*An Invariant Formulation of the New Maximum-Minimum Theory of Eigenvalues**

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1. Introduction. This paper is a complement to our previous paper in the same journal [1] and presents our criteria for the inequalities in the maximum-minimum theory in a new form. It could be called an invariant formulation of these criteria and therefore may be of some interest. Using the notation of our previous paper let A be a compact, negative definite, symmetric operator on a real Hilbert space H with the scalar product (u, v) . Denote by λ_k , $k = 1, 2, \dots$

$$\lambda_1 \leq \lambda_2 \leq \dots$$

the eigenvalues and by u_k , $(u_i, u_k) = \delta_{ik}$, the corresponding eigenfunctions (or eigenvectors) of the equation

$$(1) \quad Au = \lambda u.$$

The assumption that A be negative definite is made only for the convenience of ordering the eigenvalues in an increasing sequence. For the case of infinitely many eigenvalues we have $\lambda_k \rightarrow 0$. Let $2 \leq m \leq n$ where m is such that

$$(2) \quad \lambda_{m-1} < \lambda_m = \lambda_{m+1} = \dots = \lambda_n \leq \lambda_{n+1} \leq \dots$$

Let $p_1, p_2, \dots, p_{n-1}$ be any $n - 1$ elements in H and denote by $R(u)$ the Rayleigh quotient $R(u) = (Au, u)/(u, u)$. Let us consider

$$(3) \quad \lambda(p_1, p_2, \dots, p_{n-1}) = \min R(u)$$

under the $n - 1$ orthogonality conditions

$$(4) \quad (u, p_1) = (u, p_2) = \dots = (u, p_{n-1}) = 0.$$

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Obviously this minimum depends only on the subspace S generated by $p_1, p_2, \dots, p_{n-1}$.

2. The determinant $W(\lambda)$. Let r be the dimension of S . Omitting the trivial case $r = 0$, we assume $1 \leq r \leq n - 1$ and that the basis for S is given by r independent but not necessarily orthogonal vectors $p_1, p_2, \dots, p_r$ while $p_{r+1} = p_{r+2} = \dots = p_{n-1} = 0$. In order to formulate our result we introduce the determinant

$$W(\lambda) = W_r(\lambda) = \det \left\{ \sum_{i=1}^{\infty} \frac{(p_i, u_i)(p_k, u_i)}{\lambda_i - \lambda} \right\}, \quad i, k = 1, 2, \dots, r.$$

This meromorphic function is usually called Weinstein's determinant. Its principal minors are $W_1, W_2, \dots, W_r$. In the open interval $\lambda_{m-1} < \lambda < \lambda_n$ the r meromorphic functions $W_1, W_2, \dots, W_r$ have at most a finite number of zeros. We denote by λ_0 the largest, if any, of these zeros. Otherwise we put $\lambda_0 = \lambda_{m-1}$. Let λ be a fixed value such that $\lambda_0 < \lambda < \lambda_n$.

Denote by $a_{ik}(\lambda) (= a_{ki}(\lambda))$ the elements of $W_r(\lambda)$. Let us introduce a new basis for S , namely $p'_1, p'_2, \dots, p'_r$, defined by the equations

$$p_i = \sum_{i=1}^r \alpha_{i,i} p'_i, \quad i = 1, 2, \dots, r.$$

Denote by $a'_{ik}(\lambda)$ the elements of $W_r(\lambda)$ formed with the new basis. Consider the quadratic form

$$\sum_{i,k=1}^r a_{ik}(\lambda) \xi_i \xi_k.$$

If we put

$$\xi'_s = \sum_{i=1}^r \alpha_{i,s} \xi_i \quad s = 1, 2, \dots, r$$

we see that the $a_{ik}(\lambda)$ are transformed into a'_{ik} by the same rule as the coefficients of a quadratic form. In particular, for each λ we can choose a basis for S such that the quadratic form is diagonal. Let $\epsilon > 0$ be such that $\lambda_0 < \lambda_n - \epsilon < \lambda_n$. Such inequalities are clearly satisfied for $\epsilon > 0$ and sufficiently small, which explains the notation.

3. The main theorem. We can now formulate the main theorem of our previous paper in the following way.

The New Maximum-Minimum Principle. *For any choice of $p_1, p_2, \dots, p_{n-1}$ we have the inequality*

$$(5) \quad \lambda(p_1, p_2, \dots, p_{n-1}) \leq \lambda_n.$$

A necessary condition for the equality

$$(6) \quad \lambda(p_1, p_2, \dots, p_{n-1}) = \lambda_n$$

is that the dimension r of the subspace S generated by $p_1, p_2, \dots, p_{n-1}$ satisfies the inequality

$$(7) \quad m - 1 \leq r \leq n - 1$$

where m is defined by (2). Let $p_1, \dots, p_r$ be a basis for S . Assuming that (7) holds we have the following necessary and sufficient condition for (6): The quadratic form with the symmetric matrix of the determinant $W_r(\lambda_n - \epsilon)$ has the diagonal form

$$(8) \quad -x_1^2 - x_2^2 - \dots - x_{m-1}^2 + x_m^2 + x_{m+1}^2 + \dots + x_r^2.$$

The proof of (5) as well as the necessary condition (7) for equality (6) has been given in our previous paper [5] and will not be repeated here.

Our necessary and sufficient condition given here is equivalent to the one given in our previous papers, namely: If $W_1(\lambda_n - \epsilon) < 0$, the sequence

$$(9) \quad W_1, W_2, \dots, W_r$$

alternates in sign exactly $m - 2$ times. If $W_1(\lambda_n - \epsilon) > 0$, the sequence (9) alternates in sign exactly $m - 1$ times. These two conditions can be combined into the single condition that the sequence

$$1, W_1, W_2, \dots, W_r$$

alternates in sign exactly $m - 1$ times, which, in a classical way, proves the criterion for equality (6) [2; vol. I]. In the case $2 < m = n$, the necessary and sufficient condition for (6) is that the quadratic form with the determinant $W_{m-1}(\lambda_n - \epsilon)$ is negative definite. When $2 \leq m < n$ this condition is sufficient for (6) for any choice of the vectors $p_m, p_{m+1}, \dots, p_{n-1}$. This follows immediately from our previous paper. However, this condition is not necessary for (6) as is shown by the following counterexample communicated to the author.

4. Stenger's counterexample. Let $\lambda_1 < \lambda_2 = \lambda_3$ so that $m = 2$ and $n = 3$. Let $p_1 = u_2$ and $p_2 = u_1 + \alpha u_2$ where α is not zero but is otherwise arbitrary. Using p_1 alone we obtain

$$W_1(\lambda_3 - \epsilon) = \frac{1}{\lambda_2 - \lambda_3 + \epsilon} = \frac{1}{\epsilon} > 0,$$

for any $\epsilon > 0$. For a moment let us put $p'_1 = p_2$. Then we have

$$W'_1(\lambda_3 - \epsilon) = \frac{1}{\lambda_1 - \lambda_3 + \epsilon} + \frac{\alpha^2}{\epsilon} > 0,$$

for $\epsilon > 0$ and sufficiently small. However,

$$W_2(\lambda_3 - \epsilon) = \frac{1}{\epsilon(\lambda_1 - \lambda_3 + \epsilon)} < 0,$$

for $\epsilon > 0$ and sufficiently small. Therefore we see that $\lambda(p_1) < \lambda_2 (= \lambda_3)$ and $\lambda(p_2) < \lambda_2 (= \lambda_3)$ but $\lambda(p_1, p_2) = \lambda_3$. This shows that to obtain the equality (6) we need not have the quadratic form negative definite.

Let us also note that for a given A , depending on the choice of the $\{p_k\}$, the number r which appears in the criterion can be any value in the range (7), as can be seen by taking $p_i = u_i: i = 1, 2, \dots, r$ and $p_{r+1} = p_{r+2} = \dots = p_{n-1} = 0$.

5. The case $m = 1$. As we mentioned before, for the case $1 = m < n$, (6) holds for any choice of the $\{p_i\}$. This can be shown without using the methods of the classical maximum-minimum theory by considering as in our previous paper the equation $Au - PAu = \lambda u$ where P is the projection on the subspace S . The quantity $\lambda(p_1, p_2, \dots, p_{n-1})$ is the first (or smallest) non-vanishing eigenvalue of this equation. To prove this assertion we consider $u_0 = c_1 u_1 + c_2 u_2 + \dots + c_n u_n \neq 0$ which satisfies the conditions (4). We then have $PAu_0 = 0$ and therefore $Au_0 - PAu_0 = \lambda_1 u_0$, so that the first eigenvalue

$$\lambda(p_1, p_2, \dots, p_{n-1}) = \lambda_1 = \lambda_n.$$

Another proof follows immediately from our paper [6; p. 33]. In fact, the multiplicity μ of λ_n is by assumption greater than $n - 1$. Therefore $\lambda_1 (= \lambda_n)$ always appears in the spectrum of the equation $Au - PAu = \lambda u$. For the convenience of the reader we repeat the short proof from [6]. Our equation can be written as

$$(10) \quad Au - \lambda u = a_1 p_1 + a_2 p_2 + \dots + a_{n-1} p_{n-1}$$

where the right-hand side is PAu for certain unknown values $a_1, a_2, \dots, a_{n-1}$. This equation can be satisfied by $\lambda = \lambda_n$ for any choice of $p_1, p_2, \dots, p_{n-1}$. The Fredholm condition

$$\left(\sum_{i=1}^{n-1} a_i p_i, u_k \right) = 0; \quad k = 1, 2, \dots, \mu$$

can in fact be satisfied by putting $a_1 = a_2 = \dots = a_{n-1} = 0$ in (10). Thus (10) becomes $Au - \lambda_n u = 0$ which admits as a general solution $u = b_1 u_1 + b_2 u_2 + \dots + b_\mu u_\mu$. Since $\mu > n - 1 \geq \dim S$ there are solutions in $S^\perp$.

6. The connection with the general theory of intermediate problems. Let us add a few words about the general theory of intermediate problems as they are related to our subject.

We consider the problem of the vibrating clamped plate, see references 1, 2, and 3 in our previous paper [1]. In the language of Hilbert space it can be stated as follows. Let D be the domain of the plate and let g be the Green's function for the Laplace equation in D . Denote by G the integral operator with the kernel g . G is positive definite and completely continuous. Then the eigenvalues for the clamped plate are determined by the consideration of the Rayleigh quotient $R(u)$ where $A = -GG$ and u is restricted to the space orthogonal to the subspace of harmonic functions in $\mathcal{L}_2(D)$. If we drop this restriction and let u be any function in $\mathcal{L}_2(D)$ we obtain a base problem which is equivalent to the problem of the vibrating membrane for D and therefore can be solved explicitly for rectangular domains. This base problem gives lower bounds for the

plate problem. Let $p_1, p_2, \dots$ be a sequence of harmonic (or potential) functions in $\mathcal{L}_2(D)$. Then an intermediate problem of order s is formed by restricting u to the subspace orthogonal to $p_1, p_2, \dots, p_s$. The eigenvalues of the intermediate problem give improved lower bounds for the eigenvalues of the plate problem. As we see, this is the same setup as is in the new maximum-minimum theory, with the difference that here the functions $p_1, p_2, \dots$ are restricted to a subspace of the Hilbert space and that the purpose is to compute lower bounds for the unknown eigenvalues of the plate. The notation $\{p_k\}$ has been retained in the literature in all cases, except in [5] where a partial account of the theory is given. In the maximum-minimum theory we can, in a certain artificial way, also speak of intermediate problems. In fact, our eigenvalue equation $Au - PAu = \lambda u$ is intermediate between $Au = \lambda u$ and $Au - P'Au = \lambda u$ where P' is the projection on the subspace generated by $u_1, u_2, \dots, u_{n-1}$.

The general scheme of intermediate problems is the following. First we have to introduce a solvable base problem which gives rough lower bounds to the eigenvalues of a given problem. Second we have to define intermediate problems depending on a finite number of functions $p_1, p_2, \dots, p_s$ which give improved intermediate eigenvalues. Third we have to devise methods of solving the intermediate problems theoretically and numerically in terms of the base problem. The oldest type of problems was first considered some 30 years ago by A. Weinstein. Such problems were sometimes called intermediate problems corresponding to a change in boundary conditions. The connection with the maximum-minimum theory shows that such a name is somewhat misleading and therefore we prefer to call these problems intermediate problems of the first kind. Later other types of problems, always following the same scheme (given problem, base problem, intermediate problems) were considered by several authors. In the very important problems of the second kind, introduced by Aronszajn [3], the given problem is obtained from the base problem $Au = \lambda u$ by adding to A a positive operator B . It has been shown by S. T. Kuroda [4] and later by Fichera [5] that under certain assumptions the first kind is reducible to the second kind. However, the first kind of problems appears in a natural way not only in the theory of plates but also in the fundamental maximum-minimum theory. Therefore a reduction to the second kind would rather obscure the issue and has not been investigated in the present paper.

In conclusion let us mention that if A were a positive operator the quantity $\lambda(p_1, p_2, \dots, p_{n-1})$ would be the highest eigenvalue of the equation $Au - PAu = \lambda u$. This remark was *the starting point of our theory* and has been used later by Fichera [5] to derive another interesting criteria for the equality (6) under the restriction that the orthogonal invariants of A are finite. This assumption on A means that the eigenvalues converge sufficiently rapidly to zero. No such assumption is made in our theory. Let us emphasize that his operator is again an operator of the first kind as we have used throughout. More precisely, Fichera uses the operator $(I - P)A(1 - P)$ which is Aronszajn's modification of Weinstein's operator $A - PA = (I - P)A$. Professor Fichera

leaves the proof of his formula to the reader so that a complete discussion of his results must be postponed. Incidentally Professor Fichera uses the standard notion $\{p_k\}$ in computing the projection P .

Let us also note that our criterion allows us to prove, by inspection and without computation or limiting processes, the existence of systems $p_1, p_2, \dots, p_{n-1}$ for which the equality (6) holds and which are different from the classical choice $u_1, u_2, \dots, u_{n-1}$ (see [1]). In the simple case where $\lambda_1 < \lambda_2 < \lambda_3 \leq \lambda_4 \leq \dots$ and $n = 2$, we see immediately from our criterion that (6) holds if and only if $p_1 = c_1 u_1 + c_3 u_3 + c_4 u_4 + \dots$ with $c_1 \neq 0$ and $c_3, c_4, \dots$ satisfying an obvious inequality.

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