On A Formula of Kac and Achiezer

I. I. HIRSCHMAN, JR.*

Communicated by M. ROSENBLATT

1. Introduction. Let $\mathbf{c}_a(\xi)$ be a complex measurable function on $-\infty < \xi < \infty$ such that

$$\int_{-r}^r |\mathbf{c}_a(\xi)|^2 d\xi < \infty,$$

for every $r > 0$. Let

$$\mathbf{c}(\xi) = \delta(\xi) - \mathbf{c}_a(\xi),$$

where $\delta(\xi)$ is the Dirac delta function and consider the finite section Toeplitz operator with $\mathbf{c}(\xi)$ as kernel which takes $\mathbf{f}(\xi)$ into

$$(1) \quad \mathbf{c} * \mathbf{f}(\xi), \quad 0 \leq \xi \leq r.$$

Here $\mathbf{f}(\xi) = \mathbf{f}_s \delta(\xi) + \mathbf{f}_a(\xi)$ where $\mathbf{f}_s$ is a complex constant and $\mathbf{f}_a \in L^1(0, r)$. It follows from the theory of Fredholm integral equations that (1) is invertible if and only if the Fredholm determinant $D(r) \neq 0$ where

$$D(r) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^r \cdots \int_0^r \det [\mathbf{c}_a(\xi_i - \xi_k)] d\xi_1 \cdots d\xi_n.$$

We now make the assumptions:

- (i) $\int_{-\infty}^{\infty} |\mathbf{c}_a(\xi)| d\xi < \infty, \quad \int_{-\infty}^{\infty} |\mathbf{c}_a(\xi)|^2 (1 + |\xi|) d\xi < \infty,$
- (ii) $c(t) \neq 0, \quad -\infty < t < \infty, \quad [\arg c(t)]_{-\infty}^{\infty} = 0.$

Here

$$c(t) = \int_{-\infty}^{\infty} \mathbf{c}(\xi) e^{i\xi t} d\xi = 1 - \int_{-\infty}^{\infty} \mathbf{c}_a(\xi) e^{i\xi t} d\xi.$$

We further assume the existence of the limits

* Research supported in part by the United States Air Force Office of Scientific Research under contract No. AF-AFOSR 721-65.

$$(iii) \quad \lim_{h \rightarrow 0^+} h^{-1} \int_0^h \mathbf{c}_a(\xi) d\xi \quad \text{and} \quad \lim_{h \rightarrow 0^+} h^{-1} \int_{-h}^0 \mathbf{c}_a(\xi) d\xi,$$

and we assume throughout that

$$\mathbf{c}_a(0) = \lim_{h \rightarrow 0^+} (2h)^{-1} \int_{-h}^h \mathbf{c}_a(\xi) d\xi.$$

This normalization does not alter the finite section Toeplitz operator defined by (1), and hence involves no loss of generality. In the present paper we will show that if (i)–(iii) are satisfied then

$$(2) \quad \lim_{r \rightarrow \infty} D(r)G^{-r} = E,$$

where

$$G = \exp \left\{ \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \left(1 - \frac{t^2}{T^2} \right)^2 \log c(t) dt \right\},$$

and

$$E = \exp \left\{ \int_0^\infty \mathbf{c}_a^\lambda(\xi) \mathbf{c}_a^\lambda(-\xi) \xi d\xi \right\}.$$

Here

$$\log c(t) = \int_{-\infty}^\infty \mathbf{c}_a^\lambda(\xi) e^{i t \xi} d\xi.$$

It can be verified that under assumptions (i)–(iii) such a representation of $\log c(t)$ exists and that

$$\int_{-\infty}^\infty |\mathbf{c}_a^\lambda(\xi)|^2 |\xi| d\xi < \infty,$$

so that E is well defined.

This subject has its origin in the 1952 paper of Szegő who found the corresponding formula for the circle group. In 1954 Kac gave a derivation of Szegő's formula under different conditions and by a new method. In this same paper he also applied his method to the case where the underlying group is the line, obtaining (2) although under rather restrictive assumptions. Recently in 1964 the subject was taken up by Achiezer. Starting from Kac's assumptions Achiezer derived (2) by very different arguments, and with a different formula for E .

I wish to thank Professor M. Kac for explaining Achiezer's work to me while we were participating in the Symposium on Toeplitz Forms and Statistical Mechanics, sponsored by the Applied Mathematics Department of Brookhaven National Laboratories in the summer of 1965.

2. Introduction continued. The methods we will use are largely parallel to those employed by Baxter [2], [3], and [4] and later the author [5], and [6] to prove Szegő's formula under very general assumptions. In the present section we will quickly sketch the proof of Szegő's formula by our arguments. This is done partly to provide an introduction to what follows and partly to provide the setting for the proof of Theorem 2a, which is new even in the circle case.

Let $c(k)$ be a given complex function defined on the integers and let

$$D(n) = \det [c(j - k)]_{j,k=0,\dots,n}, \quad n = 0, 1, \dots$$

If for a given value of n , $D(n) \neq 0$, then the two systems of equations

$$\sum_{k=0}^n c(j - k)u(n, k) = \delta(j), \quad j = 0, \dots, n.$$

and

$$\sum_{k=-n}^0 c(j - k)v(n, k) = \delta(j), \quad j = -n, \dots, 0$$

both have unique solutions. In this case let

$$u^0(n, z) = \sum_{k=0}^n u(n, k)z^k,$$

$$v^0(n, z) = \sum_{k=-n}^0 v(n, k)z^k.$$

Further let

$$u(n, \theta) = u^0(n, e^{i\theta}), \quad v(n, \theta) = v^0(n, e^{i\theta}).$$

It is quite easy to show, although we omit the details, that if $D(n)$ and $D(n - 1) \neq 0$ then

$$(1) \quad \begin{aligned} u^0(n, z) &= u^0(n - 1, z) + \alpha(n)z^n v^0(n, z), \\ v^0(n, z) &= v^0(n - 1, z) + \beta(n)z^{-n} u^0(n, z), \end{aligned}$$

and also (equivalently)

$$(1') \quad \begin{aligned} [1 - \alpha(n)\beta(n)]u^0(n, z) &= u^0(n - 1, z) + \alpha(n)z^n v^0(n - 1, z), \\ [1 - \alpha(n)\beta(n)]v^0(n, z) &= v^0(n - 1, z) + \beta(n)z^{-n} u^0(n - 1, z). \end{aligned}$$

Here

$$\alpha(n)v(n, 0) = u(n, n),$$

$$\beta(n)u(n, 0) = v(n, -n).$$

We also note that as a result of the explicit formulas given for $u(n, 0)$ and $v(n, 0)$ by Cramer's rule we have

$$(2) \quad \mathbf{u}(n, 0) = \mathbf{v}(n, 0) = D(n-1)/D(n).$$

Moreover,

$$(3) \quad G(n) = G(n-1)[1 - \alpha(n)\beta(n)],$$

where $G(n) = 1/\mathbf{u}(n, 0)$. Note that $\alpha(n)\beta(n) = 1$ is impossible if $D(n), D(n-1)$, and $D(n-2) \neq 0$.

Let us assume that $D(n)$ and $D(n-1) \neq 0$. Then $\mathbf{u}(n, 0) = \mathbf{v}(n, 0) \neq 0$ and we may write

$$u^0(n, z) = \mathbf{u}(n, 0) \prod_{v=1}^n \left\{ 1 - \frac{z}{z_+(n, v)} \right\}.$$

(If the actual degree of $u^0(n, z)$ is less than n some $z_+(n, v)$'s may be ∞). Similarly we may write

$$v^0(n, z) = \mathbf{v}(n, 0) \prod_{v=1}^n \left\{ 1 - \frac{z_-(n, v)}{z} \right\}.$$

Theorem 2a. If $D(n), D(n-1)$, and $D(n-2) \neq 0$ then

$$E(n)^{-1} = E(n-1)^{-1}[1 - \alpha(k)\beta(k)]^n,$$

where

$$E(n)^{-1} = \prod_{\mu, v=1}^n \left\{ 1 - \frac{z_-(n, \mu)}{z_+(n, v)} \right\}.$$

Proof. It is easy to see that

$$E(n)^{-1} = \prod_{\mu=1}^n \frac{u^0(n, z_-(n, \mu))}{\mathbf{u}(n, 0)}.$$

It follows from (1) that

$$u^0(n, z_-(n, \mu)) = u^0(n-1, z_-(n, \mu))$$

and thus

$$\begin{aligned} E(n)^{-1} &= \prod_{\mu=1}^n \frac{u^0(n-1, z_-(n, \mu))}{\mathbf{u}(n, 0)} \\ &= \left[\frac{\mathbf{u}(n-1, 0)}{\mathbf{u}(n, 0)} \right]^n \prod_{v=1}^{n-1} \prod_{\mu=1}^n \{1 - z_-(n, \mu)z_+(n-1, v)^{-1}\} \\ &= \left[\frac{\mathbf{u}(n-1, 0)}{\mathbf{u}(n, 0)} \right]^n \prod_{v=1}^{n-1} \frac{v^0(n, z_+(n-1, v))}{\mathbf{v}(n, 0)}. \end{aligned}$$

From (1') we see that

$$[1 - \alpha(n)\beta(n)]v^0(n, z_+(n-1, v)) = v^0(n-1, z_+(n-1, v)).$$

Consequently

$$\begin{aligned} E(n)^{-1} &= \left[\frac{\mathbf{u}(n-1, 0)\mathbf{v}(n-1, 0)}{\mathbf{u}(n, 0)\mathbf{v}(n, 0)} \right]^n [1 - \alpha(n)\beta(n)]^{-n} E(n-1)^{-1} \\ &= [1 - \alpha(n)\beta(n)]^n E(n-1)^{-1}, \end{aligned}$$

and we are done.

Theorem 2b. *If $D(n)$ and $D(n-1) \neq 0$ then*

$$D(n)G(n)^{-n-1} = E(n).$$

Proof. Let

$$\mathbf{c}(\zeta, k) = (1 - \zeta)\delta(k) + \zeta\mathbf{c}(k), \quad -\infty < k < \infty,$$

where $\delta(k)$ is 1 if $k = 0$ and otherwise is 0. Here ζ is a complex variable. We set

$$D(\zeta, n) = \det [\mathbf{c}(\zeta, j - k)]_{j, k=0, \dots, n},$$

etc. We have

$$\lim_{\zeta \rightarrow 0} D(\zeta, k) = 1, \quad k = 1, 2, \dots,$$

and

$$\lim_{\zeta \rightarrow 0} u^0(\zeta, n, z) = 1, \quad \text{uniformly for } |z| \leq 1,$$

$$\lim_{\zeta \rightarrow 0} v^0(\zeta, n, z) = 1, \quad \text{uniformly for } |z| \geq 1.$$

Thus if ζ is sufficiently near 0 the identities (2) and (3) imply that

$$D(\zeta, n)G(\zeta, n)^{-n-1} = \prod_{k=1}^n [1 - \alpha(\zeta, k)\beta(\zeta, k)]^{-k},$$

while by Theorem 2a

$$E(\zeta, n) = \prod_{k=1}^n [1 - \alpha(\zeta, k)\beta(\zeta, k)]^{-k},$$

from which it follows that

$$D(\zeta, n)E(\zeta, n)^{-1} = G(\zeta, n)^{n+1}$$

for $|\zeta| < a$ for some $a > 0$.

$D(\zeta, n)$ is a polynomial of degree $n+1$ in ζ and is therefore an analytic function of ζ . $G(\zeta, n) = D(\zeta, n)/D(\zeta, n-1)$ is an analytic function of ζ for $\zeta \notin Z(n-1)$, where $Z(n-1)$ is the set of zeros of $D(\zeta, n-1)$. $E(\zeta, n)^{-1}$ is defined for $\zeta \notin Z(n-1) \cup Z(n)$. We assert that it is an analytic function of ζ . The $\{z_+(n, \nu)^{-1}\}_{\nu=0}^n$ are the zeros of

$$\frac{z^n u^0(\zeta, n, z^{-1})}{\mathbf{u}(n, 0)} = z^n + \sum_{k=1}^n A(\zeta, n, k) z^{n-k},$$

where the $A(\zeta, n, k)$'s are rational functions of ζ analytic for $\zeta \notin Z(n-1)$. Similarly the $\{z_-(n, \mu)\}_{\mu=1}^n$ are the zeros of

$$\frac{z^n v^0(\zeta, n, z)}{\mathbf{v}(n, 0)} = z^n + \sum_{k=1}^n B(\zeta, n, k) z^{n-k},$$

where, *etc.* Since $E(\zeta, n)^{-1}$ is a symmetric function of the $\{z_+(n, \nu)^{-1}\}_{\nu=1}^n$ and of the $\{z_-(n, \mu)\}_{\mu=1}^n$ it is a polynomial in the $A(\zeta, n, k)$'s and the $B(\zeta, n, k)$'s and consequently a rational function of ζ analytic for $\zeta \notin Z(n-1) \cup Z(n)$. Since $1 \notin Z(n-1) \cup Z(n)$ our desired conclusion follows by analytic continuation.

Corollary 2c. *If $D(n)$ and $D(n-1) \neq 0$ then $u^0(n, z)$ and $v^0(n, z)$ have no common zero.*

We note that if $D(n)$ and $D(n-1) \neq 0$, in which case $u^0(n, z)$ and $v^0(n, z)$ have no common zero, then

$$(4) \quad E(n) = \exp \left\{ -\frac{1}{2\pi i} \int_C \log u^0(n, z) \frac{v^0(n, z)'}{v^0(n, z)} dz \right\}.$$

Here $v^0(n, z)' = (d/dz)v^0(n, z)$ and C is any closed rectifiable curve containing all the zeros of $v^0(n, z)$ and also 0, and containing none of the zeros of $u^0(n, z)$. Since $v^0(n, z)'/v^0(n, z)$ has simple poles of residue 1 at $\{z_-(n, \nu)\}_{\nu=1}^n$, plus a simple pole of residue $-n$ at 0, we see that the right hand side of (4) is equal to

$$\left\{ \prod_{\nu=1}^n \frac{u^0(n, z_-(n, \nu))}{\mathbf{u}(n, 0)} \right\}^{-1}.$$

In the course of proving Lemma 2a we found this to be equal to $E(n)$.

We further assert that if $u^0(n, z)$ has no zeros in $|z| \leq 1$ and if $v^0(n, z)$ has no zeros in $|z| \geq 1$ then

$$(5) \quad E(n) = \exp \left\{ \sum_{k=1}^{\infty} \mathbf{c}^\lambda(n, k) \mathbf{c}^\lambda(n, -k) k \right\},$$

where

$$c(n, \theta) = \frac{\mathbf{u}(n, 0)}{u(n, \theta)v(n, \theta)},$$

$$\log c(n, \theta) = \sum_{k=1}^{\infty} \mathbf{c}^\lambda(k) e^{ik\theta}.$$

In this case we may take C as the unit circle. We see from (4) that if

$$\log u(n, \theta) = \sum_0^{\infty} \mathbf{u}^\lambda(n, k) e^{ik\theta},$$

$$\log v(n, \theta) = \sum_{-\infty}^0 \mathbf{v}^\lambda(n, k) e^{ik\theta},$$

then

$$E(n) = \exp \left\{ \sum_1^{\infty} \mathbf{u}^{\lambda}(n, k) \mathbf{v}^{\lambda}(n, -k) k \right\}.$$

Since evidently

$$\mathbf{c}^{\lambda}(n, k) = \begin{cases} -\mathbf{u}^{\lambda}(n, k), & k > 0, \\ -\mathbf{v}^{\lambda}(n, k), & k < 0, \end{cases}$$

our desired result follows.

Theorem 2b is a stronger form of a basic identity due to Szegő [11], whose proof is altogether different from the one above. The ideas developed here afford a link, previously missing, between the ideas of Szegő and those of Baxter.

In order to proceed further it is necessary to introduce the theory of finite section Wiener-Hopf equations. Let $\mathbf{A}$ be the set of all complex valued functions $\mathbf{f}(k)$, $-\infty < k < \infty$, for which $\|\mathbf{f}\| < \infty$ where

$$\|\mathbf{f}\| = 3 \left\{ \sum_{k=-\infty}^{\infty} |\mathbf{f}(k)| + \left[\sum_{k=-\infty}^{\infty} |\mathbf{f}(k)|^2 |k| \right]^{1/2} \right\}.$$

It is easily verified that $\mathbf{A}$ is a Banach algebra under convolution,

$$\mathbf{f} * \mathbf{g}(k) = \sum_{j=-\infty}^{\infty} \mathbf{f}(k-j) \mathbf{g}(j).$$

For $\mathbf{f} \in \mathbf{A}$ let

$$f(\theta) = \sum_{k=-\infty}^{\infty} \mathbf{f}(k) e^{ik\theta}.$$

The collection $\mathcal{A}$ of all such (absolutely convergent) Fourier series forms a Banach algebra of functions on T , the real numbers modulo 2π , if we set $\|f\| = \|\mathbf{f}\|$. The algebras $\mathbf{A}$ and $\mathcal{A}$ are isometrically isomorphic. We denote by $\mathbf{E}^+(0)\mathbf{A}$ all functions $\mathbf{f}(k)$ in $\mathbf{A}$ which vanish if $k < 0$. $\mathbf{E}^+(0)\mathbf{A}$ is a Banach subalgebra of $\mathbf{A}$. Let $E^+(0)\mathcal{A}$ be the set of all $f(\theta)$ for which $\mathbf{f} \in \mathbf{E}^+(0)\mathbf{A}$. Similarly we denote by $\mathbf{E}^-(0)\mathbf{A}$ all functions $\mathbf{f}(k)$ in $\mathbf{A}$ which vanish if $k > 0$, etc.

From this point on we assume that:

$$\begin{aligned} & \mathbf{c} \in \mathbf{A}, \\ (6) \quad & c(\theta) \neq 0, \quad \theta \in T, \\ & [\arg c(\theta)]_{-\pi}^{\pi} = 0. \end{aligned}$$

It follows from the Arens-Calderon form of the Wiener-Levy theorem that $\log c(\theta) \in \mathcal{A}$ and hence that

$$c(\theta) = d^2/U(\theta)V(\theta),$$

where $U(\theta) = \exp \{-E^+(0) \log c(\theta)\}$, $V(\theta) = \exp \{-E^-(0) \log c(\theta)\}$. Clearly

$U, U^{-1} \in E^+(0)A$. $V, V^{-1} \in E^-(0)A$ and $\mathbf{U}(0) = \mathbf{V}(0) = d^2$. The theory of finite section Wiener-Hopf equations now implies:

a) There is an integer n_0 such that $D(n) \neq 0$ for $n \geq n_0$. As a consequence of this $u(n, \theta), v(n, \theta), \alpha(n), \beta(n)$, and $G(n)$ are all defined for $n \geq n_0$. We further have

$$\begin{aligned} \text{b)} \quad & \lim_{n \rightarrow \infty} \|U(\theta) - u(n, \theta)\| = 0, \\ & \lim_{n \rightarrow \infty} \|V(\theta) - v(n, \theta)\| = 0, \end{aligned}$$

and

$$\begin{aligned} \text{c)} \quad & \sum_{n_0}^{\infty} |\alpha(n)| < \infty, \quad \sum_{n_0}^{\infty} |\alpha(n)|^2 n < \infty, \\ & \sum_{n_0}^{\infty} |\beta(n)| < \infty, \quad \sum_{n_0}^{\infty} |\beta(n)|^2 n < \infty. \end{aligned}$$

Since $U(\theta)$ is invertible in the algebra $E^+(0)A$ it follows from b) that $u(n, \theta)$ is invertible in $E^+(0)A$ if n is large enough, etc. Thus there exists $n_1 \geq n_0$ such that

$$\begin{aligned} \text{d)} \quad & u(n, \theta)^{-1} \in E^+(0)A, \quad n \geq n_1. \\ & v(n, \theta)^{-1} \in E^-(0)A, \end{aligned}$$

For demonstrations of these results see [7].

Theorem 2c. *If $c(\theta)$ satisfies (6) then*

$$\lim_{n \rightarrow \infty} D(n)G^{-n-1} = E,$$

where

$$\begin{aligned} G &= \exp \left\{ \frac{1}{2\pi} \int_T \log c(\theta) d\theta \right\}, \\ E &= \exp \left\{ \sum_{k=1}^{\infty} \mathbf{c}^\lambda(k) \mathbf{c}^\lambda(-k) k \right\}. \end{aligned}$$

Here

$$\log c(\theta) = \sum_{k=-\infty}^{\infty} \mathbf{c}^\lambda(k) e^{ik\theta}.$$

Note that since $\log c(\theta) \in A$, E is well defined.

Proof. It follows from d) that if $n \geq n_1$, $D(n) \neq 0$, and the zeros of $u^0(n, z)$ and $v^0(n, z)$ are in $|z| > 1$ and $|z| < 1$ respectively. Therefore by Theorem 2b

$$\frac{D(n)}{G^{n+1}} = \left[\frac{G(n)}{G} \right]^{n+1} E(n), \quad n > n_1.$$

We see from the formula above defining G that $G = \mathbf{U}(0)^{-1}$ and thus, by b), that

$$G = \lim_{k \rightarrow \infty} \mathbf{u}(k, 0)^{-1} = \lim_{k \rightarrow \infty} G(k).$$

It follows from (3) that

$$G(n)/G = \prod_{k=n+1}^{\infty} [1 - \alpha(k)\beta(k)]^{-1}.$$

Since

$$\sum_{n_0}^{\infty} n |\alpha(n)|^2 < \infty, \quad \sum_{n_0}^{\infty} n |\beta(n)|^2 < \infty,$$

by c) it is apparent that

$$\lim_{n \rightarrow \infty} [G(n)/G]^{n+1} = 1.$$

For $n \geq n_0$, let

$$c(n, \theta) = \frac{\mathbf{u}(n, 0)}{u(n, \theta)v(n, \theta)}.$$

It follows from b) that $\|c(n, \theta) - c(\theta)\| \rightarrow 0$ as $n \rightarrow \infty$ and by a “uniform” version of the Wiener–Levy theorem this implies that $\|\log c(n, \theta) - \log c(\theta)\| \rightarrow 0$ as $n \rightarrow \infty$. Thus if

$$\log c(n, \theta) = \sum_{k=-\infty}^{\infty} \mathbf{c}^{\lambda}(n, k) e^{ik\theta},$$

we have

$$\sum_{k=-\infty}^{\infty} |\mathbf{c}^{\lambda}(k) - \mathbf{c}^{\lambda}(n, k)|^2 |k| \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

and therefore

$$\lim_{n \rightarrow \infty} E(n) = E.$$

Our proof is now complete.

3. The elements of structure. In what follows we shall work with measures on the line which are absolutely continuous except at $\xi = 0$. For greater convenience we represent any such measure $\mathbf{f}$ as a generalized function,

$$\mathbf{f}(\xi) = \mathbf{f}_* \delta(\xi) + \mathbf{f}_a(\xi),$$

where $\mathbf{f}_*$ is a complex constant, $\delta(\xi)$ is the Dirac delta function, and $\mathbf{f}_a(\xi)$ is a complex locally integrable function.

We assume as given once and for all a generalized function

$$(1) \quad \mathbf{c}(\xi) = \delta(\xi) - \mathbf{c}_a(\xi),$$

where in sections 3 and 4 we shall suppose that

$$(iii)' \quad c_a(\xi) \in C(-\infty, \infty).$$

Note that in this one case we write $-c_a$ instead of c_a . For $0 \leq r < \infty$ we define

$$D(r) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^r \cdots \int_0^r \begin{vmatrix} c_a(\xi_1 - \xi_1) & \cdots & c_a(\xi_1 - \xi_n) \\ \vdots & & \vdots \\ c_a(\xi_n - \xi_1) & \cdots & c_a(\xi_n - \xi_n) \end{vmatrix} d\xi_1 \cdots d\xi_n,$$

$$D^+(r; \eta, \xi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^r \cdots \int_0^r \begin{vmatrix} c_a(\eta - \xi) c_a(\eta - \xi_1) & \cdots & c_a(\eta - \xi_n) \\ c_a(\xi_1 - \xi) c_a(\xi_1 - \xi_1) & \cdots & c_a(\xi_1 - \xi_n) \\ \vdots & & \vdots \\ c_a(\xi_n - \xi) c_a(\xi_n - \xi_1) & \cdots & c_a(\xi_n - \xi_n) \end{vmatrix} d\xi_1 \cdots d\xi_n,$$

and

$$D^-(r; \eta, \xi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-r}^0 \cdots \int_{-r}^0 \begin{vmatrix} c_a(\eta - \xi) c_a(\eta - \xi_1) & \cdots & c_a(\eta - \xi_n) \\ c_a(\xi_1 - \xi) c_a(\xi_1 - \xi_1) & \cdots & c_a(\xi_1 - \xi_n) \\ \vdots & & \vdots \\ c_a(\xi_n - \xi) c_a(\xi_n - \xi_1) & \cdots & c_a(\xi_n - \xi_n) \end{vmatrix} d\xi_1 \cdots d\xi_n.$$

In $D^+(r; \eta, \xi)$, η and ξ are restricted to $0 \leq \eta, \xi \leq r$ and in $D^-(r; \eta, \xi)$ to $-r \leq \eta, \xi \leq 0$.

Theorem 3a. Let $c_a(\xi)$ be continuous, and $D(r)$ and $D^+(r; \eta, \xi)$ be defined as above, then

$$(2') \quad D^+(r; \eta, \xi) = D(r) c_a(\eta - \xi) + \int_0^r c_a(\zeta - \xi) D^+(r; \eta, \zeta) d\zeta,$$

$$(2'') \quad D^+(r; \eta, \xi) = D(r) c_a(\eta - \xi) + \int_0^r D^+(r; \zeta, \xi) c_a(\eta - \zeta) d\zeta,$$

for $0 \leq \eta, \xi \leq r$.

Proof. This is a standard identity from the theory of Fredholm integral equations, see [12, pp. 213-220]. There is a corresponding result for $[-r, 0]$ and $D^-(r; \eta, \xi)$. We regard Theorem 3a as including both cases.

Corollary 3b. If $D(r) \neq 0$ then given $g(\xi) = g_a(\xi) + g_b(\xi)$, $g_a(\xi) \in L^1(0, r)$, there exists a unique $f(\xi) = f_a(\xi) + f_b(\xi)$, $f_a(\xi) \in L^1(0, r)$, such that

$$c * f(\xi) = g(\xi), \quad 0 \leq \xi \leq r.$$

If $D(r) = 0$ then there exists $f(\xi) = f_a(\xi)$, where $f_a(\xi)$ is continuous for $0 \leq \xi \leq r$, $f(\xi) \neq 0$, and such that

$$\mathbf{c} * \mathbf{f} \cdot (\xi) = 0, \quad 0 \leq \xi \leq r.$$

Proof. The equation $\mathbf{c} * \mathbf{f} \cdot (\xi) = \mathbf{g}(\xi)$ is equivalent to the system

$$\mathbf{f}_s = \mathbf{g}_s,$$

$$\mathbf{f}_a(\xi) - \int_0^r \mathbf{c}_a(\xi - \eta) \mathbf{f}_a(\eta) d\eta = \mathbf{g}_a(\xi) + \mathbf{g}_s \mathbf{c}_a(\xi).$$

Now apply the theory of Fredholm equations; see the reference above.

Suppose $D(r) \neq 0$. We define

$$(3) \quad \begin{aligned} \mathbf{u}_a(r, \xi) &= D^+(r; \xi, 0)/D(r), \quad 0 \leq \xi \leq r, \\ \mathbf{u}(r, \xi) &= \delta(\xi) + \mathbf{u}_a(r, \xi), \end{aligned}$$

and

$$(3') \quad \begin{aligned} \mathbf{v}_a(r, \xi) &= D^-(r; \xi, 0)/D(r), \quad -r \leq \xi \leq 0, \\ \mathbf{v}(r, \xi) &= \delta(\xi) + \mathbf{v}_a(r, \xi). \end{aligned}$$

Corollary 3c. *If $D(r) \neq 0$ then*

$$(4) \quad \begin{aligned} \mathbf{c} * \mathbf{u}(r) \cdot (\xi) &= \delta(\xi), \quad 0 \leq \xi \leq r, \\ \mathbf{c} * \mathbf{v}(r) \cdot (\xi) &= \delta(\xi), \quad -r \leq \xi \leq 0. \end{aligned}$$

Proof. This follows easily from (2). We see from Corollary 3b that $\mathbf{u}(r, \xi)$ and $\mathbf{v}(r, \xi)$ are uniquely determined by (4). Compare in these connections §2.

We set $\mathbf{u}(r, \xi)$ equal to 0 on $(-\infty, 0)$ and (r, ∞) so that $\mathbf{u}(r, \xi)$ is defined on the entire line.

Let $D(r) \neq 0$; then

$$\begin{aligned} u(r, t) &= \int_{-\infty}^{\infty} \mathbf{u}(r, \xi) e^{it\xi} d\xi \\ &= 1 + \int_0^r \mathbf{u}_a(r, \xi) e^{it\xi} d\xi, \end{aligned}$$

and

$$\begin{aligned} v(r, t) &= \int_{-\infty}^{\infty} \mathbf{v}(r, \xi) e^{it\xi} d\xi \\ &= 1 + \int_{-r}^0 \mathbf{v}_a(r, \xi) e^{it\xi} d\xi. \end{aligned}$$

Note that $u(r, z)$ and $v(r, z)$ are entire functions of exponential type.

It is easy to see that if $D(r) \neq 0$ then

$$\mathbf{u}_a(r, 0) = \frac{D^+(r; 0, 0)}{D(r)} = \frac{D^-(r; 0, 0)}{D(r)} = \mathbf{v}_a(r, 0).$$

We may therefore set

$$(5) \quad \gamma(r) = \mathbf{u}_a(r, 0) = \mathbf{v}_a(r, 0), \quad G(r) = e^{-\gamma(r)}.$$

We further define

$$(6) \quad \begin{aligned} \alpha(r) &= \mathbf{u}_a(r, r), \\ \beta(r) &= \mathbf{v}_a(r, -r). \end{aligned}$$

Suppose that $D(r) \neq 0$ for $r \in [a, b]$. It follows from the formulas for $D(r)$ and $D^+(r; \xi, 0)$ that not only is $\mathbf{u}_a(r, \xi)$ continuous for $a \leq r \leq b$, $0 \leq \xi \leq r$, but that $\partial/\partial r \mathbf{u}_a(r, \xi)$ is also continuous in the same region. Similar remarks apply to $\mathbf{v}_a(r, \xi)$.

Theorem 3d. *Let $D(r) \neq 0$. Then*

$$\begin{aligned} \frac{\partial}{\partial r} \mathbf{u}_a(r, \xi) &= \alpha(r) \mathbf{v}_a(r, \xi - r), & 0 \leq \xi \leq r, \\ \frac{\partial}{\partial r} \mathbf{v}_a(r, \xi) &= \beta(r) \mathbf{u}_a(r, \xi + r), & -r \leq \xi \leq 0. \end{aligned}$$

Proof. We have

$$\mathbf{u}_a(r, \xi) = \int_0^r \mathbf{u}_a(r, \eta) \mathbf{c}_a(\xi - \eta) d\eta = \mathbf{c}_a(\xi).$$

Differentiating with respect to r , as we may, we find that

$$\frac{\partial}{\partial r} \mathbf{u}_a(r, \xi) = \int_0^r \frac{\partial}{\partial r} \mathbf{u}_a(r, \eta) \mathbf{c}_a(\xi - \eta) d\eta = \alpha(r) \mathbf{c}_a(\xi - r).$$

Now

$$\begin{aligned} \alpha(r) \mathbf{v}_a(r, \xi - r) &= \int_0^r \alpha(r) \mathbf{v}_a(r, \eta - r) \mathbf{c}_a(\xi - \eta) d\eta \\ &= \alpha(r) \mathbf{v}_a(r, \xi - r) - \int_{-r}^0 \alpha(r) \mathbf{v}_a(r, \eta_1) \mathbf{c}_a(\xi - r - \eta_1) d\eta_1 \\ &= \alpha(r) \mathbf{c}_a(\xi - r). \end{aligned}$$

Combining these two results we see that if

$$\varphi(\xi) = \frac{\partial}{\partial r} \mathbf{u}_a(r, \xi) - \alpha(r) \mathbf{v}_a(r, \xi - r), \quad (0 \leq \xi \leq r),$$

then

$$\mathbf{c} * \varphi = 0, \quad 0 \leq \xi \leq r.$$

By Corollary 3b $\varphi = 0$ for $0 \leq \xi \leq r$, etc.

The argument above, due to Krein, is borrowed from [1], where in fact a somewhat more general result is proved; see also [8, §8].

Corollary 3e. Under the above assumptions

$$\begin{aligned}\frac{\partial}{\partial r} u(r, z) &= \alpha(r) v(r, z) e^{irz}, \\ \frac{\partial}{\partial r} v(r, z) &= \beta(r) u(r, z) e^{-irz}.\end{aligned}$$

Corollary 3f. Under the above assumptions

$$\frac{d}{dr} \gamma(r) = \alpha(r) \beta(r).$$

The formula below is due to Achiezer, whose proof is different and much more complicated.

Theorem 3g. Let $c \in C(-\infty, \infty)$. If $D(r) \neq 0$ then

$$\frac{d}{dr} D(r) = -D(r) \gamma(r).$$

Proof. It is evident that we can differentiate the series defining $D(r)$ term by term. We find after a slight simplification that

$$\begin{aligned}\frac{d}{dr} D(r) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^r \dots \int_0^r \begin{vmatrix} c_a(r-r) & c_a(r-\xi_2) & \dots & c_a(r-\xi_n) \\ c_a(\xi_2-r) & c_a(\xi_2-\xi_2) & \dots & c_a(\xi_2-\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ c_a(\xi_n-r) & c_a(\xi_n-\xi_2) & \dots & c_a(\xi_n-\xi_n) \end{vmatrix} d\xi_2 \dots d\xi_n, \\ (7) \quad &= - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-r}^0 \dots \int_{-r}^0 \begin{vmatrix} c_a(0) & c_a(-\eta_1) & \dots & c_a(-\eta_n) \\ c_a(\eta_1) & c_a(\eta_1-\eta_1) & \dots & c_a(\eta_1-\eta_n) \\ \vdots & \vdots & \ddots & \vdots \\ c_a(\eta_n-\eta_1) & c_a(\eta_n-\eta_1) & \dots & c_a(\eta_n-\eta_n) \end{vmatrix} d\eta_1 \dots d\eta_n, \\ &= -D(r; 0, 0) = -\gamma(r) D(r).\end{aligned}$$

4. The elements of structure. II. We continue to assume in this section that $c_a(\xi) \in C(-\infty, \infty)$. Suppose that $D(t) \neq 0$. It follows from the formulas

$$\begin{aligned}(1) \quad u(t, z) &= 1 + \int_0^t \mathbf{u}_a(t, \xi) e^{i\xi z} d\xi, \\ v(t, z) &= 1 + \int_{-t}^0 \mathbf{v}_a(t, \xi) e^{i\xi z} d\xi,\end{aligned}$$

that $u(t, z)$ and $v(t, z)$ are entire functions of exponential type. Since $u(t, z) \rightarrow 1$ as $z \rightarrow \infty$ in $\text{Im } z \geq 0$, and $v(t, z) \rightarrow 1$ as $z \rightarrow \infty$ in $\text{Im } z \leq 0$, $u(t, z)$ has only finitely many zeros in $\text{Im } z \geq 0$, and $v(t, z)$ only finitely many zeros in $\text{Im } z \leq 0$.

We now assume that $u(t, z)$ and $v(t, z)$ have no common zeros. Let C be a simple rectifiable curve coinciding with the real axis except for a finite portion and such that the zeros of $u(t, z)$ are all below C and the zeros of $v(t, z)$ all be above C .

We set

$$E(t) = \exp \left\{ -\frac{1}{2\pi i} \int_C \log u(t, z) \frac{v(t, z)'}{v(t, z)} dz \right\}.$$

Since $\mathbf{u}_a(t, \xi)$ and $\mathbf{v}_a(t, \xi) \in L^2 \cap L^1$ it is easy to see from (1) that the integral above is absolutely convergent and that it is independent of C provided C satisfies the conditions specified.

Lemma 4a. *Under the above assumptions*

$$\frac{d}{dt} E(t) = E(t) t \alpha(t) \beta(t).$$

Proof. Continuity considerations show that $D(r) \neq 0$ in some neighborhood of t . Also if r and s lie in a suitable neighborhood of t the zeros of $u(r, z)$ will be below and the zeros of $v(s, z)$ above C . We set

$$e(r, s) = \frac{-1}{2\pi i} \int_C \log u(r, z) \frac{v(s, z)'}{v(s, z)} dz,$$

$$e_T(r, s) = \frac{-1}{2\pi i} \int_{C(T)} \left(1 - \frac{z^2}{T^2} \right) \log u(r, z) \frac{v(s, z)'}{v(s, z)} dz.$$

Here T is a large positive number and $C(T)$ is the portion of C lying in the strip $-T \leq \operatorname{Re} z \leq T$. It follows from the recursion formulas, Corollary 3e, that $e_T(r, s)$ has continuous partial derivatives with respect to r and s . It is also easy to see that

$$e(r, s) = \lim_{T \rightarrow \infty} e_T(r, s)$$

uniformly in r and s . By Corollary 3e

$$\begin{aligned}\frac{\partial}{\partial r} e_T(r, s) &= \frac{-\alpha(r)}{2\pi i} \int_{C(T)} \left(1 - \frac{z^2}{T^2}\right) \frac{v(r, z) e^{irs}}{u(r, z)} \frac{v(s, z)'}{v(s, z)} dz, \\ \frac{\partial}{\partial r} e_T(r, s) \Big|_{s=-r} &= \frac{-\alpha(r)}{2\pi i} \int_{C(T)} \left(1 - \frac{z^2}{T^2}\right) \frac{v(r, z)' e^{irs}}{u(r, z)} dz \\ &= \frac{\alpha(r)}{2\pi i} \int_{P(T)} \left(1 - \frac{z^2}{T^2}\right) \frac{v(r, z)' e^{irs}}{u(r, z)} dz,\end{aligned}$$

where $P(T)$ is the part of the circle $|z| = T$ lying in $\text{Im } z \geq 0$ and the integration proceeds from T to $-T$. Since

$$v(r, z)' e^{irs} = \int_0^r i(\xi - r) \mathbf{v}_a(r, \xi - r) e^{i\xi z} d\xi,$$

we have

$$(2) \quad v(r, z)' e^{irs} = \frac{r\beta(r)}{z} + o(1) \min \{1, y^{-1}\},$$

as $z \rightarrow \infty$ in $\text{Im } z \geq 0$. We also have

$$(3) \quad u(r, z) = 1 + o(1)$$

as $z \rightarrow \infty$. Thus

$$\begin{aligned}\lim_{T \rightarrow \infty} \frac{\partial}{\partial r} e_T(r, s) \Big|_{s=-r} &= \frac{\alpha(r)}{2\pi i} \lim_{T \rightarrow \infty} \int_{P(T)} \left(1 - \frac{z^2}{T^2}\right) \frac{r\beta(r)}{z} dz \\ &= \frac{1}{2} \alpha(r) \beta(r) r.\end{aligned}$$

If I is a sufficiently small interval about t then there exists a constant A such that for all large enough T , $T \geq T_0$,

$$\begin{aligned}|u(r, z)|^{-1} &\leq A, & r \in I, & z \in P(T), \\ |(\xi - r) \mathbf{v}_a(r, \xi - r)| &\leq A, & r \in I, & 0 \leq \xi \leq r, \\ |\alpha(r)| &\leq A, & r \in I.\end{aligned}$$

We have

$$\frac{\partial}{\partial r} e_T(r, s) \Big|_{s=-r} = \frac{\alpha(r)}{2\pi i} \int_{P(T)} \left(1 - \frac{z^2}{T^2}\right) \frac{dz}{u(r, z)} \int_0^r (\xi - r) \mathbf{v}_a(r, \xi - r) e^{is\xi} d\xi,$$

and thus if $z = x(z) + iy(z)$

$$\begin{aligned}\left| \frac{\partial}{\partial r} e_T(r, s) \Big|_{s=-r} \right| &\leq \frac{A^3}{2\pi} \int_{P(T)} \left| y(z)^{-1} \left(1 - \frac{z^2}{T^2}\right) dz \right| \int_0^\infty y(z) e^{-y(z)\xi} d\xi, \\ &\leq \frac{1}{2} A^3, & r \in I,\end{aligned}$$

so that $(\partial/\partial r) e_T(r, s) \Big|_{s=-r}$ approaches $\frac{1}{2} \alpha(r) \beta(r) r$ boundedly for $r \in I$.

If in the formula defining $e_T(r, s)$ we integrate by parts we obtain

$$e_T(r, s) = \frac{1}{2\pi i} \int_{C(T)} \left(1 - \frac{z^2}{T^2}\right) \log v(s, z) \frac{u(r, z)'}{u(r, z)} dz + R_T(r, s),$$

where

$$R_T(r, s) = \frac{-1}{2\pi i} \int_{C(T)} \log v(s, z) \log u(r, z) \frac{2z}{T^2}.$$

We may show in the same way that

$$\lim_{T \rightarrow \infty} \frac{\partial}{\partial s} \frac{1}{2\pi i} \int_{C(T)} \left(1 - \frac{z^2}{T^2}\right) \log v(s, z) \frac{u(r, z)'}{u(r, z)} dz \Big|_{s=r} = \frac{1}{2} \alpha(r) \beta(r) r,$$

boundedly for $r \in I$. Next

$$\frac{\partial}{\partial s} R_T(r, s) \Big|_{s=r} = \frac{-1}{2\pi i} \int_{C(T)} \frac{\beta(r) e^{-ir t} u(r, t)}{v(r, t)} \log u(r, t) \frac{2t}{T^2} dt.$$

There exist constants M and a such that

$$\int_{|x| \geq a} \left| \frac{\beta(r) e^{-ir x} u(r, x)}{v(r, x)} \log u(r, x) \right|^2 dx \leq M, \quad r \in I.$$

Using this and Schwarz's inequality we see that

$$\frac{\partial}{\partial s} R_T(r, s) \Big|_{s=r} = O(T^{-1/2})$$

uniformly for $r \in I$. Collecting our results we find that

$$\lim_{T \rightarrow \infty} \frac{d}{dr} e_T(r, r) = \alpha(r) \beta(r) r,$$

boundedly for $r \in I$. We have

$$\begin{aligned} (4) \quad e(r_2, r_2) - e(r_1, r_1) &= \lim_{T \rightarrow \infty} [e_T(r_2, r_2) - e_T(r_1, r_1)] \\ &= \lim_{T \rightarrow \infty} \int_{r_1}^{r_2} \frac{d}{dr} e_T(r, r) dr \\ &= \int_{r_1}^{r_2} \alpha(r) \beta(r) r dr. \end{aligned}$$

Our desired result follows.

Theorem 4b. Let $c_a(\xi) \in C(-\infty, \infty)$. If $D(t) \neq 0$ then

$$(5) \quad D(t)G(t)^{-t} = E(t).$$

Proof. We first assume that $D(r) \neq 0$ for $0 \leq r \leq t$, and that $u(t, z)$ and $v(t, z)$ are without common zeros. We assert that then $u(r, z)$ and $v(r, z)$ are without common zeros for $0 \leq r \leq t$. Suppose that $u(r', z') = v(r', z') = 0$; then by Corollary 3e

$$u(r', z') = \frac{d}{dr} u(r', z') = v(r', z') = \frac{d}{dr} v(r', z') = 0,$$

and

$$\frac{d}{dr} u(r, z') = \alpha(r) v(r, z') e^{irz'},$$

$$\frac{d}{dr} v(r, z') = \beta(r) u(r, z') e^{-irz'},$$

for $0 \leq r \leq r'$. The unique solution of this system with these initial conditions is $u(r, z') = v(r, z') = 0$. Thus we would have $u(0, z') = v(0, z') = 0$, but since $u(0, z') = v(0, z') = 1$ this is impossible.

By Theorem 3g

$$\frac{d}{dr} D(r) = -D(r)\gamma(r),$$

and thus, since $D(0) = 1$, we have

$$D(t) = \exp \left[- \int_0^t \gamma(\xi) d\xi \right].$$

By Corollary 3f

$$- \int_0^t \gamma(\xi) d\xi = [-\xi\gamma(\xi)]_0^t + \int_0^t \xi\alpha(\xi)\beta(\xi) d\xi,$$

and hence

$$D(t)G(t)^{-t} = \exp \left[\int_0^t \xi\alpha(\xi)\beta(\xi) d\xi \right].$$

It follows from Lemma 4a, since $E(0) = 1$, that

$$E(t) = \exp \left[\int_0^t \xi\alpha(\xi)\beta(\xi) d\xi \right].$$

Combining these formulas we obtain (5).

Now assume only that $D(t) \neq 0$. Let

$$\mathbf{c}(\zeta, \xi) = (1 - \zeta) \delta(\xi) + \zeta \mathbf{c}(\xi),$$

where ζ is a complex variable, and let

$$\begin{aligned} D(\zeta, r) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int \cdots \int_0^r \left| \begin{array}{c} \mathbf{c}_a(\zeta, \xi_1 - \xi_1) \cdots \mathbf{c}_a(\zeta, \xi_1 - \xi_n) \\ \vdots \\ \mathbf{c}_a(\zeta, \xi_n - \xi_1) \cdots \mathbf{c}_a(\zeta, \xi_n - \xi_n) \end{array} \right| d\xi_1 \cdots d\xi_n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-\zeta)^n}{n!} \int \cdots \int_0^r \left| \begin{array}{c} \mathbf{c}_a(\xi_1 - \xi_1) \cdots \mathbf{c}_a(\xi_1 - \xi_n) \\ \vdots \\ \mathbf{c}_a(\xi_n - \xi_1) \cdots \mathbf{c}_a(\xi_n - \xi_n) \end{array} \right| d\xi_1 \cdots d\xi_n, \end{aligned}$$

etc. We have

$$\lim_{\zeta \rightarrow 0} D(\zeta, r) = 1,$$

uniformly for $0 \leq r \leq t$. Furthermore

$$\lim_{\zeta \rightarrow 0} u(\zeta, r, z) = 1,$$

uniformly for $\text{Im } z \geq 0$ and $0 \leq r \leq t$, and

$$\lim_{\zeta \rightarrow 0} v(\zeta, r, z) = 1,$$

uniformly for $\text{Im } z \leq 0$ and $0 \leq r \leq t$. Consequently

$$(6) \quad D(\zeta, t)G(\zeta, t)^{-t} = E(\zeta, t),$$

for $|\zeta| < a$ for some a .

Let Z_1 be the set of values of ζ for which $D(\zeta, t) = 0$. It follows from the theory of Fredholm integral equations that Z_1 has no finite limit points, see [12, Chapter 11]. Let Z_2 be the set of those ζ in the complement of Z_1 for which $u(\zeta, t, z)$ and $v(\zeta, t, z)$ have common zeros. We will ultimately prove that Z_2 is empty; however, initially all we can show is that the set of limit points of Z_2 in Z'_1 , the complement of Z_1 , is empty. To demonstrate this we will show first that Z_2 is relatively closed in Z'_1 , and second that if $\zeta_0 \in Z'_1$ is a limit point of Z_2 then Z_2 contains a neighborhood of ζ_0 . It follows easily from this that either Z_2 is all of Z'_1 or Z_2 has no limit points in Z'_1 . However, the first possibility is excluded since we know there is a neighborhood of $\zeta = 0$ which does not meet $Z_2 \cup Z_1$.

We proceed to carry out the program outlined above. Suppose that ζ_0 is a limit point of Z_2 in Z'_1 and let K be a compact neighborhood of ζ_0 in Z'_1 . It follows from (1) that there is a constant $M > 0$ such that

$$\zeta \in K, \quad |z| > M \Rightarrow |u(\zeta, t, z)| + |v(\zeta, t, z)| > 0.$$

By assumption there exists a sequence $\{(\zeta_k, z_k)\}_{k=1}^\infty$ with $\lim_{k \rightarrow \infty} \zeta_k = \zeta_0$ such that $u(\zeta_k, t, z_k) = v(\zeta_k, t, z_k) = 0$ $k = 1, 2, \dots$. As a consequence of the above we may assume without loss of generality that $\lim_{k \rightarrow \infty} z_k = z_0$, as $k \rightarrow \infty$, for some z_0 , $|z_0| \leq M$. It follows that $u(\zeta_0, t, z_0) = v(\zeta_0, t, z_0) = 0$ so that $\zeta_0 \in Z_2$.

By the Weierstrass preparation theorem

$$u(\zeta, t, z) = \Phi(\zeta, z) \prod_{i=1}^r \left\{ (z - z_0)^{s_i} + \sum_{j=1}^{s_i} \varphi_{ij}(\zeta) (z - z_0)^{s_i - j} \right\},$$

$$v(\zeta, t, z) = \Psi(\zeta, z) \prod_{i=1}^p \left\{ (z - z_0)^{q_i} + \sum_{j=1}^{q_i} \psi_{ij}(\zeta) (z - z_0)^{q_i - j} \right\},$$

where $\Phi(\zeta, z)$, $\Psi(\zeta, z)$, the $\varphi_{ij}(\zeta)$, and the $\psi_{ij}(\zeta)$ are all analytic in a neighborhood N of (ζ_0, z_0) . Moreover $\Phi(\zeta, z)$ and $\Psi(\zeta, z)$ do not vanish in N and $\varphi_{ij}(\zeta_0)$, $\psi_{ij}(\zeta_0) = 0$ for all i, j . Furthermore the factors in these products are irreducible

in the ring of polynomials in $z - z_0$ with coefficients which are functions of ζ defined and analytic in some neighborhood of ζ_0 .

It follows from this that there is a uniformizing parameter $\zeta - \zeta_0 = \omega^n$ and an $\epsilon > 0$ such that if $\zeta = \zeta_0 + \omega^n$ and if $|\omega| < \epsilon$ then

$$u(\zeta, t, z)/\Phi(\zeta, z) = \prod_{\nu=1}^R \{z - z_0 - \tau_\nu(\omega)\},$$

$$v(\zeta, t, z)/\Psi(\zeta, z) = \prod_{\mu=1}^S \{z - z_0 - \sigma_\mu(\omega)\}.$$

Here $\tau_\nu(\omega)$ and $\sigma_\mu(\omega)$ are analytic for $|\omega| < \epsilon$. Our assumptions imply that for some ν_0 and μ_0 , $\tau_{\nu_0}(\omega)$ and $\sigma_{\mu_0}(\omega)$ are equal for infinitely many values of ω having $\omega = 0$ as a limit point. This implies $\tau_{\nu_0}(\omega) \equiv \sigma_{\mu_0}(\omega)$ and thus that $Z_2 \supset \{|\zeta - \zeta_0| < \epsilon^n\}$; that is, Z_2 contains a neighborhood of ζ_0 .

For the Weierstrass preparation theorem, *etc.* see S. Bochner and W. T. Martin, *Several Complex Variables*, Princeton, 1948, Chapter IX.

The function $D(\zeta, t)$ is entire and $G(\zeta, t)$ is analytic for $\zeta \notin Z_1$. We assert that $E(\zeta, t)$ is analytic for $\zeta \notin Z_1 \cup Z_2$. Let $\zeta_0 \notin Z_1 \cup Z_2$ and let C be as described at the beginning of this section. In particular the zeros of $u(\zeta_0, t, z)$ lie below and the zeros of $v(\zeta_0, t, z)$ lie above C . Choose a disk K about ζ_0 such that the closure of K does not meet $Z_1 \cup Z_2$, and such that for $\zeta \in K$ the zeros of $u(\zeta, t, z)$ lie below and the zeros of $v(\zeta, t, z)$ lie above C . Let $L^2(C)$ be the Hilbert space with inner product

$$\int_C f(z)g(z) * |dz| = (f | g).$$

The functions $\mathbf{u}_a(\zeta, t, \xi)$ and $\mathbf{v}_a(\zeta, t, \xi)$ are analytic functions of $\zeta \in K$ with values in $L^2(0, t)$ and $L^2(-t, 0)$ respectively. Using (1) it is easy to see that $\log u(\zeta, t, z)$ and $v(\zeta, t, z)' / v(\zeta, t, z)$ are analytic functions of $\zeta \in K$ with values in $L^2(C)$. It follows that $E(\zeta, t)$ is analytic in K , and thus more generally in the complement of $Z_1 \cup Z_2$.

It now follows by analytic continuation that (6) holds for all $\zeta \notin Z_2 \cup Z_1$.

Let $\zeta_0 \in Z_2 - Z_1$ and suppose that $\{z_k\}_{k=1}^r$ are the zeros of $v(\zeta_0, t, z)$ which are also zeros of $u(\zeta_0, t, z)$, repeated according to their multiplicities as zeros of $v(\zeta_0, t, z)$. Let C be a curve of the usual type such that all the zeros of $u(\zeta_0, t, z)$ lie below C and all of the zeros of $v(\zeta_0, t, z)$ except, of course, $\{z_k\}_1^r$ lie above C . Using Cauchy's theorem we see that if ζ is sufficiently near ζ_0 but not equal to ζ_0 then

$$E(\zeta, t)^{-1} = \prod_{k=1}^r u(\zeta, t, z_k(\zeta)) \exp \left\{ \frac{1}{2\pi i} \int_C \log u(\zeta, t, z) \frac{v(\zeta, t, z)'}{v(\zeta, t, z)} dz \right\}.$$

Here $\{z_k(\zeta)\}_1^r$ are the zeros of $v(\zeta, t, z)$ below C . By Hurwitz's theorem the $z_k(\zeta)$'s appropriately numbered approach the z_k 's as $\zeta \rightarrow \zeta_0$. It follows that $E(\zeta, t)^{-1}$ approaches 0 as $\zeta \rightarrow \zeta_0$. On the other hand $E(\zeta, t)^{-1} = D(\zeta, t)^{-1}G(\zeta, t)^t$

for all ζ in some deleted neighborhood of ζ_0 . Letting $\zeta \rightarrow \zeta_0$ we see that $E(\zeta, t)^{-1}$ approaches $D(\zeta_0, t)^{-1}G(\zeta_0, t)^t \neq 0$. This contradiction shows that $Z_2 - Z_1$ is empty; that is $Z_2 \subset Z_1$.

Thus (6) holds for all $\zeta \notin Z_1$ and in particular for $\zeta = 1$. We have also shown that if $D(t) \neq 0$ then $u(t, z)$ and $v(t, z)$ have no common zeros.

5. The elements of structure. III. In this section we will derive the formulas of §3 and §4 but with the assumption $\mathbf{c}_a(\xi) \in C(-\infty, \infty)$ replaced by the assumptions:

$$\begin{aligned} \text{a)} \quad & \int_{-r}^r |\mathbf{c}_a(\xi)|^2 d\xi < \infty, \quad \text{for all } r > 0; \\ \text{b)} \quad & \lim_{h \rightarrow 0+} h^{-1} \int_0^h \mathbf{c}_a(\xi) d\xi = \mathbf{c}_a(0+), \\ & \lim_{h \rightarrow 0+} h^{-1} \int_{-h}^0 \mathbf{c}_a(\xi) d\xi = \mathbf{c}_a(0-). \end{aligned}$$

In b) what is asserted is that the limits on the left exist; $\mathbf{c}_a(0+)$ and $\mathbf{c}_a(0-)$ are by *definition* the values of these limits. We *always* suppose that

$$\mathbf{c}_a(0) = \frac{1}{2}[\mathbf{c}_a(0+) + \mathbf{c}_a(0-)].$$

As explained in §1 this is a normalization and involves no loss of generality.

All of the statements below on Fredholm integral equations are either contained in Smithies' paper [10] or can be deduced from his results.

Let condition a) hold. It follows from [10] that $D(r)$, $D^+(r; \xi, \eta)$, and $D^-(r; \xi, \eta)$ are well defined by the formulas of §1 and §3. $D(r)$ is a continuous function of r and

$$\begin{aligned} (1) \quad D^+(r; \xi, \eta) &= D(r)\mathbf{c}_a(\xi - \eta) + E^+(r; \xi, \eta), & 0 \leq \xi, \eta \leq r, \\ D^-(r; \xi, \eta) &= D(r)\mathbf{c}_a(\xi - \eta) + E^-(r; \xi, \eta), & -r \leq \xi, \eta \leq 0, \end{aligned}$$

where $E^+(r; \xi, \eta)$ is continuous in r , ξ , and η for $0 \leq \xi, \eta \leq r$. Moreover, if $\{\mathbf{c}_a(n, \xi)\}_{n=0}^\infty$ are such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-r}^r |\mathbf{c}_a(n, \xi) - \mathbf{c}_a(\xi)|^2 d\xi &= 0, \quad \text{for all } r > 0, \\ \lim_{n \rightarrow \infty} \mathbf{c}_a(n, 0) &= \mathbf{c}_a(0), \end{aligned}$$

then (with an evident notation)

$$(2) \quad \lim_{n \rightarrow \infty} D(n, r) = D(r), \quad 0 \leq r < \infty,$$

uniformly for r in any finite interval, and

$$(3) \quad \lim_{n \rightarrow \infty} E^+(n, r; \xi, \eta) = E^+(r; \xi, \eta),$$

uniformly for $0 \leq \xi, \eta \leq r$, and r in any finite interval. There are, of course, analogous statements for $E^-(n, r; \xi, \eta)$ and we regard (2) and (3) as including both cases. Theorem 3a and Corollary 3b hold in the present context.

For $D(r) \neq 0$ we define $\mathbf{u}(r, \xi)$ and $\mathbf{v}(r, \xi)$ by (3) and (3') of §3. Corollary 3c is then valid.

We now assume that both assumptions a) and b) hold. We assert that the limits

$$(4) \quad \mathbf{u}_a(r, 0+) = \lim_{h \rightarrow 0+} h^{-1} \int_0^h \mathbf{u}_a(r, \xi) d\xi,$$

and

$$(4') \quad \mathbf{v}_a(r, 0-) = \lim_{h \rightarrow 0+} h^{-1} \int_{-h}^0 \mathbf{v}_a(r, \xi) d\xi.$$

exist. This is apparent from the formula

$$(5) \quad \mathbf{u}_a(r, \xi) = \mathbf{c}_a(\xi) + E^+(r; \xi, 0)/D(r),$$

etc. Note that it is not in general true that $\mathbf{u}_a(r, 0+) = \mathbf{v}_a(r, 0-)$. We set

$$\gamma(r) = \frac{1}{2}[\mathbf{u}_a(r, 0+) + \mathbf{v}_a(r, 0-)],$$

and also

$$\alpha(r) = \mathbf{u}_a(r, r), \quad \beta(r) = \mathbf{v}_a(r, -r).$$

We will obtain in the general case the structural formulas proved in §3 by approximating the general case by the continuous one. We define for $n = 1, 2, \dots$

$$\mathbf{c}_a(n, \xi) = \frac{1}{2}n \int_{-1/n}^{1/n} \mathbf{c}_a(\xi + \eta) d\eta.$$

Then $\mathbf{c}_a(n, \xi)$ is continuous on $-\infty < \xi < \infty$ for each n , and

$$\lim_{n \rightarrow \infty} \int_{-r}^r |\mathbf{c}_a(n, \xi) - \mathbf{c}_a(\xi)|^2 d\xi = 0, \quad \text{for all } r > 0,$$

$$\lim_{n \rightarrow \infty} \mathbf{c}_a(n, 0) = \mathbf{c}_a(0).$$

We define $D(n, r)$, $D^+(n, r; \xi, \eta)$, $\mathbf{u}(n, r, \xi)$, $\gamma(n, r)$, $\alpha(n, r)$, etc. in the obvious way.

Lemma 5a. *Let assumptions a) and b) hold and let $I = \{r \mid r' \leq r \leq r''\}$ be a finite closed interval such that $D(r) \neq 0$ for $r \in I$. Then*

$$(6) \quad \lim_{n \rightarrow \infty} \gamma(n, r) = \gamma(r)$$

uniformly for $r \in I$, and

$$(7) \quad \lim_{n \rightarrow \infty} \int_I |\alpha(n, r) - \alpha(r)|^2 dr = 0,$$

$$\lim_{n \rightarrow \infty} \int_I |\beta(n, r) - \beta(r)|^2 dr = 0.$$

Proof. It follows from (2) that $D(n, r) \neq 0$ for $r \in I$ if n is sufficiently large. Consequently $\gamma(n, r)$, $\alpha(n, r)$, and $\beta(n, r)$ are defined for $r \in I$ if n is "large". From (4) and (5) together with the corresponding formulas for $\mathbf{v}_a(r, \xi)$ we see that

$$\gamma(r) = \frac{1}{2}[\mathbf{c}_a(0+) + E^+(r; 0, 0)/D(r)] + \frac{1}{2}[\mathbf{c}_a(0-) + E^-(r; 0, 0)/D(r)].$$

Similarly since $\gamma(n, r) = \mathbf{u}_a(n, r, 0) = \mathbf{v}_a(n, r, 0)$ we have

$$\gamma(n, r) = \frac{1}{2}[\mathbf{c}_a(n, 0) + E^+(n, r; 0, 0)/D(n, r)]$$

$$+ \frac{1}{2}[\mathbf{c}_a(n, 0) + E^-(n, r; 0, 0)/D(n, r)].$$

Using (2) and (3) we see that (6) holds.

Similarly we have

$$\alpha(n, r) = \mathbf{c}_a(n, r) + E^+(n, r; r, 0)/D(n, r),$$

$$\alpha(r) = \mathbf{c}_a(r) + E^+(r; r, 0)/D(r),$$

and from these formulas and (2) and (3) the first relation in (7) follows, *etc.*

Theorem 5b. *Under the same assumptions if $r_1, r_2 \in I$ then*

$$\gamma(r_2) - \gamma(r_1) = \int_{r_1}^{r_2} \alpha(r)\beta(r) dr,$$

and

$$\log D(r_2) - \log D(r_1) = - \int_{r_1}^{r_2} \gamma(r) dr.$$

Proof. By Corollary 3f we have for "large" n

$$\gamma(n, r_2) - \gamma(n, r_1) = \int_{r_1}^{r_2} \alpha(n, r)\beta(n, r) dr.$$

Now use Lemma 5a to pass to the limit, *etc.*

Theorem 5c. *Let assumptions a) and b) hold. If I is a closed interval of the non-negative real axis such that $D(r) \neq 0$ for $r \in I$, then*

$$\log E(r_2) - \log E(r_1) = \int_{r_1}^{r_2} \alpha(r)\beta(r)r dr,$$

for any $r_1, r_2 \in I$.

Proof. It is easy to see that if n is sufficiently large then $D(n, r) \neq 0$ for

$r \in I$ and $u(n, r, z)$ and $v(n, r, z)$ have no common zero for $r \in I$. Therefore by Lemma 4a

$$(8) \quad \log E(n, r_2) - \log E(n, r_1) = \int_{r_1}^{r_2} \alpha(n, r) \beta(n, r) r \, dr.$$

From the formulas

$$u_a(r, \xi) = c_a(\xi) + E^+(r; \xi, 0)/D(r),$$

$$u_a(n, r, \xi) = c_a(n, \xi) + E^+(n, r; \xi, 0)/D(n, r),$$

it follows that if $r \in I$,

$$\lim_{n \rightarrow \infty} \int_0^r |u_a(r, \xi) - u_a(n, r, \xi)|^2 d\xi = 0,$$

and similarly

$$\lim_{n \rightarrow \infty} \int_{-r}^0 |v_a(r, \xi) - v_a(n, r, \xi)|^2 d\xi = 0.$$

These formulas imply that

$$\lim_{n \rightarrow \infty} E(n, r) = E(r), \quad r \in I.$$

Using this and (7) in (8) we obtain our desired result.

Theorem 5d. *Let assumptions a) and b) hold. If $D(t) \neq 0$ then*

$$D(t)G(t)^{-t} = E(t).$$

Proof. This follows from Theorem 4b by an argument similar to that used to prove Theorem 5c.

Corollary 5e. *If $D(t) \neq 0$ then $u(t, z)$ and $v(t, z)$ have no common zeros.*

Proof. See the argument at the end of §4.

6. The finite section norm inequality. The results stated here are very special cases of more general theorems proved in [8]. We will borrow exactly what we need. The reader who wishes a more comprehensive picture should consult this reference.

Let $\mathbf{A}$ be the set of all generalized functions $\mathbf{f}(\xi)$ on $(-\infty, \infty)$ for which $\|\mathbf{f}\| < \infty$ where

$$\|\mathbf{f}\| = 3 \left\{ \int_{-\infty}^{\infty} |\mathbf{f}(\xi)| \, d\xi + \left[\int_{-\infty}^{\infty} |\mathbf{f}_a(\xi)|^2 (1 + |\xi|) \, d\xi \right]^{1/2} \right\}.$$

$\mathbf{A}$ is a commutative Banach algebra under convolution. The set of all Fourier-Stieltjes transforms

$$f(t) = \int_{-\infty}^{\infty} \mathbf{f}(\xi) e^{it\xi} \, d\xi = \mathbf{f}_* + \int_{-\infty}^{\infty} \mathbf{f}_a(\xi) e^{it\xi} \, d\xi$$

of measures $\mathbf{f}$ in $\mathbf{A}$ is denoted by A . If we take $||f|| = ||\mathbf{f}||$ then A is a commutative Banach algebra of functions, and in the terminology of [8] A is of type $\mathcal{S}$.

Every homomorphism of $\mathbf{A}$ into the complex numbers is of the form

$$\mathbf{f} \rightarrow f(t), \quad -\infty < t < \infty,$$

or

$$\mathbf{f} \rightarrow \mathbf{f}_s.$$

Let us denote by $\mathbf{E}^+(0)\mathbf{A}$ all measures $\mathbf{f}$ in $\mathbf{A}$ whose support lies in $[0, \infty)$. It is evident that $\mathbf{E}^+(0)\mathbf{A}$ is a Banach subalgebra of $\mathbf{A}$. We denote by $E^+(0)A$ the set of all Fourier-Stieltjes transforms of measures in $\mathbf{E}^+(0)\mathbf{A}$. Every homomorphism of $\mathbf{A}$ into the complex numbers is of the form

$$\mathbf{f} \rightarrow f(z), \quad \text{Im } z \geq 0,$$

or

$$\mathbf{f} \rightarrow \mathbf{f}_s.$$

Similarly we denote $\mathbf{E}^-(0)\mathbf{A}$ all measures $\mathbf{f}$ in $\mathbf{A}$ whose support lies in $(-\infty, 0]$, etc.

Theorem 6a. *Let c satisfy assumptions (i) and (ii) of §1; then*

$$c(t) = \frac{1}{U(t)V(t)},$$

where $U, U^{-1} \in E^+(0)A, V, V^{-1} \in E^-(0)A$.

Proof. It follows from the Arens-Calderon form of the Wiener-Levy theorem and our assumptions that $\log c(t) \in A$ and thus that

$$\log c(t) = \int_{-\infty}^{\infty} \mathbf{c}_a^\lambda(\xi) e^{it\xi} d\xi,$$

where $\mathbf{c}_a^\lambda(\xi) \in \mathbf{A}$. (Since $\log c(t) \rightarrow 0$, as $t \rightarrow \pm\infty$, $\mathbf{c}_s^\lambda = 0$.) Set

$$U(t) = \exp \left[- \int_0^\infty \mathbf{c}_a^\lambda(\xi) e^{it\xi} d\xi \right],$$

$$V(t) = \exp \left[- \int_{-\infty}^0 \mathbf{c}_a^\lambda(\xi) e^{it\xi} d\xi \right],$$

etc.

As a consequence of this result and Theorem 4a of [8] the theory developed in [8] is applicable here.

Theorem 6b. *Let c satisfy assumptions (i) and (ii) of §1. Then there exists $r_0 \geq 0$ such that*

$$\text{a) } D(r) \neq 0, \quad r \geq r_0;$$

$$\begin{aligned}
 & \lim_{r \rightarrow \infty} \|U(t) - u(r, t)\| = 0, \\
 & \lim_{r \rightarrow \infty} \|V(t) - v(r, t)\| = 0; \\
 & \int_{r_0}^{\infty} |\alpha(\xi)| d\xi + \int_{r_0}^{\infty} |\alpha(\xi)|^2 \xi d\xi < \infty, \\
 & \int_{r_0}^{\infty} |\beta(\xi)| d\xi + \int_{r_0}^{\infty} |\beta(\xi)|^2 \xi d\xi < \infty.
 \end{aligned}$$

Proof. By Theorem 5b of [8] there exists $r_0 \geq 0$ such that if $r \geq r_0$

$$c * f(\xi) = g(\xi), \quad 0 \leq \xi \leq r,$$

has a unique solution $f(\xi)$ in $\mathbf{A}$ with support in $[0, r]$ for each $g(\xi) \in \mathbf{A}$ with support in $[0, r]$. Referring to the theory of Fredholm equations we see that this is equivalent to a). Conclusions b) and c) follow from Corollary 5c and Theorem 8a of [8] respectively. Actually assumption (iii) of §1 is not needed for b) and c).

Theorem 6c. *Let $c \in \mathbf{A}$ satisfy (i) and (ii) of §1. Then there exists $r_1 \geq r_0$ such that if $r \geq r_1$ $u(r, t)^{-1} \in E^+(0)A$, and $v(r, t)^{-1} \in E^-(0)A$.*

Proof. We know that $U(t)$ is invertible in $E^+(0)A$. Conclusion b) of Theorem 6b shows that $u(r, t)$ is invertible $E^+(0)A$ if r is large enough, etc.

7. The Kac-Achiezer formula. In this section we prove our principal result. We begin by assembling various facts we will need.

Lemma 7a. *Let (i), (ii), and (iii) hold. Then*

$$G = \lim_{r \rightarrow \infty} G(r),$$

exists and

$$(1) \quad \lim_{r \rightarrow \infty} [G(r)/G]^r = 1.$$

Proof. $G(r) = e^{-\gamma(r)}$ is well defined for all $r \geq r_1$ where r_1 is as in §6. By Theorem 5b if $s > r \geq r_1$,

$$G(s) = G(r) \exp \left\{ - \int_r^s \alpha(\xi) \beta(\xi) d\xi \right\}.$$

Theorem 6b implies that $\alpha(\xi) \beta(\xi) \in L^1(r_1, \infty)$ from which it follows that

$$G = \lim_{s \rightarrow \infty} G(s) = G(r) \exp \left\{ - \int_r^{\infty} \alpha(\xi) \beta(\xi) d\xi \right\}.$$

Thus G is well defined. Furthermore

$$[G(r)/G]^r = \exp \left\{ -r \int_r^{\infty} \alpha(\xi) \beta(\xi) d\xi \right\}.$$

Since

$$\left| r \int_r^\infty \alpha(\xi) \beta(\xi) d\xi \right| \leq \int_r^\infty |\xi \alpha(\xi) \beta(\xi)| d\xi,$$

and since $\xi \alpha(\xi) \beta(\xi) \in L^1(r_1, \infty)$ (also by Theorem 6b) we have

$$\lim_{r \rightarrow \infty} r \int_r^\infty \alpha(\xi) \beta(\xi) d\xi = 0,$$

etc.

We now proceed to identify G .

Lemma 7b. *If (i), (ii), and (iii) hold then*

$$G = \exp \left\{ \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \left(1 - \frac{t^2}{T^2} \right)^2 \log c(t) dt \right\}.$$

Proof. We have if $\xi > r_1$,

$$\begin{aligned} \int_0^\infty \mathbf{c}_a(\xi - \eta) \mathbf{U}_a(\eta) d\eta &= \mathbf{U}_a(\xi) - \mathbf{c}_a(\xi), & 0 \leq \xi < \infty, \\ \int_0^\infty \mathbf{c}_a(\xi - \eta) \mathbf{u}_a(r, \eta) d\eta &= \mathbf{u}_a(r, \xi) - \mathbf{c}_a(\xi), & 0 \leq \xi < r. \end{aligned}$$

Consequently for $0 \leq \xi \leq r$, we have

$$\mathbf{U}_a(\xi) - \mathbf{u}_a(r, \xi) = \int_0^\infty \mathbf{c}_a(\xi - \eta) \{ \mathbf{U}_a(\eta) - \mathbf{u}_a(r, \eta) \} d\eta,$$

which implies that

$$\begin{aligned} h^{-1} \int_0^h \mathbf{U}_a(\xi) d\xi - h^{-1} \int_0^h \mathbf{u}_a(r, \xi) d\xi \\ = h^{-1} \int_0^h d\xi \int_0^\infty \mathbf{c}_a(\xi - \eta) \{ \mathbf{U}_a(\eta) - \mathbf{u}_a(r, \eta) \} d\eta, \end{aligned}$$

and hence, since

$$\lim_{h \rightarrow 0} \int_0^\infty \left| h^{-1} \int_0^h [\mathbf{c}_a(\xi - \eta) - \mathbf{c}_a(-\eta)] d\xi \right|^2 d\eta = 0,$$

the limit

$$\lim_{h \rightarrow 0+} h^{-1} \int_0^h \mathbf{U}_a(\xi) d\xi = \mathbf{U}_a(0+)$$

exists and

$$\mathbf{U}_a(0+) - \mathbf{u}_a(r, 0+) = \int_0^\infty \mathbf{c}_a(-\eta) \{ \mathbf{U}_a(\eta) - \mathbf{u}_a(r, \eta) \} d\eta.$$

Since by Theorem 6b

$$\lim_{r \rightarrow \infty} \int_0^\infty |\mathbf{U}_a(\eta) - \mathbf{u}_a(r, \eta)|^2 d\eta = 0,$$

we have

$$\mathbf{U}_a(0+) = \lim_{r \rightarrow \infty} \mathbf{u}_a(r, 0+),$$

and similarly

$$\mathbf{V}_a(0-) = \lim_{r \rightarrow \infty} \mathbf{v}_a(r, 0-).$$

Thus

$$\log G = -\lim_{r \rightarrow \infty} \frac{1}{2}[\mathbf{u}_a(r, 0+) + \mathbf{v}_a(r, 0-)] = -\frac{1}{2}[\mathbf{U}_a(0+) + \mathbf{V}_a(0-)].$$

Starting from

$$U(z) = 1 + \int_0^\infty \mathbf{U}_a(\xi) e^{iz\xi} d\xi,$$

it is easily seen that

$$U(z) = 1 - \frac{\mathbf{U}_a(0+)}{iz} + o(1) \min \{1, |z|^{-2}\},$$

as $z \rightarrow \infty$, in $\text{Im } z \geq 0$. It follows that

$$\log U(z) = \frac{-\mathbf{U}_a(0+)}{iz} + o(1) \min \{1, |z|^{-2}\}.$$

By Cauchy's theorem

$$-\frac{1}{2\pi} \int_{-T}^T \left(1 - \frac{t^2}{T^2}\right)^2 \log U(t) dt = \frac{1}{2\pi} \int_{P(T)} \left(1 - \frac{z^2}{T^2}\right)^2 \log U(z) dz,$$

where $P(T)$ is the part of the circle $|z| = T$ lying in $\text{Im } z \geq 0$. On $P(T)$ the integration proceeds from T to $-T$. Inserting the estimate for $\log U(z)$ given above we find that

$$\lim_{T \rightarrow \infty} -\frac{1}{2\pi} \int_{-T}^T \left(1 - \frac{t^2}{T^2}\right)^2 \log U(t) dt = -\frac{1}{2}[\mathbf{U}_a(0+)].$$

Similarly we can show that

$$\lim_{T \rightarrow \infty} -\frac{1}{2\pi} \int_{-T}^T \left(1 - \frac{t^2}{T^2}\right)^2 \log V(t) dt = -\frac{1}{2}[\mathbf{V}_a(0-)].$$

Since $c(t) = 1/U(t)V(t)$ it follows that

$$\begin{aligned}\lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \left(1 - \frac{t^2}{T^2}\right)^2 \log c(t) dt &= -\frac{1}{2}[\mathbf{U}_a(0+) + \mathbf{V}_a(0-)] \\ &= \log G,\end{aligned}$$

and our proof is complete.

Lemma 7c. *If (i), (ii), and (iii) hold then*

$$\lim_{r \rightarrow \infty} E(r) = E$$

exists where

$$E = \exp \left\{ \int_0^\infty \mathbf{c}_a^\lambda(\xi) \mathbf{c}_a^\lambda(-\xi) \xi d\xi \right\}.$$

Here

$$\log c(t) = \int_{-\infty}^\infty \mathbf{c}_a^\lambda(\xi) e^{i\xi t} d\xi.$$

Proof. By Theorem 6c if $r \geq r_1$ the zeros of $u(r, z)$ lie in $\text{Im } z < 0$ and the zeros of $v(r, z)$ in $\text{Im } z > 0$. Thus C can be taken to be the real axis and

$$\log E(r) = \frac{-1}{2\pi i} \int_{-\infty}^\infty \log u(r, x) \frac{v(r, x)'}{v(r, x)} dx.$$

By the Wiener-Levy theorem $\log u(t, x) \in E^+(0)A$ and thus since $\log u(t, x) \rightarrow 0$ as $x \rightarrow \pm \infty$,

$$\log u(r, x) = \int_0^\infty \mathbf{u}_a^\lambda(r, \xi) e^{ix\xi} d\xi.$$

Similarly

$$\log v(r, x) = \int_{-\infty}^0 \mathbf{v}_a^\lambda(r, \xi) e^{ix\xi} d\xi,$$

$$\frac{v(r, x)'}{v(r, x)} = \int_{-\infty}^0 i\xi \mathbf{v}_a^\lambda(r, \xi) e^{ix\xi} d\xi.$$

We know from §5 that both $\log u(r, x)$ and $v(r, x)'/v(r, x) \in L^2(-\infty, \infty)$. Thus we may apply Plancherel's formula to obtain

$$\log E(r) = \int_0^\infty \xi \mathbf{u}_a^\lambda(r, \xi) \mathbf{v}_a^\lambda(r, -\xi) d\xi, \quad r \geq r_1.$$

We define

$$c(r, x) = \frac{1}{u(r, x)v(r, x)}.$$

Since $\log u(r, x)$ and $\log v(r, x) \in A$ so does $\log c(r, x)$ and

$$\log c(r, x) = \int_{-\infty}^{\infty} \mathbf{c}_a^\lambda(r, \xi) e^{i\xi x} d\xi.$$

But

$$\log c(r, x) = -\log u(r, x) - \log v(r, x)$$

from which it follows that

$$\mathbf{c}_a^\lambda(r, \xi) = \begin{cases} -\mathbf{u}_a(r, \xi), & \xi > 0, \\ -\mathbf{v}_a(r, \xi), & \xi < 0, \end{cases}$$

and thus

$$\log E(r) = \int_0^\infty \xi \mathbf{c}_a^\lambda(\xi) \mathbf{c}_a^\lambda(-\xi) d\xi, \quad r \geq r_1.$$

Since

$$c(x) = \frac{1}{U(x)V(x)},$$

it follows from Theorem 6b that

$$||c(r, x) - c(x)|| \rightarrow 0, \quad \text{as } r \rightarrow \infty.$$

By what is essentially a “uniform” Wiener–Levy theorem this implies that

$$||\log c(r, x) - \log c(x)|| \rightarrow 0, \quad \text{as } r \rightarrow \infty,$$

and thus *a fortiori* that

$$\int_{-\infty}^{\infty} |\xi| |\mathbf{c}_a^\lambda(r, \xi) - \mathbf{c}_a^\lambda(\xi)|^2 d\xi \rightarrow 0, \quad \text{as } r \rightarrow \infty.$$

It follows that

$$\lim_{r \rightarrow \infty} E(r) = E,$$

where

$$\log E = \int_0^\infty \xi \mathbf{c}_a^\lambda(\xi) \mathbf{c}_a^\lambda(-\xi) d\xi.$$

Combining Lemmas 7a–7c with Theorem 5d we obtain our principal results.

Theorem 7d. *If (i), (ii), and (iii) hold then*

$$\lim_{r \rightarrow \infty} D(r)G^{-r} = E,$$

where

$$E = \exp \left\{ \int_0^\infty \xi c_a^\lambda(\xi) c_a^\lambda(-\xi) d\xi \right\},$$

$$G = \exp \left\{ \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \left(1 - \frac{t^2}{T^2} \right)^2 \log c(t) dt \right\}.$$

Theorem 7e. If $D(r) \neq 0$, $0 \leq r < \infty$, then

$$E = \exp \left[\int_0^\infty \xi \alpha(\xi) \beta(\xi) d\xi \right],$$

$$G = \exp \left[- \int_0^\infty \alpha(\xi) \beta(\xi) d\xi \right].$$

In particular these formulas are valid if $c(t) > 0$, for $-\infty < t < \infty$.

Theorem 7d is Kac's formula and Theorem 7e is Achiezer's.

REFERENCES

- [1] N. I. ACHIEZER, The continuous analogue of some theorems on Toeplitz matrices, *Ukrainian Math. J.*, **16** (1964) 445-462.
- [2] GLEN BAXTER, Polynomials defined by a difference system, *J. Math. Anal. and App.*, **2** (1961) 223-263.
- [3] ———, A convergence equivalence related to polynomials on the unit circle, *Trans. Amer. Math. Soc.*, **99** (1961) 471-487.
- [4] ———, A norm inequality for a "finite section" Wiener-Hopf equation, *Illinois J. Math.*, **7** (1963) 97-103.
- [5] I. I. HIRSCHMAN, JR., Finite sections of Wiener-Hopf equations and Szegő polynomials, *J. Math. Anal. and App.*, **11** (1965) 290-320.
- [6] ———, On a theorem of Kac, Szegő and Baxter, *J. d'Anal. Math.*, **14** (1965) 225-234.
- [7] ———, Szegő polynomials on a compact group with ordered dual, *Canadian Math. J.*, **18** (1966) 538-560.
- [8] ———, Szegő functions on a locally compact Abelian group with ordered dual, *Trans. Amer. Math. Soc.*, **121** (1966) 133-159.
- [9] M. KAC, Toeplitz matrices, translation kernels and a related problem in probability theory, *Duke Math. J.*, **21** (1954) 501-509.
- [10] F. SMITHIES, The Fredholm theory of integral equations, *Duke Math. J.*, **8** (1941) 107-130.
- [11] G. SZEGÖ, On certain Hermitian forms associated with the Fourier series of a positive function, *Commun. du seminaire math. de l'Univ. de Lund*, tome supp. (1952) 228-237.
- [12] E. T. WHITTAKER & G. N. WATSON, *A Course of Modern Analysis*, 4th edition, Cambridge, 1952.

Washington University

Date Communicated: MARCH 7, 1966