A Remark on the Singular Cauchy Problem, for all Values of the Time, for the Euler-Poisson-Darboux Equation

J. B. DIAZ* & ABDUL R. KIWAN

Abstract. The purpose of the present note is to simplify the treatment of J. B. Diaz and G. S. S. Ludford for obtaining a solution, for all values of the time, to the singular Cauchy problem for the Euler-Poisson-Darboux equation for values of the parameter $k > m - 1$. It is verified directly that a certain multiple integral involving the initial value function $f(x_1, x_2, \dots, x_m)$ provides a solution to the problem. The treatment given here is a modification of an argument due to A. Weinstein. Weinstein's proof requires that f be of class C^3 , while the present discussion requires only that f be of class C^2 .

1. Introduction. The singular second order partial differential equation

$$(1) \quad \Delta u = \frac{\partial^2 u}{\partial t^2} + \frac{k}{t} \frac{\partial u}{\partial t},$$

where k is a parameter,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_m^2}$$

is the Laplacian, and $u(x_1, x_2, \dots, x_m, t)$ is a real valued function, is known as the Euler-Poisson-Darboux equation (abbreviated EPD). This singular partial differential equation has occurred, for special values of k and m , in many important and classical problems since the time of Euler.

The singular Cauchy problem (one sided, *i.e.*, for $t > 0$) for the EPD equation requires the determination of a real valued function $u(x_1, \dots, x_m, t)$ which satisfies (1) for $t > 0$, and meets the initial conditions

$$(2) \quad u(x_1, \dots, x_m, 0) = f(x_1, \dots, x_m), \quad \frac{\partial u}{\partial t}(x_1, \dots, x_m, 0) = 0,$$

* The research of this author was supported by Grant AFOSR 400-64 from the Air Force Office of Scientific Research to the University of Maryland and by Grant AFOSR 1122-66 to the University of California, Riverside, which is his present address.

where f is a given real valued function defined for all values of the real variables $x_1, \dots, x_m$. (When $k = 0$ the EPD equation (1) reduces to the wave equation in m space dimensions; while for $k \neq 0$ the coefficient kt^{-1} is infinite on the plane $t = 0$, the carrier of the Cauchy data.) For k any real number, this singular Cauchy problem (1), (2) was first solved by A. Weinstein [1], [2], who employed what he termed the "method of recurrence" and a generalized method of descent. Weinstein's method is based upon the known solution of the (one sided) Cauchy problem (1), (2) for $k = m - 1$:

$$(3) \quad u(x, t) = \frac{1}{\omega_m} \iint_{\sum_{i=1}^m \alpha_i^2 = 1} f(x + t\alpha) d\omega_m,$$

where $x = (x_1, \dots, x_m)$, $\alpha = (\alpha_1, \dots, \alpha_m)$, $d\omega_m$ is the surface element of the unit sphere in m dimensional Euclidean space, and

$$\omega_m = \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}$$

is the surface area of this sphere; and on the explicit formula

$$(4) \quad u(x, t) = \frac{\omega_{k+1-m}}{\omega_{k+1}} \iint_{\sum_{i=1}^m \alpha_i^2 < 1} f(x + \alpha t)(1 - \alpha^2)^{(k-m-1)/2} d\alpha,$$

where $\alpha^2 = \sum_{i=1}^m \alpha_i^2$ and $d\alpha = d\alpha_1 \cdots d\alpha_m$, for the solution of the problem (1), (2) for $k > m - 1$.

J. B. Diaz and H. F. Weinberger [3] obtained a solution of the same Cauchy problem (1), (2), using again as a starting point the same explicit formula (4) for the solution for $k > m - 1$ which was employed by Weinstein, and then continuing this definite integral (4) analytically with respect to the parameter k .

The singular Cauchy problem (two sided, *i.e.*, for $-\infty < t < +\infty$) for the EPD equation (1) consists in the determination of a real valued function $u(x_1, \dots, x_m, t)$ which satisfies (1) for $t \neq 0$, satisfies the initial conditions (2), and which, on the singular plane $t = 0$, satisfies "the limiting form, as t approaches zero, of (1)," namely

$$(5) \quad \Delta u = (k + 1) \frac{\partial^2 u}{\partial t^2}.$$

This problem was formulated by J. B. Diaz and G. S. S. Ludford [4]. The method of solution of (1), (2), for $-\infty < t < +\infty$, given in [4], is based upon showing that the formula (4) yields a solution, for all time, for $k > m - 1$. This is done (see p. 78 of [4]), by first transforming the definite integral with fixed limits of integration appearing in (4) to an integral with variable limits of integration (see equation (17) on p. 78 of [4]).

The purpose of the present note is to simplify the treatment of [4] by verifying directly that the formula (4) gives a solution of the singular Cauchy problem (1), (2), for all time, for $k > m - 1$, thus avoiding all the possible complications arising from the detailed justification of the differentiation of a multiple integral with variable limits of integration, such as that in equation (17) on page 78 of [4]. The direct verification for $k > m - 1$ given here is a modification of that given by Weinstein in section 6 of [2]. However, Weinstein's argument requires that f be of class C^3 , while the present argument requires only that f be of class C^2 . Intuitively speaking, the present verification bears the same relation to Weinstein's verification as Du Bois Reymond's Lemma bears to the "Fundamental Lemma" in the calculus of variations (see *e.g.*, Bolza [5, pp. 20-25]).

2. Verification for $k > m - 1$. Formula (4) may be written:

$$(6) \quad u(x, t) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{m/2}\Gamma\left(\frac{k+1-m}{2}\right)} \iint_{\sum_{i=1}^m \alpha_i^2 < 1} f(x + \alpha t)(1 - \alpha^2)^{(k-m-1)/2} d\alpha,$$

where the integral in (6) converges for all $k > m - 1$. Furthermore, if, given f , one defines $u(x, t)$ by means of the right hand side of (6) for these values of k and all values of (x, t) , then $u(x, t) \in C^2(x, t)$ whenever $f \in C^2(x)$. In fact, if $f \in C^r(x)$, then $u(x, t) \in C^r(x, t)$ and is an even function of t . Putting $t = 0$ in (6), and performing the integration, one obtains:

$$u(x, 0) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{m/2}\Gamma\left(\frac{k+1-m}{2}\right)} \int_0^1 f(x)\omega_m(1 - \rho^2)^{(k-m-1)/2} \rho^{m-1} d\rho,$$

i.e.,

$$u(x, 0) = f(x).$$

Furthermore, since $u(x, t) \in C^2(x, t)$, and u is an even function of t ,

$$u_t(x, 0) = 0.$$

Thus, the initial conditions (2) are satisfied. It remains to verify that, for $t \neq 0$, (1) is satisfied, and that for $t = 0$ the limiting form (5) holds. Let

$$(7) \quad v(x, t) = \iint_{\sum_{i=1}^m \alpha_i^2 < 1} f(x + \alpha t)(1 - \alpha^2)^{(k-m-1)/2} d\alpha,$$

hence

$$u(x, t) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{m/2}\Gamma\left(\frac{k+1-m}{2}\right)} v(x, t).$$

Differentiating (7) with respect to t gives

$$(8) \quad v_i(x, t) = \sum_{i=1}^m \iint_{\alpha^2 < 1} f_i \cdot \alpha_i (1 - \alpha^2)^{(k-m-1)/2} d\alpha,$$

where

$$f_i = \frac{\partial f}{\partial x_i} = \frac{1}{t} \frac{\partial f}{\partial \alpha_i} \quad t \neq 0.$$

Letting $R^2 = \alpha^2 - \alpha_i^2$, and integrating (8) by parts with respect to α_i , one obtains

$$\begin{aligned} v_i(x, t) &= \frac{1}{t} \sum_{i=1}^m \iint_{R^2 < 1} d\alpha_1 \cdots d\alpha_{i-1} d\alpha_{i+1} \cdots d\alpha_m \int_{-\sqrt{1-R^2}}^{\sqrt{1-R^2}} \frac{\partial f}{\partial \alpha_i} \alpha_i (1 - \alpha^2)^{(k-m-1)/2} d\alpha_i \\ &= \frac{1}{t} \sum_{i=1}^m \iint_{R^2 < 1} d\alpha_1 \cdots d\alpha_{i-1} d\alpha_{i+1} \cdots d\alpha_m \left[f \alpha_i (1 - \alpha^2)^{(k-m-1)/2} \right]_{\alpha_i = -\sqrt{1-R^2}}^{\alpha_i = \sqrt{1-R^2}} \\ &\quad - \int_{-\sqrt{1-R^2}}^{\sqrt{1-R^2}} f \cdot (1 - \alpha^2)^{(k-m-1)/2} d\alpha_i + (k-m-1) \int_{-\sqrt{1-R^2}}^{\sqrt{1-R^2}} f \cdot \alpha_i^2 (1 - \alpha^2)^{(k-m-3)/2} d\alpha_i \Big]. \end{aligned}$$

For $k > m + 1$, the “boundary” integrals vanish, and one gets

$$v_i(x, t) = -\frac{m}{t} \iint_{\alpha^2 < 1} f (1 - \alpha^2)^{(k-m-1)/2} d\alpha + \frac{k-m-1}{t} \iint_{\alpha^2 < 1} f \alpha^2 (1 - \alpha^2)^{(k-m-3)/2} d\alpha.$$

Substituting $-(1 - \alpha^2) - 1$ for α^2 in the second integral, and using (7), one obtains

$$(9) \quad v_i(x, t) = -\frac{k-1}{t} v(x, t) + \frac{k-m-1}{t} \iint_{\alpha^2 < 1} f (1 - \alpha^2)^{(k-m-3)/2} d\alpha.$$

Another differentiation with respect to t gives

$$\begin{aligned} v_{ii}(x, t) &= \frac{k-1}{t^2} v(x, t) - \frac{k-1}{t} v_i(x, t) - \frac{k-m-1}{t^2} \iint_{\alpha^2 < 1} f \cdot (1 - \alpha^2)^{(k-m-3)/2} d\alpha \\ &\quad + \frac{k-m-1}{t} \iint_{\alpha^2 < 1} \sum_{i=1}^m f_i \alpha_i (1 - \alpha^2)^{(k-m-3)/2} d\alpha. \end{aligned}$$

Using (9), it follows that

$$v_{ii}(x, t) + \frac{k}{t} v_i(x, t) = \sum_{i=1}^m \frac{k-m-1}{t} \iint_{\alpha^2 < 1} f_i \alpha_i (1 - \alpha^2)^{(k-m-3)/2} d\alpha.$$

This equation may be rewritten thus

$$\begin{aligned}
v_{it} + \frac{k}{t} v_i &= - \sum_{i=1}^m \frac{1}{t} \iint_{\alpha^2 < 1} \frac{\partial f}{\partial x_i} \cdot \frac{\partial}{\partial \alpha_i} (1 - \alpha^2)^{(k-m-1)/2} d\alpha \\
&= - \sum_{i=1}^m \frac{1}{t} \iint_{R^2 < 1} d\alpha_1 \cdots d\alpha_{i-1} d\alpha_{i+1} \cdots d\alpha_m \int_{-\sqrt{1-R^2}}^{\sqrt{1-R^2}} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial \alpha_i} (1 - \alpha^2)^{(k-m-1)/2} d\alpha_i.
\end{aligned}$$

An integration by parts with respect to α_i yields

$$\begin{aligned}
v_{it} + \frac{k}{t} v_i &= - \sum_{i=1}^m \frac{1}{t} \iint_{R^2 < 1} d\alpha_1 \cdots d\alpha_{i-1} d\alpha_{i+1} \cdots d\alpha_m \\
&\quad \cdot \left[\frac{\partial f}{\partial x_i} (1 - \alpha^2)^{(k-m-1)/2} \right]_{-\sqrt{1-R^2}}^{\sqrt{1-R^2}} - \int_{-\sqrt{1-R^2}}^{\sqrt{1-R^2}} \frac{\partial^2 f}{\partial \alpha_i \partial x_i} (1 - \alpha^2)^{(k-m-1)/2} d\alpha_i.
\end{aligned}$$

Again, the "boundary" integrals vanish, and one obtains

$$v_{it} + \frac{k}{t} v_i = \sum_{i=1}^m \iint_{\alpha^2 < 1} \frac{1}{t} \frac{\partial^2 f}{\partial \alpha_i \partial x_i} (1 - \alpha^2)^{(k-m-1)/2} d\alpha.$$

However, since

$$\frac{1}{t} \frac{\partial^2 f}{\partial \alpha_i \partial x_i} = \frac{\partial^2 f}{\partial x_i^2},$$

one obtains

$$v_{it} + \frac{k}{t} v_i = \sum_{i=1}^m \iint_{\alpha^2 < 1} \frac{\partial^2 f}{\partial x_i^2} (1 - \alpha^2)^{(k-m-1)/2} d\alpha,$$

i.e.,

$$(10) \quad v_{it} + \frac{k}{t} v_i = \Delta v.$$

But,

$$u(x, t) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{m/2} \Gamma\left(\frac{k+1-m}{2}\right)} v(x, t),$$

hence

$$-\Delta u(x, t) + \frac{\partial^2 u(x, t)}{\partial t^2} + \frac{k}{t} \frac{\partial u(x, t)}{\partial t} = 0.$$

Thus, it follows from (6) that the function $(-\Delta + \partial^2/\partial t^2 + (k/t)(\partial/\partial t))u(x, t)$ is continuous for all (x, t) with $t \neq 0$ and that for any fixed (x, t) with $t \neq 0$, the function $(-\Delta + \partial^2/\partial t^2 + (k/t)(\partial/\partial t))u(x, t)$ is analytic in k for $k > m - 1$.

Since, however, this function vanishes for $k > m + 1$, it follows that

$$(11) \quad \left(-\Delta + \frac{\partial^2}{\partial t^2} + \frac{k}{t} \frac{\partial}{\partial t} \right) u(x, t) = 0$$

for all $k > m - 1$, in view of the analyticity in k . Consider now $t = 0$. One has that

$$\lim_{t \rightarrow 0} \left(-\Delta + \frac{\partial^2}{\partial t^2} + \frac{k}{t} \frac{\partial}{\partial t} \right) u(x, t) \quad (t \neq 0)$$

exists, because $u(x, t) \in C^2(x, t)$ for all (x, t) , and also that (see (2)):

$$\lim_{t \rightarrow 0} \frac{k}{t} \frac{\partial u(x, t)}{\partial t} = k \frac{\partial^2 u(x, 0)}{\partial t^2} \quad (t \neq 0).$$

Hence, the limit of (11) as $t \rightarrow 0$ is:

$$-\Delta u(x, 0) + (k + 1)u_{,tt}(x, 0) = 0.$$

Thus, it has been proved that if $k > m - 1$, and the given function f of (2) is of class $C^2(x)$ for all $(x_1, \dots, x_m)$ then the function $u(x, t)$, as defined by (6), is of class $C^2(x, t)$, satisfies the initial conditions (2), and for $t \neq 0$ satisfies (1), while for $t = 0$ it satisfies the limiting form (5).

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Institute for Fluid Dynamics and Applied Mathematics,
University of Maryland

and

U. S. Army Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland

Date Communicated: FEBRUARY 7, 1966