrak{S}V)_p$ thus contradicting the minimality of $\dim W_i$. Hence in both cases TW_i is real and the proof of the lemma is complete.

It will now be shown that a unique smallest W can be found with the properties of Lemma 11.2.

Lemma 11.4. *Let W_m be a component of $\mathfrak{B}_m$ such that $T_m W_m$ is real but is not a component of $(\mathfrak{S}V)_p$ and let W_i be a real component of $(\mathfrak{S}V)_p$ such that $T_m W_m \subset W_i$ and TW_i is a real component of $(\mathfrak{S}V)_p$ (this exists by Lemma 11.2). Then if W_i , satisfying these properties, is taken of lowest possible dimension, it is unique and smallest with these properties.*

Proof. The statement of the lemma means that, if W_i is also a real component of $(\mathfrak{S}V)_p$ such that $T_m W_m \subset W_i$ and TW_i is a real component of $(\mathfrak{S}V)_p$, then it is to be shown that $W_i \subset W_i$, with equality of course in the case of equal dimensions. Suppose on the contrary that $W_i \subsetneq W_i$. All the components of $W_i \cap W_i$ thus have dimension less than $\dim W_i$. Also $T_m W_m$ is contained in some component W_h of $W_i \cap W_i$ and TW_h is a component of $TW_i \cap TW_i$ (Lemma 9.1). W_h and TW_h are components of $(\mathfrak{S}V)_p$, since W_i , W_i and their transforms under T are. And so $T_m W_m$ is contained in a component W_h of $(\mathfrak{S}V)_p$ such that TW_h is a component of $(\mathfrak{S}V)_p$ and $\dim W_h < \dim W_i$. If W_h is taken minimal with these properties Lemma 11.3 implies that it is real, contradicting the minimal property of $\dim W_i$. It follows that $W_i \subset W_i$ as required.

Continuing with the notation of the last lemma, take W_i as the unique smallest real component of $(\mathfrak{S}V)_p$ such that TW_i is also a real component of $(\mathfrak{S}V)_p$ and $T_m W_m \subset W_i$. Some properties of W_i will now be studied.

Lemma 11.5. *In the notation of Lemma 11.4 take W_i as the unique smallest component of $(\mathfrak{S}V)_p$ with the properties stated there.*

- (1) *If $TW_i \neq W_i$, then $T = T_m$ (that is W_i and W_m are in $\mathfrak{B}_1$) and $W_m \subsetneq W_i$.*
- (2) *If $TW_i = W_i$ and W_i is a component of $\mathfrak{B}_i (\neq \mathfrak{B}_1)$ then, if $T_i W_i$ is not real, T_m precedes T_i in the ordering of $\mathfrak{T}$ and W_m is not contained in any component of SW_i .*
- (3) *If $TW_i = W_i$ and W_i is a component of $\mathfrak{B}_i (\neq \mathfrak{B}_1)$ then if $T_i W_i$ is a real component of $(\mathfrak{S}V)_p$, $T_m W_m \subsetneq T_i W_i$ and T_i does not precede T_m in the ordering of $\mathfrak{T}$.*
- (4) *If W_i is not a component of $\mathfrak{B}$, W_m is not in any component of SW_i .*

Proof. (1) Here the hypothesis is that $TW_i \neq W_i$. Suppose that $T_m \neq T$, or in other words that T precedes T_m . Then $TT_m W_m = T_m W_m$ and so $T_m W_m \subset TW_i$. Thus TW_i would have the same properties as W_i , and the same dimension, contradicting the unique minimality of W_i . Hence $T_m = T$.

Next suppose that $W_m \subset W_i$. Since $T_m = T$ this means that $T_m W_m \subset T_m W_i$ which is the same as TW_i . Again this contradicts the unique minimality of W_i . And so $W_m \subsetneq W_i$. This completes the proof of part (1).

(2) The hypothesis now is that $TW_i = W_i$ so that W_i is a component of a bunch $\mathfrak{B}_i \neq \mathfrak{B}_1$; T_i is the member of $\mathfrak{T}$ operating non-trivially on W_i . Also it is assumed here that $T_i W_i$ is not real. There are three possible relations between T_m and T_i , of which it will now be shown that the first two lead to contradictions.

(a) Suppose that $T_m = T_i$. Then the inclusion $T_m W_m \subset W_i$ implies that $W_m \subset T_m W_i = T_i W_i$. The hypothesis here is that $T_i W_i$ is not real and so W_m is contained in some component of the bunch $ST_i W_i$ and so $T_m W_m = T_i W_m$ is contained in some component W_h of the bunch SW_i . W_h is a component of $(\mathfrak{S}V)_p$ and, since TW_i is a component of this bunch, so also is TW_h (Lemma 9.2). Thus $T_m W_m$ is contained in a component W_h of $(\mathfrak{S}V)_p$ such that TW_h is also a component of $(\mathfrak{S}V)_p$ and $\dim W_h < \dim W_i$. If W_h is taken minimal with these properties it is real (Lemma 11.3) and this contradicts the minimal property of W_i . Hence the case $T_m = T_i$ is impossible.

(b) Suppose T_i precedes T_m in the ordering of $\mathfrak{T}$. Then T_i operates trivially on both W_m and $T_m W_m$. The inclusion $T_m W_m \subset W_i$ implies that $T_i T_m W_m = T_m W_m \subset T_i W_i$. $T_m W_m$ is real but $T_i W_i$ is assumed not real and so $T_m W_m$ is in some component of the bunch $ST_i W_i$ and so, operating with T_i again, $T_m W_m$ is contained in some component of the bunch SW_i . Then arguing as in (a) a contradiction is reached. Hence the case T_i preceding T_m is impossible.

(c) Thus the only remaining possibility is that T_m precedes T_i , so that T_m operates trivially on W_i and $T_i W_i$. It must be checked that W_m is not contained in any component of SW_i . If it were, $T_m W_m$ would be contained in a component of $T_m SW_i = ST_m W_i$ (Lemma 9.2) and this is equal to SW_i by hypothesis. An argument as in (a) and (b) would then lead to a contradiction. And so part (2) of the lemma is completely proved.

(3) Here the hypothesis is as in part (2) except that $T_i W_i$ is now assumed to be a real component of $(\mathfrak{S}V)_p$. First it is to be shown that $T_m W_m \subsetneq T_i W_i$. Suppose on the contrary that $T_m W_m \subset T_i W_i$. By hypothesis T precedes T_i and so $TT_i W_i = T_i W_i$. But then this means that $T_m W_m$ is contained in the real component $T_i W_i$ of $(\mathfrak{S}V)_p$ and $TT_i W_i$ is also a real component of $(\mathfrak{S}V)_p$. This contradicts the unique minimality of W_i . And so in fact $T_m W_m \subsetneq T_i W_i$ as required. Finally suppose that T_i precedes T_m . Then T_i operates trivially on $T_m W_m$ and so the inclusion $T_m W_m \subset W_i$ implies that $T_m W_m \subset T_i W_i$, contradicting the result just proved. Hence T_i does not precede T_m , as was to be shown.

(4) Condition (F) implies that if W is not in $\mathfrak{B}$ then it has a continuation along a path $\tilde{C}$ projecting onto C . In this case T_m operates trivially on W_i (Lemma 5.4) and so if W_m is contained in a component of SW_i so is $T_m W_m$. Arguing as in case (2) a contradiction is reached and the lemma is proved.

12. The inductive construction. From now on assume Theorem 3 to be false. It was seen at the beginning of Section 11 that in this case the minimum component of $(\mathfrak{S}V)_p$ containing Z' (properly of course) is a component W_1 of some bunch $\mathfrak{B}_{m(1)}$ such that $T_{m(1)}W_1$ is real but is not a component of $(\mathfrak{S}V)_p$. W_1 can then be taken as W_m in the lemmas of §11. Hence there is a unique smallest component W_2 (namely the W_i of §11) of $(\mathfrak{S}V)_p$ such that $T_{m(1)}W_1 \subset W_2$ and TW_2 is also a real component of $(\mathfrak{S}V)_p$.

Now, remembering that Z' has a continuation along C to the final interval of C , while the path corresponding to the transformation $T_{m(1)}$ projects into C , it is clear that $T_{m(1)}Z' = Z'$ and so $T_{m(1)}W_1 \supset Z'$ (Lemma 8.1) and so $Z' \subset W_2$. By hypothesis (F), W_2 has an effective continuation along a path $\tilde{C}$ which is either closed or projects onto C . Suppose that $\tilde{C}$ projects onto C . The continuation of W_2 cannot be real along $\tilde{C}$, since then, by Lemma 8.2, its continuation to the final interval would be a real subvariety of V of dimension greater than $\dim Z$ at q in the final interval of C , contrary to the hypothesis on V . And so if $\tilde{C}$ projects onto C the continuation of W_2 along $\tilde{C}$ is not real and this means (Lemma 9.3) that Z' is contained in some component of SW_2 . The initial choice of W_1 as the smallest component of $(\mathfrak{S}V)_p$ containing Z' then implies that W_1 is in some component W of SW_2 . W of course is a component of $(\mathfrak{S}V)_p$. Also TW is a component of $STW_2 = SW_2$ (Lemma 9.2) and so is a component of $(\mathfrak{S}V)_p$. Thus W_1 is contained in a component W of $(\mathfrak{S}V)_p$ such that TW is also a component of $(\mathfrak{S}V)_p$ and $\dim W < \dim W_2$. Take W minimal with this property and Lemma 11.3 implies that it is real, so contradicting the minimal property of W_2 . It follows from this that W_2 is contained in a bunch $\mathfrak{B}_{m(2)}$.

There are now several cases to consider, all except one of which will be ruled out by using Lemma 11.5. Remember that in any case $Z' \subset W_1$ and Z' is invariant under $T_{m(1)}$, whence $Z' \subset W_2$.

(1) Suppose first that $TW_2 \neq W_2$. According to Lemma 11.5 part (1) this implies that $W_1 \subsetneq W_2$. But $Z' \subset W_2$ and W_1 is the smallest component of $(\mathfrak{S}V)_p$ containing Z' , and so should be contained in W_2 . This contradiction rules out Case (1).

(2) Suppose that $TW_2 = W_2$ and that W_2 is a component of $\mathfrak{B}_{m(2)}$ such that $T_{m(2)}W_2$ is not real. Note that the remarks made above imply that W_2 is certainly in the bunch $\mathfrak{B}$. Lemma 11.5, part (2), implies now that W_1 is not contained in any component of SW_2 . On the other hand $Z' \subset W_2$ and so $Z' \subset T_{m(2)}W_2$ which is not real. Hence Z' is contained in some component of the bunch $ST_{m(2)}W_2$ and so in some component of SW_2 (Lemma 9.3). This contradiction rules out Case (2).

(3) Suppose that $TW_2 = W_2$ and that W_2 is a component of some bunch $\mathfrak{B}_{m(2)}$ such that $T_{m(2)}W_2$ is real component of $(\mathfrak{S}V)_p$. Part (3) of Lemma 11.5 implies that $T_{m(1)}W_1 \subsetneq T_{m(2)}W_2$ and that $T_{m(2)}$ does not precede $T_{m(1)}$. There are then two possibilities:

(a) $T_{m(1)} = T_{m(2)}$: In this case the relation $T_{m(1)}W_1 \subsetneq T_{m(2)}W_2$ implies

that $W_1 \subset W_2$. On the other hand W_2 is a component of $(\mathfrak{S}V)_p$ containing Z' and so should contain W_1 . Hence this case is ruled out.

(b) $T_{m(1)}$ precedes $T_{m(2)}$: In this case W_2 and $T_{m(2)}W_2$ are invariant under $T_{m(1)}$ and so $T_{m(1)}W_1 \subset T_{m(2)}W_2$ implies that $W_1 \subset T_{m(2)}W_2$. But $T_{m(2)}W_2$ is a component of $(\mathfrak{S}V)_p$ by hypothesis and c contains Z' (since $W_2 \supset Z'$) and so should contain W_1 . This contradiction rules out case (b). Thus case (3) is ruled out.

The only remaining possibility is that $T_{m(2)}W_2$ is real and is not a component of $(\mathfrak{S}V)_p$. That is to say it has a property similar to that of W_1 . This gives the start of the induction which will now be set up.

The induction will involve constructing a sequence $W_1, W_2, \dots$ of components of $(\mathfrak{S}V)_p$, each W_i being in fact a component of a bunch $\mathfrak{W}_{m(i)}$ with corresponding transformation $T_{m(i)}$. To simplify the notation throughout this argument, $\mathfrak{W}_{m(i)}$ and $T_{m(i)}$ will be denoted by $\mathfrak{W}_i$ and T_i respectively. Thus the index i here is not the same as in the original indexing of the members of $\mathfrak{T}$ and sub-bunches of $\mathfrak{W}$, but is only a short-hand for $m(i)$.

With this convention in mind the induction hypothesis can be stated. As above W_1 is to be the smallest component of $(\mathfrak{S}V)$ containing Z' . Suppose that there is a sequence $W_1, W_2, \dots, W_k$ of real components of $(\mathfrak{S}V)_p$ of strictly ascending dimensions, W_i being a component of a bunch $\mathfrak{W}_i$, operated on non-trivially by a transformation T_i and satisfying the conditions.

(1) In the ordering of $\mathfrak{T}$ all the T_i follow T (the first transformation in the ordering of $\mathfrak{T}$) except that possibly $T_1 = T$.

(2) $T_i W_i$ is real and is not a component of $(\mathfrak{S}V)_p$ for $1 \leq i \leq k$.

(3) W_i is the unique smallest component of $(\mathfrak{S}V)_p$ such that $T_{i-1}W_{i-1} \subset W_i$ and W_i, TW_i are both real components of $(\mathfrak{S}V)_p$ for $1 \leq i \leq k$.

(4) All the W_i contain Z' .

It has already been seen at the beginning of this section that all this holds for $k = 2$. Note also that in any case condition (2) along with Lemma 11.4 implies that condition (3) makes sense.

It will now be shown that the sequence can be continued to W_{k+1} .

Since W_k is a real component of $(\mathfrak{S}V)_p$ with $T_k W_k$ real but not a component of $(\mathfrak{S}V)_p$, there exists a unique smallest real component W_{k+1} of $(\mathfrak{S}V)_p$ such that $T_k W_k \subset W_{k+1}$ and TW_{k+1} is a real component of $(\mathfrak{S}V)_p$ (Lemma 11.4).

Lemma 12.1. $Z' \subset W_{k+1}$.

Proof. $Z' \subset W_k$ by the induction hypothesis part (4) and Z' is invariant under T_k . The result follows at once.

Lemma 12.2. $TW_{k+1} = W_{k+1}$.

Proof. If $TW_{k+1} \neq W_{k+1}$ then Lemma 11.5 part (1) implies that $T_k = T$, which is false for $k \geq 2$ (induction hypothesis part (1)).

Lemma 12.3. W_{k+1} is a component of $\mathfrak{W}$.

Proof. The condition (F) says that W_{k+1} has an effective continuation along a path $\tilde{C}$ over C which is either closed or projects onto C . Suppose that $\tilde{C}$ projects onto C . Then the continuation of W_{k+1} cannot be real along $\tilde{C}$, since otherwise the dimension condition of V would be contradicted. It follows that if $\tilde{C}$ projects onto C , that is if W_{k+1} is not contained in $\mathfrak{B}$, then Z' is contained in some component of SW_{k+1} . Since W_1 is the smallest component of $(\mathfrak{S}V)_p$ containing Z' this implies that W_1 is contained in some component of SW_{k+1} and so T_1W_1 is contained in some component of SW_{k+1} (W_{k+1} being invariant under T_1 by Lemma 5.4). It will now be shown by induction that T_jW_j is contained in a component of SW_{k+1} for $j = 1, 2, \dots, k$. Suppose that it is already known that T_jW_j is contained in a component of SW_{k+1} ; this has just been proved for $j = 1$. Since W_{k+1} is invariant under T (Lemma 5.4) this means that T_jW_j is contained in a component W of $(\mathfrak{S}V)_p$ such that TW is a component of $(\mathfrak{S}V)_p$ and in fact W is a component of SW_{k+1} . Take W minimal with these properties and Lemma 11.3 implies that it is real. Then, since W_{j+1} is the unique smallest component of $(\mathfrak{S}V)_p$ containing T_jW_j and such that TW_{j+1} is a component of $(\mathfrak{S}V)_p$ it follows that $W_{j+1} \subset W$. Thus W_{j+1} is contained in a component of SW_{k+1} . And since W_{k+1} is invariant under T_{j+1} it follows that $T_{j+1}W_{j+1}$ is contained in a component of SW_{k+1} .

This completes the induction, and so in particular T_kW_k is contained in a component of SW_{k+1} . Again since $T_kW_{k+1} = W_{k+1}$ this means that W_k is contained in a component of SW_{k+1} . But Lemma 11.5 part (4) says that W_k is not contained in any component of SW_{k+1} . This contradiction shows that W_{k+1} is in $\mathfrak{B}$ as was to be shown.

Let $\mathfrak{B}_{k+1}$ denote the sub-bunch of $\mathfrak{B}$ containing W_{k+1} , T_{k+1} the operation of $\mathfrak{T}$ acting non-trivially on $\mathfrak{B}_{k+1}$.

Lemma 12.4. $T_{k+1}W_{k+1}$ is real.

Proof. Suppose that $T_{k+1}W_{k+1}$ is not real. Then Lemma 11.5 part (2) says that T_k precedes T_{k+1} and W_k is not contained in any component of SW_{k+1} . That is to say, the case $j = k$ of the following statement holds: (A_i) T_i precedes T_{k+1} and W_i is not in any component of SW_{k+1} , $j \leq i \leq k$.

Assuming that (A_i) holds it will now be shown that (A_{i-1}) holds.

It is to be shown in particular that W_{i-1} is not contained in any component of SW_{k+1} . First it will be shown that $T_{i-1}W_{i-1}$ is not in any component of SW_{k+1} . Suppose on the contrary that $T_{i-1}W_{i-1}$ is contained in a component W of SW_{k+1} . Since $TW_{k+1} = W_{k+1}$ (Lemma 12.2) TW is a component of SW_{k+1} . Thus $T_{i-1}W_{i-1} \subset W$, where W and TW are both components of $(\mathfrak{S}V)_p$. If W is taken minimal it is real (Lemma 12.2) and so contains W_i . Hence W_i is contained in a component of SW_{k+1} , contradicting the hypothesis (A_i) . It follows that $T_{i-1}W_{i-1}$ is not contained in any component of SW_{k+1} .

It will now be shown that T_{i-1} precedes T_{k+1} . Consider the two contrary cases:

(a) Suppose $T_{i-1} = T_{k+1}$. In this case the fact that $T_{i-1}W_{i-1}$ is not in any

component of SW_{k+1} (proved above) implies that W_{i-1} is not in any component of $ST_{k+1}W_{k+1}$ (using Lemma 9.2). But this means that $W_{i-1} \subsetneq T_{k+1}W_{k+1}$ since $T_{k+1}W_{k+1}$ is being assumed not real. Applying $T_{i-1} = T_{k+1}$ this gives $T_{i-1}W_{i-1} \subsetneq W_{k+1}$.

(b) Suppose T_{k+1} precedes T_{i-1} . In this case $T_{i-1}W_{i-1}$ is invariant under T_{k+1} and so $T_{i-1}W_{i-1}$ is not in any component of $ST_{k+1}W_{k+1}$. Since $T_{k+1}W_{k+1}$ is being assumed not real this means that $T_{i-1}W_{i-1} \subsetneq T_{k+1}W_{k+1}$ and so $T_{i-1}W_{i-1} \subsetneq W_{k+1}$.

Hence the assumption that T_{i-1} does not precede T_{k+1} implies that $T_{i-1}W_{i-1} \subsetneq W_{k+1}$. This relation is now to be taken as the start of an induction. Suppose it is known that $(B_h) \quad T_h W_h \subsetneq W_{k+1}, j-1 \leq h$.

The above argument shows that, if T_{i-1} does not precede T_{k+1} , then (B_{i-1}) holds.

Since $T_h W_h \subsetneq W_{h+1}$, (B_h) implies that $W_{h+1} \subsetneq W_{k+1}$. If $h \geq j-1$, (A_{h+1}) holds (part of the initial induction hypothesis) and so T_{h+1} precedes T_{k+1} . Hence W_{k+1} is invariant under T_{h+1} and so $W_{h+1} \subsetneq W_{k+1}$ implies $T_{h+1}W_{h+1} \subsetneq W_{k+1}$. Hence (B_h) implies (B_{h+1}) and so, by induction (B_k) holds. That is to say, $T_k W_k \subsetneq W_{k+1}$. But this is a contradiction. It arises from the assumption that T_{i-1} does not precede T_{k+1} and so in fact T_{i-1} does precede T_{k+1} , as was to be shown. The relation that $T_{i-1}W_{i-1}$ is not contained in any component of SW_{k+1} then implies that W_{i-1} is not contained in any component of SW_{k+1} . That is to say (A_{i-1}) has been proved.

Thus by induction (A_j) holds for all j . In particular this means that W_1 is not contained in any component of SW_{k+1} . On the other hand, Z' is contained in W_{k+1} and so also in $T_{k+1}W_{k+1}$. The latter is being assumed not real and so Z' is contained in some component of $ST_{k+1}W_{k+1}$ and so in some component of SW_{k+1} . But then the minimal property of W_1 implies that W_1 is contained in a component of SW_{k+1} , contradicting the result obtained a few lines above. Hence the assumption that $T_{k+1}W_{k+1}$ is not real is false, as was to be shown.

Lemma 12.5. $T_{k+1}W_{k+1}$ is not a component of $(\mathfrak{S}V)_p$.

Proof. Like the last lemma, this will be proved by a double induction starting with the assumption that the result is false.

Suppose then that $T_{k+1}W_{k+1}$ is a component of $(\mathfrak{S}V)_p$. By the last Lemma it is real. Then Lemma 11.5 part (3) says that $T_k W_k \subsetneq T_{k+1}W_{k+1}$ and T_{k+1} does not precede T_k . This means that, in particular, $T_k W_k \subsetneq W_{k+1} \cap T_{k+1}W_{k+1}$. This is the case $j = k$ of the following: $(C_i) \quad T_i W_i \subsetneq W_{k+1} \cap T_{k+1}W_{k+1}$ and T_{k+1} does not precede T_i .

Assume that this holds for $i \leq j \leq k$. It will be shown that (C_{i-1}) holds.

Suppose that the first part of (C_{i-1}) is false. Namely suppose $T_{i-1}W_{i-1} \subsetneq W_{k+1} \cap T_{k+1}W_{k+1}$. The right hand side is invariant under T . Hence $T_{i-1}W_{i-1}$ is contained in a component W of $W_{k+1} \cap T_{k+1}W_{k+1}$ and both W and TW are components of $(\mathfrak{S}V)_p$ since $T_{k+1}W_{k+1}$ is assumed to be in $(\mathfrak{S}V)_p$. Taking W minimal it must be real (Lemma 11.3) and so the minimal property of W_i

implies that $W_i \subset W$. Hence $W_i \subset W_{k+1} \cap T_{k+1}W_{k+1}$. (C_i) says that $T_{k+1} = T_i$ of T_i precedes T_{k+1} . In either case the last inclusion relation implies $T_iW_i \subset W_{k+1} \cap T_{k+1}W_{k+1}$. This contradicts (C_i) and so $T_{i-1}W_{i-1} \subsetneq W_{k+1} \cap T_{k+1}W_{k+1}$, the first half of (C_{i-1}) .

Now suppose that the second part of (C_{i-1}) is false. Namely suppose that T_{k+1} precedes T_{i-1} . Then $T_{i-1}W_{i-1} \subset W_{k+1}$ if and only if $T_{i-1}W_{i-1} \subset T_{k+1}W_{k+1}$. And so, by the part of (C_{i-1}) already proved, the following is true for $j = i - 1$: (D_j) $T_jW_j \subsetneq W_{k+1}$ and $T_jW_j \subsetneq T_{k+1}W_{k+1}$.

Assuming (D_j) for $i - 1 \leq j \leq h$ it will be shown that (D_{h+1}) holds. Since $T_hW_h \subset W_{h+1}$, (D_h) implies that $W_{h+1} \subsetneq W_{k+1}$, and $W_{h+1} \subsetneq T_{k+1}W_{k+1}$. Since $h + 1 \geq i$, (C_{h+1}) holds and so either $T_{k+1} = T_{h+1}$ or T_{h+1} precedes T_{k+1} . In either case it follows that $T_{h+1}W_{h+1} \subsetneq W_{k+1}$ and $T_{h+1}W_{h+1} \subsetneq T_{k+1}W_{k+1}$. That is to say (D_{h+1}) holds. And so, by induction (D_k) holds. But this says that $T_kW_k \subsetneq W_{k+1}$, which is a contradiction. It follows that T_{k+1} cannot precede T_{i-1} , and so (C_{i-1}) has been completely proved.

Thus by induction (C_j) holds for all j . In particular $T_1W_1 \subsetneq W_{k+1} \cap T_{k+1}W_{k+1}$ and T_{k+1} does not precede T_1 . Hence $T_1 = T_{k+1}$ or T_1 precedes T_{k+1} , and so in either case $W_1 \subsetneq W_{k+1} \cap T_{k+1}W_{k+1}$. But, by the supposition at the beginning of the proof, $T_{k+1}W_{k+1}$ is a component of $(\mathfrak{S}V)_p$ and so the components of $W_{k+1} \cap T_{k+1}W_{k+1}$ are components of $(\mathfrak{S}V)_p$, and all contain Z' . The minimal property of W_1 then implies that $W \subset W_{k+1} \cap T_{k+1}W_{k+1}$. A contradiction has thus been obtained and so the initial assumption that $T_{k+1}W_{k+1}$ is a component of $(\mathfrak{S}V)_p$ is false and the proof of the lemma is complete.

Thus all the parts (1) through (4) of the induction hypothesis carry over to W_{k+1} and so the sequence $W_1, W_2, \dots$ can be continued indefinitely. However this gives a contradiction since the dimensions of the W_i are strictly increasing and are bounded above by $\dim V$ at p . This contradiction arises from the assumption that Theorem 3 is false and so in fact that theorem has been proved.

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