Some Remarks on Axially-Symmetric Potential Theory

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1. Introduction. These remarks are intended to simplify considerably the derivation of some integral equations which K. Srivastav and P. Nairain [1] gave recently in this journal and to make further comments on the source of such integral equations. We recall that amongst other problems, these writers were concerned with the following problem of potential theory. A circular disk is contained in a sphere which is maintained at zero potential. It is desired to find the charge density which is induced upon the disk from which, with the other boundary data we assign, it is possible to determine the potential inside of the sphere. Let it be noted that the disk was placed at the center of the sphere in [1], and use was made of the spherical harmonics to meet the boundary conditions on the disk. The present method avoids the use of spherical harmonics completely and the disk may be located at any point on the axis of the sphere. In fact, we could formulate the problem for a finite number of parallel disks with centers on the same axis or that of a spherical cap by the methods we shall employ.

The mathematical interest in this problem is that there are two competing sets of coordinates, that is, those of the sphere and those of the disk. We are mainly concerned here with the formulation of certain potential theory problems involving the sphere. It is to be remarked that the medium of special functions is neither the easiest way to get a useable integral equation nor to justify its derivation. Indeed, the recent efforts in axially-symmetric potential theory [2] would show once again that an appropriate Helmholtz representation combined with the Poisson representation would give us the desired integral equation with little effort. The term appropriate is to be construed to mean that we have used a Green's function in the Helmholtz representation that eliminates as many boundary conditions as possible. In the problems which we shall discuss, the boundary conditions on the sphere will be eliminated.

II. The disk inside of a sphere. Let us suppose that a disk of radius c is perpendicular to the z axis with its center on this axis and that this disk is

located inside of a sphere of radius a , where $c < a$. The sphere is maintained at potential zero, while the disk is maintained at an "arbitrary potential" which we shall assume for simplicity is independent of the polar angle with respect to the z axis. The geometry of the sphere makes it possible to give a particularly simple integral equation for the charge density on such a disk (or for that part on a spherical cap). As we have remarked, the effect of the spherical boundary may be absorbed into the Green's function for the sphere which vanishes on the sphere of radius a . Such a Green's function is known explicitly from the Kelvin method of inversion. A Green's function for Laplace's equation in three dimensions which vanishes on a sphere of radius $\rho = a$ is

$$G(x, y, z, \alpha, \beta, \gamma) = \frac{1}{4\pi} \left[\{(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2\}^{-1/2} - \frac{a}{\rho} \left\{ \left(x - \frac{a^2 \alpha}{\rho^2} \right)^2 + \left(y - \frac{a^2 \beta}{\rho^2} \right)^2 + \left(z - \frac{a^2 \gamma}{\rho^2} \right)^2 \right\}^{-1/2} \right]$$

where $\rho = (\alpha^2 + \beta^2 + \gamma^2)^{1/2}$. For a point $P(x, y, z)$ inside of the sphere, we have because of the boundary condition on the sphere, (that is, the potential vanishes on it) that the potential $\Phi(x, y, z)$ which satisfies the partial differential equation

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$$

has the representation

$$(2.1) \quad \Phi(x, y, z) = \iint [\Phi_n] G(x, y, z, \alpha, \beta, b) dA.$$

Here dA is the element of area on the disk and $[\Phi_n]$ is the discontinuity of the normal derivative of Φ on the disk. We have located the disk of radius c , perpendicular to the z axis at the point $x = 0, y = 0$ and $z = b$ and have assumed that $c^2 + b^2 < a^2$ (that is, the disk does not intersect the sphere). Since a regular axially-symmetric potential can be determined from a knowledge of this function on the axis of symmetry, let us examine (2.1) with this in mind. On the z axis, $x = 0, y = 0$ and we have for $|z| < a, z \neq b$

$$(2.2) \quad \Phi(0, 0, z) = \frac{1}{4\pi} \int_0^c \int_0^{2\pi} t dt d\phi [\Phi_n] \cdot \left\{ \frac{1}{[t^2 + (z - b)^2]^{1/2}} - \frac{a}{[a^4 - 2za^2b + z^2(t^2 + b^2)]^{1/2}} \right\}.$$

If, as we have assumed for simplicity, the boundary condition on the disk is independent of the polar angle with respect to the z axis, the function $\Phi(0, 0, z)$ can be identified with a Fourier coefficient [3]. We write $x = r \cos \phi$ and $y = r \sin \phi$ where ϕ is the polar angle. We have then

$$\Phi(x, y, z) = \tilde{\Phi}(r, \phi, z) = \sum_{n=1}^{\infty} \tilde{\Phi}_c^{(n)}(r, z) \cos n\phi + \sum_{n=1}^{\infty} \tilde{\Phi}_s^{(n)}(r, z) \sin n\phi + \tilde{\Phi}^{(0)}(r, z)$$

where $\tilde{\Phi}_n^{(n)}(r, z)$ and $\tilde{\Phi}_n^{(n)}(r, z) \rightarrow 0$, $r \rightarrow 0^+$, $n \geq 1$ and therefore $\tilde{\Phi}^{(0)}(0, z) = \Phi(0, 0, z)$. Similarly, the function $[\tilde{\Phi}_n^{(0)}(t)]$ is the term in the discontinuity of the normal derivative on the disk which is independent of the angle ϕ . (2.2) then simplifies to

$$\tilde{\Phi}^{(0)}(r, z) = \frac{1}{2} \int_0^c t \, dt [\Phi_n^{(0)}(t)] \left\{ \frac{1}{[t^2 + (z - b)^2]^{1/2}} - \frac{a}{[a^4 - 2a^2b + z^2(t^2 + b^2)]^{1/2}} \right\}.$$

The Poisson representation states that a regular axially-symmetric potential which is independent of the angle ϕ may be written in terms of axis data, that is

$$\tilde{\Phi}^{(0)}(r, z) = \frac{1}{\pi} \int_0^\pi \tilde{\Phi}^{(0)}(0, z + ir \cos \psi) \, d\psi.$$

Hence if $\tilde{\Phi}^{(0)}(r, b)$ is assigned on the disk, we have

$$\begin{aligned} (2.3) \quad \tilde{\Phi}^{(0)}(r, b) &= \frac{1}{\pi} \int_0^\pi \tilde{\Phi}^{(0)}(0, b + ir \cos \psi) \, d\psi \\ &= \frac{1}{\pi} \int_{-r}^r \frac{\tilde{\Phi}^{(0)}(0, b + it) \, dt}{(r^2 - t^2)^{1/2}} \\ &= \frac{1}{\pi} \int_0^r \frac{[\tilde{\Phi}^{(0)}(0, b + it) + \tilde{\Phi}^{(0)}(0, b - it)] \, dt}{(r^2 - t^2)^{1/2}} \\ &= \frac{2}{\pi} \int_0^r \frac{\operatorname{Re} \tilde{\Phi}^{(0)}(0, b + it) \, dt}{(r^2 - t^2)^{1/2}} \end{aligned}$$

since $\tilde{\Phi}^{(0)}(0, \lambda)$ is real for real λ . Upon inverting (2.3) we get that

$$(2.4) \quad \operatorname{Re} \tilde{\Phi}^{(0)}(0, b + ir) = \frac{d}{dr} \int_0^r \frac{\tilde{\Phi}^{(0)}(t, b) \, t \, dt}{(r^2 - t^2)^{1/2}}$$

which is known when $0 < r < c$.

On the other hand when $0 < r < c$, we have that if $z \rightarrow ir + b^+$

$$\begin{aligned} \tilde{\Phi}^{(0)}(0, b + ir) &= \frac{1}{2} \int_0^r \frac{it[\tilde{\Phi}_n^{(0)}(t)] \, dt}{(r^2 - t^2)^{1/2}} + \frac{1}{2} \int_r^c \frac{t[\tilde{\Phi}_n^{(0)}(t)] \, dt}{(t^2 - r^2)^{1/2}} \\ &\quad - \frac{a}{2} \int_0^c \frac{t[\tilde{\Phi}_n^{(0)}(t)] \, dt}{[a^4 - 2a^2b(b + ir) + (b + ir)^2(t^2 + b^2)]^{1/2}} \end{aligned}$$

while for $0 < -r < c$, $z \rightarrow ir + b^+$, we get

$$\begin{aligned} \tilde{\Phi}^{(0)}(0, b + ir) &= -\frac{i}{2} \int_0^{-r} \frac{t[\tilde{\Phi}_n^{(0)}(t)] \, dt}{(r^2 - t^2)^{1/2}} + \frac{1}{2} \int_{-r}^c \frac{t[\tilde{\Phi}_n^{(0)}(t)] \, dt}{(t^2 - r^2)^{1/2}} \\ &\quad - \frac{a}{2} \int_0^c \frac{t[\tilde{\Phi}_n^{(0)}(t)] \, dt}{[a^4 - 2a^2b(b + ir) + (b + ir)^2(t^2 + b^2)]^{1/2}}. \end{aligned}$$

Hence

$$(2.5) \quad \tilde{\Phi}^{(0)}(0, b + ir) + \tilde{\Phi}^{(0)}(0, b - ir) = \int_r^c \frac{t[\tilde{\Phi}_n^{(0)}(t)] dt}{(t^2 - r^2)^{1/2}} \\ - a \int_0^c t[\tilde{\Phi}_n^{(0)}(t)] \operatorname{Re} \frac{dt}{[a^4 - 2a^2b(b + ir) + (b + ir)^2(t^2 + b^2)]^{1/2}}$$

$0 < r < c$. The same equation arises when $z \rightarrow ir + b^-$, although there are some changes in signs in the component parts from which it is derived.

The left side of this equation is known from (2.4) and we shall simplify its right side. Let us put

$$\int_r^c \frac{t[\tilde{\Phi}_n^{(0)}(t)] dt}{(t^2 - r^2)^{1/2}} = \Psi(r).$$

Then

$$[\tilde{\Phi}_n^{(0)}(r)] = -\frac{2}{\pi} \int_r^c \frac{\Psi'(t) dt}{(t^2 - r^2)^{1/2}} + \frac{2}{\pi} \frac{\Psi(c)}{(c^2 - r^2)^{1/2}}, \quad 0 < r < c.$$

(Here we have made essential use of the edge condition for the function $[\tilde{\Phi}_n^{(0)}]$ at $r = c^-$. That is, its leading term is of the form $(c^2 - r^2)^{-1/2}$, $r \rightarrow c^-$ and successive terms are of lower order in the neighborhood of this point. Though we used this notion in [2], we did not state there the reason why $\Psi(c) \neq 0$.) Hence (2.5) becomes

$$2 \operatorname{Re} \tilde{\Phi}^{(0)}(0, b + ir) \\ = \Psi(r) + \frac{2a}{\pi} \int_0^c \operatorname{Re} \frac{t dt}{[a^4 - 2a^2b(b + ir) + (t^2 + b^2)(b + ir)^2]^{1/2}} \\ \cdot \left[\int_r^c \frac{\Psi'(\lambda) d\lambda}{(\lambda^2 - t^2)^{1/2}} - \frac{\Psi(c)}{(c^2 - t^2)^{1/2}} \right] \\ = \Psi(r) + \frac{2a}{\pi} \operatorname{Re} \int_0^c \Psi'(\lambda) d\lambda \\ \cdot \int_0^\lambda \frac{t dt}{[(\lambda^2 - t^2)(a^4 - 2a^2b(b + ir) + (t^2 + b^2)(b + ir)^2)]^{1/2}} \\ - \frac{2a}{\pi} \operatorname{Re} \int_0^c \frac{\Psi(c) t dt}{[(c^2 - t^2)(a^4 - 2a^2b(b + ir) + (t^2 + b^2)(b + ir)^2)]^{1/2}} \\ = \Psi(r) + \frac{2a}{\pi} \operatorname{Re} \int_0^c \frac{\Psi'(\lambda) d\lambda}{b + ir} \arctan \frac{\lambda(b + ir)}{a^2 - b^2 - bir} \\ - \frac{2a}{\pi} \operatorname{Re} \frac{\Psi(c)}{b + ir} \arctan \frac{c(b + ir)}{a^2 - b^2 - bir}.$$

Upon integrating the last integral by parts we get

$$2 \operatorname{Re} \tilde{\Phi}^{(0)}(0, b + ir) = \Psi(r) - \frac{2a}{\pi} \operatorname{Re} \int_0^c \frac{[a^2 - b^2 - bir]\Psi(\lambda) d\lambda}{(a^2 - b^2 - bir)^2 + \lambda^2(b + ir)^2}.$$

This integral contains the one found by Srivastav and Nairain in the limiting case $b = 0$, that is

$$2 \operatorname{Re} \tilde{\Phi}^{(0)}(0, ir) = \Psi(r) - \frac{2a^3}{\pi} \int_0^c \frac{\Psi(\lambda) d\lambda}{a^4 - \lambda^2 r^2}, \quad 0 \leq r \leq c.$$

The left side of this equation will be a continuous function of r if we assume that $\tilde{\Phi}^{(0)}(r, b)$ possesses a first continuous derivative in $0 \leq r \leq c$.

III. A mixed boundary-value problem involving a hemi-sphere. Let us now examine the problem in which a hemi-sphere of radius a is maintained at potential zero, while the intersecting plane $z = 0$ is maintained at potential zero in the ring $b < r < a$ while the charge density on the circle $z = 0$, $0 < r < b$ is assigned (that is, essentially the normal derivative on that circle). As we noted in Section II

$$(3.1) \quad \Phi(x, y, z) = \iint_A [G\Phi_n - \Phi G_n] dA$$

where A is the surface of the hemi-sphere of radius a and the intersecting plane $z = 0$. The outer normal derivative is understood. For G , the Green's function, we take

$$\begin{aligned} G(x, y, z, \alpha, \beta, \gamma) = \frac{1}{4\pi} & \left[\{(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2\}^{-1/2} \right. \\ & - \{(x - \alpha)^2 + (y - \beta)^2 + (z + \gamma)^2\}^{-1/2} \\ & - \frac{a}{\rho} \left\{ \left(x - \frac{a^2\alpha}{\rho^2}\right)^2 + \left(y - \frac{a^2\beta}{\rho^2}\right)^2 + \left(z - \frac{a^2\gamma}{\rho^2}\right)^2 \right\}^{-1/2} \\ & \left. + \frac{a}{\rho} \left\{ \left(x - \frac{a^2\alpha}{\rho^2}\right)^2 + \left(y - \frac{a^2\beta}{\rho^2}\right)^2 + \left(z + \frac{a^2\gamma}{\rho^2}\right)^2 \right\}^{-1/2} \right] \end{aligned}$$

where now this Green's function vanishes on the sphere $(\alpha^2 + \beta^2 + \gamma^2)^{1/2} = a$ and on the plane $\gamma = 0$. In the light of the boundary conditions on the function Φ , there are no contributions from the hemi-sphere and we are only left with an integral over the plane $\gamma = 0$. Since $G = 0$ on this plane, (3.1) reduces to

$$\Phi(x, y, z) = \iint \Phi(t, 0, \phi') t \frac{\partial G}{\partial \gamma} dt d\phi'$$

where $\alpha = t \cos \phi'$, $\beta = t \sin \phi'$ and $\partial G / \partial \gamma$ is evaluated at $\gamma = 0$. Hence (3.1) becomes on the axis $r = (x^2 + y^2)^{1/2} = 0$

$$(3.2) \quad \begin{aligned} & \Phi(0, 0, z) \\ &= \frac{1}{2\pi} \int_0^b \int_0^{2\pi} [z(z^2 + t^2)^{-3/2} - a^3 z(a^4 + z^2 t^2)^{-3/2}] \Phi(t \cos \phi', t \sin \phi') dt d\phi', \end{aligned}$$

since now $t^2 = \alpha^2 + \beta^2$ and $\Phi(t \cos \phi', t \sin \phi') = 0$ for $b < t < a$. In terms of the Fourier coefficients which we described in Section II, (3.2) simplifies to

$$(3.3) \quad \tilde{\Phi}^{(0)}(0, z) = \int_0^b tz[(t^2 + z^2)^{-3/2} - a^3(a^4 + z^2 t^2)^{-3/2}] \tilde{\Phi}^{(0)}(t, 0) dt.$$

Before we can continue (3.3) into the domain of the complex variable $z = \xi + i\eta$, it is necessary to reduce the order of the singularity induced in the first term under the integral sign, so that integrability properties will be available. This can be accomplished by integrating the first term in (3.3) by parts to get (Here again, we make essential use of an edge condition, that is, $d\tilde{\Phi}^{(0)}/dt$ has a property similar to $[\tilde{\Phi}_n^{(0)}]$ in the neighborhood of the point $t \rightarrow c^-$.)

$$(3.4) \quad \begin{aligned} \tilde{\Phi}^{(0)}(0, z) &= -z(z^2 + b^2)^{-1/2} \tilde{\Phi}^{(0)}(b, 0) + \tilde{\Phi}^{(0)}(0, 0) \\ &+ z \int_0^b (t^2 + z^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt - a^3 z \int_0^b t(a^4 + z^2 t^2)^{-3/2} \tilde{\Phi}^{(0)}(t, 0) dt. \end{aligned}$$

Now if we put $z = \xi + i\eta$ and let $\xi \rightarrow 0^+$, $0 < \eta < b$ we obtain

$$(3.5) \quad \begin{aligned} \tilde{\Phi}^{(0)}(0, i\eta) &= -i\eta \tilde{\Phi}^{(0)}(b, 0)(b^2 - \eta^2)^{-1/2} + \tilde{\Phi}^{(0)}(0, 0) \\ &- \int_0^\eta \eta(\eta^2 - t^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt + i\eta \int_\eta^b (t^2 - \eta^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt \\ &- ia^3 \eta \int_0^b t(a^4 - t^2 \eta^2)^{-3/2} \tilde{\Phi}^{(0)}(t, 0) dt. \end{aligned}$$

On the other hand if $z = \xi + i\eta$, $0 < -\eta < b$, $\xi \rightarrow 0^+$ we get

$$(3.6) \quad \begin{aligned} \tilde{\Phi}^{(0)}(0, i\eta) &= -i\eta(b^2 - \eta^2)^{-1/2} \tilde{\Phi}^{(0)}(b, 0) + \tilde{\Phi}^{(0)}(0, 0) \\ &+ \int_0^{-\eta} \eta(\eta^2 - t^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt + \int_{-\eta}^b i\eta(t^2 - \eta^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt \\ &- ia^3 \eta \int_0^b t(a^4 - t^2 \eta^2)^{-3/2} \tilde{\Phi}^{(0)}(t, 0) dt. \end{aligned}$$

We now replace η by $-\eta$ in (3.6) and subtract it from (3.5) to get

$$(3.7) \quad \begin{aligned} \tilde{\Phi}^{(0)}(0, i\eta) - \tilde{\Phi}^{(0)}(0, -i\eta) &= -2i\eta(b^2 - \eta^2)^{-1/2} \tilde{\Phi}^{(0)}(b, 0) \\ &+ 2i\eta \int_\eta^b (t^2 - \eta^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}}{dt} dt - 2ia^3 \eta \int_0^b t(a^4 - t^2 \eta^2)^{-3/2} \tilde{\Phi}^{(0)}(t, 0) dt. \end{aligned}$$

Two tasks now remain. The first is to account for the left side of equation (3.7) and the second is to convert (3.7) into a more useable form. We proceed with the first task by noting that we have not invoked the boundary condition on the disk, that is, that $\partial\tilde{\Phi}^{(0)}/\partial t = m(r)$ is assigned at $z = 0$, $0 \leq r \leq b$. Let us recall that the Poisson representation states that

$$\begin{aligned} \tilde{\Phi}^{(0)}(r, z) &= \frac{1}{\pi} \int_0^\pi \tilde{\Phi}^{(0)}(0, z + ir \cos \psi) d\psi \\ &= \frac{1}{\pi} \int_0^r \frac{[\tilde{\Phi}^{(0)}(0, z + it) + \tilde{\Phi}^{(0)}(0, z - it)]}{(r^2 - t^2)^{1/2}} dt. \end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial \tilde{\Phi}^{(0)}}{\partial z} &= \frac{1}{\pi} \int_0^r \left[\frac{\partial \tilde{\Phi}^{(0)}(0, z + it)}{\partial z} + \frac{\partial \tilde{\Phi}^{(0)}(0, z - it)}{\partial z} \right] (r^2 - t^2)^{-1/2} dt \\ &= \frac{1}{i\pi} \int_0^r \left[\frac{\partial \tilde{\Phi}^{(0)}(0, z + it)}{\partial t} - \frac{\partial \tilde{\Phi}^{(0)}(0, z - it)}{\partial t} \right] (r^2 - t^2)^{-1/2} dt.\end{aligned}$$

This last integral may be inverted to give at $z = 0$

$$-i[\tilde{\Phi}^{(0)}(0, ir) - \tilde{\Phi}^{(0)}(0, -ir)] = 2 \int_0^r tm(t)(r^2 - t^2)^{-1/2} dt,$$

if we observe that $\tilde{\Phi}^{(0)}(0, ir)$ is continuous on the line $r = 0$. The left side of equation (3.6) is therefore known for $0 \leq r \leq b$.

In order to simplify the right side of equation (3.7), we put

$$-\tilde{\Phi}^{(0)}(b, 0)(b^2 - \eta^2)^{-1/2} + \int_\eta^b (t^2 - \eta^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt = \Psi(\eta).$$

From this, we can derive by inversion of the above integral

$$\tilde{\Phi}^{(0)}(\eta, 0) = -\frac{2}{\pi} \int_\eta^b t\Psi(t)(t^2 - \eta^2)^{-1/2} dt.$$

Hence (3.7) becomes

$$\begin{aligned}\tilde{\Phi}^{(0)}(0, i\eta) - \tilde{\Phi}^{(0)}(0, -i\eta) &= 2i\eta\Psi(\eta) + \frac{4ia^3\eta}{\pi} \int_0^b \frac{t dt}{(a^4 - \eta^2 t^2)^{3/2}} \int_t^b \frac{\Psi(\tau)\tau d\tau}{(\tau^2 - t^2)^{1/2}} \\ &= 2i\eta\Psi(\eta) + \frac{4ia^3\eta}{\pi} \int_0^b \frac{\tau^2\Psi(\tau) d\tau}{(a^4 - \eta^2 \tau^2)}\end{aligned}$$

and therefore

$$\int_0^\eta \frac{tm(t) dt}{(\eta^2 - t^2)^{1/2}} = \eta\Psi(\eta) + \frac{2a\eta}{\pi} \int_0^b \frac{\tau^2\Psi(\tau) d\tau}{(a^4 - \eta^2 \tau^2)}, \quad 0 \leq \eta \leq b,$$

which is a regular Fredholm integral equation of the second kind with a symmetric kernel for the function $\eta\Psi(\eta)$. Let us note that if $m(t)$ is a continuous function in $0 \leq t \leq b$, the integral on the left side of this equation will also be a continuous function of η in the same interval and indeed it will be $O(\eta)$, $\eta \rightarrow 0$.

IV. A hemi-sphere on an infinite plane. The following problem leads to much the same integral equation which we encountered in Section III, although a modification of the Poisson representation is required. Let us suppose that the sphere of radius a and center at the origin is maintained at zero potential and that on a portion of the plane $z = 0$, $(x^2 + y^2)^{1/2} = r > a$ the following boundary conditions are obeyed. For $a < r < b$, $\tilde{\Phi}^{(0)}(r, 0) = 0$ and for $b < r$, $\partial \tilde{\Phi}^{(0)}/\partial z = m(r)$ is assigned as a function of r at $z = 0$. For simplicity we shall only discuss the case for which the potential is independent of the angular

dependence ϕ about the z axis. With the assumption that $\tilde{\Phi}^{(0)}(r, 0) = O(1/r)$ $r \rightarrow \infty$, we get the following representation on the axis $r = 0$ upon using the same Green's function which we employed in Section III; that is,

$$\tilde{\Phi}^{(0)}(0, z) = \int_b^\infty z t \tilde{\Phi}^{(0)}(t, 0) [(t^2 + z^2)^{-3/2} - a^3(a^4 + t^2 z^2)^{-3/2}] dt, \quad z > 0.$$

We integrate the first term by parts for the same reason which we gave in Section III and note that $\lim_{t \rightarrow \infty} \tilde{\Phi}^{(0)}(t, 0) = 0$ to get

$$(4.1) \quad \begin{aligned} \tilde{\Phi}^{(0)}(0, z) &= z \tilde{\Phi}^{(0)}(b, 0)(b^2 + z^2)^{-1/2} \\ &+ \int_b^\infty z(t^2 + z^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt - a^3 z \int_b^\infty t(a^4 + t^2 z^2)^{-3/2} \tilde{\Phi}^{(0)}(t, 0) dt. \end{aligned}$$

Now we continue (4.1) into the domain of the complex variable $z = \xi + i\eta$, $\xi \rightarrow 0^+$, $\eta > b$ and obtain

$$(4.2) \quad \begin{aligned} \tilde{\Phi}^{(0)}(0, i\eta) &= -\eta \tilde{\Phi}^{(0)}(b, 0)(\eta^2 - b^2)^{-1/2} \\ &- \int_b^\eta \eta(\eta^2 - t^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt \\ &+ i\eta \int_\eta^\infty (t^2 - \eta^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt + a^3 \eta \int_b^\infty t(t^2 \eta^2 - a^4)^{-3/2} \tilde{\Phi}^{(0)}(t, 0) dt \end{aligned}$$

while for $\xi \rightarrow 0^+$, $\eta < -b$ we get

$$(4.3) \quad \begin{aligned} \tilde{\Phi}^{(0)}(0, i\eta) &= \eta \tilde{\Phi}^{(0)}(b, 0)(\eta^2 - b^2)^{-1/2} + \int_b^{-\eta} \eta(\eta^2 - t^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt \\ &+ i\eta \int_{-\eta}^\infty (t^2 - \eta^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt - a^3 \eta \int_b^\infty t(t^2 \eta^2 - a^4)^{-3/2} \tilde{\Phi}^{(0)}(t, 0) dt, \end{aligned}$$

Upon replacing η by $-\eta$ in (4.3) and adding it to (4.2) we get

$$(4.4) \quad \begin{aligned} \tilde{\Phi}^{(0)}(0, i\eta) + \tilde{\Phi}^{(0)}(0, -i\eta) &= -2\eta(\eta^2 - b^2)^{-1/2} \tilde{\Phi}^{(0)}(b, 0) \\ &- 2\eta \int_b^\eta (\eta^2 - t^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt + 2a^3 \int_b^\infty \eta t(t^2 \eta^2 - a^4)^{-3/2} \tilde{\Phi}^{(0)}(t, 0) dt. \end{aligned}$$

If we put

$$\Psi(\eta) = \tilde{\Phi}^{(0)}(b, 0)(\eta^2 - b^2)^{-1/2} + \int_b^\eta (\eta^2 - t^2)^{-1/2} \frac{d\tilde{\Phi}^{(0)}(t, 0)}{dt} dt$$

we find that

$$\tilde{\Phi}^{(0)}(\eta, 0) = \frac{2}{\pi} \int_b^\eta t(\eta^2 - t^2)^{-1/2} \Psi(t) dt$$

and hence

$$\begin{aligned}
(4.5) \quad \tilde{\Phi}^{(0)}(0, i\eta) + \tilde{\Phi}^{(0)}(0, -i\eta) &= -2\eta\Psi(\eta) \\
&\quad + \frac{4a^3\eta}{\pi} \int_b^\infty t(t^2\eta^2 - a^4)^{-3/2} dt \int_b^t \lambda(t^2 - \lambda^2)^{-1/2} \Psi(\lambda) d\lambda \\
&= -2\eta\Psi(\eta) + \frac{4a^3}{\pi} \int_b^\infty \lambda\Psi(\lambda) d\lambda \int_\lambda^\infty t(t^2 - \lambda^2)^{-1/2} (t^2\eta^2 - a^4)^{-3/2} dt \\
&= -2\eta\Psi(\eta) + \frac{4a^3}{\pi} \int_b^\infty \lambda(\lambda^2\eta^2 - a^4)^{-1} \Psi(\lambda) d\lambda, \quad \eta > b.
\end{aligned}$$

This leaves us with the problem of determining $\tilde{\Phi}^{(0)}(0, i\eta) + \tilde{\Phi}^{(0)}(0, -i\eta)$, $\eta > b$. Let us recall that we can express $\Phi^{(0)}(r, z)$ in terms of the discontinuity of the normal derivative of $\tilde{\Phi}^{(0)}(r, z)$ by changing the form of the Green's function. That is, if we take a Green's function of the form

$$\begin{aligned}
G^{(1)}(x, y, z, \alpha, \beta, \gamma) &= \frac{1}{4\pi} \{[(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2]^{-1/2} \\
&\quad + [(x - \alpha)^2 + (y - \beta)^2 + (z + \gamma)^2]^{-1/2} \\
&\quad - a\rho^{-1}[(x - a^2\rho^{-2}\alpha)^2 + (y - a^2\rho^{-2}\beta)^2 + (z - a^2\rho^{-2}\gamma)^2]^{-1/2} \\
&\quad - a\rho^{-1}[(x - a^2\rho^{-2}\alpha)^2 + (y - a^2\rho^{-2}\beta)^2 + (z + a^2\rho^{-2}\gamma)^2]^{-1/2}\},
\end{aligned}$$

we have, since $G = 0$ on the hemi-sphere of radius a and $\partial G/\partial z$ on the plane $z = 0$, that

$$\tilde{\Phi}^{(0)}(0, z) = -\int_a^\infty tm(t)[(t^2 + z^2)^{-1/2} - a(t^2z^2 + a^4)^{-1/2}] dt$$

where $m(t) = \partial\tilde{\Phi}^{(0)}/\partial\gamma$, $\gamma = 0$. Now we recognize that for $z = x + iy$, $x \rightarrow 0^+$, $y > b$, we have

$$\begin{aligned}
\tilde{\Phi}^{(0)}(0, iy) &= -i \int_b^y t(y^2 - t^2)^{-1/2} m(t) dt \\
&\quad - \int_y^\infty t(t^2 - y^2)^{-1/2} m(t) dt - ai \int_b^\infty t(t^2y^2 - a^4)^{-1/2} m(t) dt
\end{aligned}$$

from which we derive

$$\operatorname{Re} \tilde{\Phi}^{(0)}(0, iy) = -\int_y^\infty t(t^2 - y^2)^{-1/2} m(t) dt.$$

If $y > b$, $m(t)$ is known and therefore the left side of (4.5). Conditions on $m(t)$ which will insure the convergence of this integral can be given without any trouble.

We have tried to emphasize in the past [2] that the formation of integral equations of the type which have been described above, is due to two main

features. The first is that we have the Poisson representation, while the second is due to the fact that if we use a Helmholtz representation with a Green's function which eliminates as many boundary terms as possible, both representations may be combined as we have described here and elsewhere. Let us observe that the form of the Green's function we require, is the one which displays explicitly the fundamental solution.

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