Integration over Non-Euclidean Geometries of Infinite Dimension

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Introduction. The main object of this paper is to treat integration over an instance of a curved infinite-dimensional manifold whose finite-dimension analogues are not compact. Segal [9] has treated integration over flat manifolds of infinite dimension; that is over real Hilbert space. One of us in a forthcoming paper [13] treats integration over some instances of closed infinite-dimensional manifolds; that is, over manifolds whose finite-dimensional analogues are compact. We have chosen here the hyperbolic non-Euclidean geometry as the simplest example of a curved manifold with non-compact finite-dimensional analogues.

We have three motives for considering integration over manifolds of infinite dimension. Firstly, these integration theories are natural and interesting in their own right. Secondly, finite-dimensional experience has indicated how crucial integration is for many analytic developments, such as, for example, harmonic analysis. Thirdly, one of Segal's proposals for quantum field theory [11] will require integration over the manifold of solutions of a Lorentz-invariant differential equation.

Invariant integration over the sphere. Although integration over the sphere is aside from the main object stated in the introduction, nevertheless we include it here for clarification. Our discovery of invariant integration over the sphere inspired Shale [13] to treat other closed manifolds and encouraged us to tackle the hyperbolic geometry. Integration over the sphere is subsumed in [13] by identifying the sphere with all isometries of the real line into Hilbert space sending 0 into 0, but its simplicity stands out best in a direct treatment.

Let $\mathcal{H}$ be a real Hilbert space of dimension κ_0 , and let S be the sphere of all

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vectors in $\mathcal{H}$ with norm 1. By an integration theory over S we do not mean a countably-additive measure on a σ -ring of subsets of S , but rather a probability algebra in the sense of Segal [12] suitably connected with S . We form an algebra of starting random variables that are actual functions on S (though not everywhere defined) together with an expectation functional on this algebra. To illustrate, in Segal's theory [9] of the normal distribution on $\mathcal{H}$, the starting random variables may be taken as tame functions on $\mathcal{H}$. By a *tame function* on $\mathcal{H}$ we mean in this paper a function of the form $f \circ P$ where P is orthogonal projection of $\mathcal{H}$ onto a finite-dimensional subspace M and f is a bounded Baire function defined almost everywhere on M . We shall call the intersection of S with a closed linear subspace of $\mathcal{H}$ a *great sphere* on S . If G is a great sphere on S , there is a natural squashing of S onto G given by the map $\pi_G : S - G^\perp \rightarrow G$ that sends a point of S into the nearest point of G when this is unique. In other words,

$$\pi_G(x) = Px/||Px||$$

where P is orthogonal projection of $\mathcal{H}$ onto the linear subspace spanned by G . It is in the nature of things that π_G tears S at the poles $G^\perp \cap S$ of G . The algebra $\mathfrak{A}(S)$ of starting random variables shall consist of all functions of the form $f \circ \pi_G$ where G is a *finite-dimensional* great sphere on S and f is a bounded Baire function defined almost everywhere on G . The expected value of this random variable is to be the average of f over G . Since the random variable does not determine G uniquely, it is necessary to check that this expectation is well-defined. This can be computed directly or else seen very easily from an alternate description given in the next paragraph.

Projective Hilbert space is commonly thought of in two ways: either as S with antipodal points identified or as the set of all one-dimensional subspaces of $\mathcal{H}$. So too, the sphere can be thought of as S or as the set of all rays in $\mathcal{H}$ issuing from 0. In the latter picture certain (not everywhere defined) functions on $\mathcal{H}$ naturally become functions on the sphere; namely, positively homogeneous functions of degree 0; that is, functions invariant under homotheties λI where λ is a positive real number. The algebra of starting random variables can be taken to be the algebra of positively homogeneous tame functions of degree 0. The expected value of such a function is its integral with respect to an isotropic normal distribution on $\mathcal{H}$. (Any variance gives the same answer.) This obviously gives the same integration theory as the one in the last paragraph. In some cases, however, the present description has advantages. Whenever we have a probability algebra $\mathfrak{A}(X)$ attached to a space X , we shall write $L_p(X)$ ($1 \leq p < \infty$) for the associated Banach space of all p th power integrable random variables, $L_\infty(X)$ for the associated Banach algebra of all bounded random variables, and $\mathfrak{M}(X)$ for the associated algebra of all random variables. The present picture embeds $L_p(S)$, $L_\infty(S)$ and $\mathfrak{M}(S)$ as substructures of $L_p(\mathcal{H})$, $L_\infty(\mathcal{H})$ and $\mathfrak{M}(\mathcal{H})$ respectively. As a result the sphere inherits Segal's ergodicity result [10]: if an element of $\mathfrak{M}(S)$ is invariant under a subgroup of the orthogonal

group of $\mathcal{H}$ that leaves invariant no finite-dimensional subspaces of $\mathcal{H}$, then it is constant. If G is a great sphere, then $L_p(G)$, $L_\infty(G)$, and $\mathfrak{M}(G)$ are naturally contained in $L_p(S)$, $L_\infty(S)$, and $\mathfrak{M}(S)$ respectively. Taking this into account, we can state the ergodicity result more generally: if M is a closed subspace of $\mathcal{H}$ and if an element of $\mathfrak{M}(S)$ is invariant under a subgroup of the orthogonal group of M that leaves invariant no finite-dimensional subspaces of M , then the element of $\mathfrak{M}(S)$ is actually in $\mathfrak{M}(S \cap M^\perp)$. Integration over projective Hilbert spaces can be discussed similarly with obvious relations to integration over spheres. The ergodicity results hold for projective Hilbert spaces.

One bizarre feature of the present integration theory is that except for constants none of the starting random variables is a continuous function on S . It is reassuring to observe among the starting random variables functions that are reasonably well related to classical contexts. Let $x_1, x_2, \dots$ be a coordinate system in $\mathcal{H}$. We can form a coordinate system $\theta_2, \theta_3, \theta_4, \dots$ on S analogous to finite-dimensional spherical coordinates as follows:

$$\theta_n = \cos^{-1} (x_n / [x_1^2 + x_2^2 + \dots + x_n^2]^{1/2}), \quad n = 3, 4, 5 \dots,$$

and

$$\theta_2 = (x_1 / |x_1|) \cos^{-1} (x_2 / [x_1^2 + x_2^2]^{1/2}).$$

Then each θ_n is an element of $\mathfrak{A}(S)$, and an element of $\mathfrak{A}(S)$ is just a function expressible in terms of finitely many angles in some such coordinate system. The random variable

$$F = \frac{n}{m} \cdot \frac{x_1^2 + x_2^2 + \dots + x_m^2}{x_{m+1}^2 + x_{m+2}^2 + \dots + x_{m+n}^2}$$

has for its probability distribution R. A. Fisher's F -distribution [8] with m degrees of freedom in the denominator. On the other hand, as mentioned in [13], no element of $\mathfrak{M}(S)$ has an interpretation as a continuous homogeneous quadratic function on S except for constants. To make this assertion precise we must explain what we mean by interpretation. Since a homogeneous quadratic function has equal values at antipodal points of S , it is convenient to discuss this on projective space. If M is a linear subspace of $\mathcal{H}$, we shall denote the corresponding projective space by $\tilde{M}$. A continuous homogeneous quadratic function on $\mathcal{H}$ is of the form $x \rightarrow (Ax, x)$ where A is a bounded self-adjoint operator on $\mathcal{H}$. When we say that f in $\mathfrak{M}(\mathcal{H})$ can be interpreted as $x \rightarrow (Ax, x)$, we shall demand at least that:

- (1) f is invariant under any orthogonal transformation that commutes with A ;
- (2) for every closed subspace M invariant under A , f is the sum of something in $\mathfrak{M}(\tilde{M})$ and something in $\mathfrak{M}(\tilde{M}^\perp)$.

Take M to be the closed subspace spanned by all eigenvectors of A , and write $M = M_1 \oplus M_2 \oplus \dots$ where $M_1, M_2, \dots$ are orthogonal one-dimensional eigenspaces of A . Since f is invariant under all orthogonal transformations of $M^\perp$ that commute with A , the ergodicity result implies that f is in

$\mathfrak{M}(\tilde{M})$. By (2), $f = f_1 + \cdots + f_n + g$ where f_i is in $\mathfrak{M}(\tilde{M}_i)$ and g is in $\mathfrak{M}([M_{n+1} \oplus M_{n+2} \oplus \cdots]^\sim)$. But $\mathfrak{M}(\tilde{M}_i)$ is just the constants, so f is in $\mathfrak{M}([M_{n+1} \oplus M_{n+2} \oplus \cdots]^\sim)$ for all n . It follows from the ergodicity result that f is constant.

Let T be a bounded linear transformation on $\mathcal{H}$. If f is a positively homogeneous tame function of degree 0, then so is $f \circ T$. Consequently, we can transform our integration theory under T by saying that the new integral of f is the old integral of $f \circ T$. Of course, if T is orthogonal, then new integral coincides with the old. Conversely, if the new integral is equivalent to the old, in the sense of mutual absolute continuity, then T is a scalar multiple of an orthogonal transformation. In the finite-dimensional case, of course, any linear T transforms the old integral into an equivalent new integral; in fact, the Randon-Nikodym derivative of the new with respect to the old is just:

$$||x||^n ||Tx||^{-n} \det(T),$$

where n is the dimension of the space. To prove the infinite-dimensional assertion, let $T = AU$ where A is positive self-adjoint and U is orthogonal. Then A transforms the old integral into the same new integral as T does. Let M be the subspace spanned by all the eigenvectors of A , and let f in $L_1(S)$ be the Randon-Nikodym derivative of the new integral with respect to the old. Since f is invariant under all orthogonal transformations of $M^\perp$ that commute with A , the ergodicity result implies that f is in $L_1(M \cap S)$. In other words, $A \upharpoonright M^\perp$ preserves the integral on $M^\perp \cap S$. If F is any subspace of $M^\perp$ of finite dimension n , then

$$||x||^n ||Ax||^{-n} \det(A \upharpoonright F) = 1$$

for all $x \neq 0$ in F . Hence $A \upharpoonright M^\perp$ is a scalar multiple of an orthogonal transformation, and since $A \geq 0$ it must be a scalar operator. Therefore $M = \mathcal{H}$. Let $e_1, e_2, \cdots$ be a basis of eigenvectors of A , and set $Ae_i = \lambda_i e_i$. Let V be the orthogonal transformation that interchanges e_1 and e_2 and preserves $e_3, e_4, \cdots$. If A sends the old integral into an equivalent new integral, so does $VA^{-1}VA$. Let g in $L_1(S)$ be the derivative arising from $VA^{-1}VA$. By the ergodicity result g is in $L_1(M_2 \cap S)$ where M_2 is the linear span of e_1 and e_2 . Hence, as x varies over M_3 , the linear span of e_1, e_2 , and e_3 , the expression

$$||x||^3 ||VA^{-1}VAx||^{-3} \det(VA^{-1}VA)$$

does not depend on x_3 where $x = x_1 e_1 + x_2 e_2 + x_3 e_3$. Since

$$VA^{-1}VAx = \lambda_1 \lambda_2^{-1} x_1 e_1 + \lambda_1^{-1} \lambda_2 x_2 e_2 + x_3 e_3,$$

this means that $\lambda_1 \lambda_2^{-1} = 1$. Similarly $\lambda_i = \lambda_j$ for all i and j , so A is a scalar operator.

The hyperbolic geometry. The new problem involved in integrating over an infinite-dimensional hyperbolic non-Euclidean geometry is that the in-

variant measure on a finite-dimensional hyperbolic geometry is infinite. Experience has shown that attempts to put infinite measures on manifolds of infinite dimension fail. Segal, for example, succeeded in integrating over a Hilbert space by ignoring ordinary volume and using the normal distribution [9]. An analogue to the normal distribution in the present case is obtained by choosing a distinguished point Ω , putting a unit mass at Ω , and letting it diffuse.

Let $\mathcal{H}$ be a real Hilbert space of dimension κ_0 , let R be the real line, and let $\mathcal{K} = R \oplus \mathcal{H}$. If $\xi = \{x_0, x\}$ and $\eta = \{y_0, y\}$ are in $\mathcal{K}$, define a bilinear form on $\mathcal{K}$ by setting:

$$\xi \cdot \eta = x_0 y_0 - (x, y).$$

The space $\mathcal{L}$ of an infinite-dimensional hyperbolic geometry is one sheet of a hyperboloid given by:

$$\mathcal{L} = \{\xi = \{x_0, x\} \in \mathcal{K} : \xi \cdot \xi = 1 \text{ and } x_0 > 0\}.$$

Some of the subspaces of $\mathcal{K}$ (for example, $\mathcal{H}$ itself) are Hilbert spaces: namely, any subspace such that $\xi \cdot \xi < 0$ for every non-zero vector ξ in it is a pre-Hilbert subspace. If $\eta \cdot \eta > 0$, then

$$\eta^\perp = \{\xi : \xi \cdot \eta = 0\}$$

is a Hilbert subspace. The tangent hyperplane to $\mathcal{L}$ at the point η is just $\eta + \eta^\perp$. Hence $\mathcal{L}$ is an infinite-dimensional Riemannian manifold. The geodesic distance between the points ξ and η of $\mathcal{L}$ is just $\cosh^{-1} \xi \cdot \eta$.

As base point on $\mathcal{L}$ we shall choose the point $\Omega = \{1, 0\}$. Any n -dimensional subgeometry of $\mathcal{L}$ containing Ω is given by $\mathcal{L} \cap [R \oplus M]$ where M is an n -dimensional subspace of $\mathcal{H}$. By projection on $R \oplus M$ we shall mean $P : \mathcal{K} \rightarrow R \oplus M$ given by:

$$P\{x_0, x\} = \{x_0, Qx\},$$

where Q is orthogonal projection of $\mathcal{H}$ onto M . If G is a finite-dimensional subgeometry of $\mathcal{L}$ containing Ω , there is a natural squashing of $\mathcal{L}$ onto G given by the map $\pi_G : \mathcal{L} \rightarrow G$ that sends a point of $\mathcal{L}$ into the nearest point of G . If ξ is in $\mathcal{L}$, then

$$\pi_G(\xi) = P\xi / [P\xi \cdot P\xi]^{1/2}$$

where P is the projection of $\mathcal{K}$ onto the linear subspace spanned by G . Unlike the case of the sphere, π_G here does not tear at any point. To integrate over $\mathcal{L}$, the algebra $\mathfrak{A}(\mathcal{L})$ of starting random variables shall consist of all functions of the form of $f \circ \pi_G$ where G is a finite-dimensional subgeometry containing Ω and f is a bounded Baire function on G . The expected value of this random variable is given by fixing a variance parameter $t > 0$ and taking the expectation to be

$$\int_G f dN_t,$$

where $\{N_t: t > 0\}$ is the one-parameter family of probability measures on G that satisfies the heat equation:

$$(HE) \quad \frac{d}{dt} \int_G \varphi dN_t = \int_G \Delta \varphi dN_t, \quad \text{for } t > 0,$$

with initial condition:

$$(0) \quad \lim_{t \rightarrow 0^+} \int_G \varphi dN_t = \varphi(\Omega).$$

In these equations Δ is the Laplace–Beltrami operator (see, for example, [3], pp. 386–387) on the Riemannian manifold G , and φ varies over $C_0^\infty(G)$, the family of all infinitely differentiable functions on G with compact support. It is easy to see that although the expectation does depend on the choice of base point Ω , nonetheless the algebra $\mathfrak{A}(\mathfrak{L})$ is independent of the choice of Ω . The existence and uniqueness of the family $\{N_t: t > 0\}$ may be established in various ways, one of which is given in the appendix. Since the random variable does not determine G uniquely, it is necessary to check that this expectation is well-defined. The fact to be verified is the following: if G' is a subgeometry of G containing Ω , and if $\{N'_t: t > 0\}$ is defined on G' just as $\{N_t: t > 0\}$ is on G , then $N'_t = N_t \circ \pi_{G',G}^{-1}$ where $\pi_{G',G} = \pi_{G'} \mid G$ is the natural squashing of G onto G' . By the uniqueness of $\{N'_t\}$, we need only check that

$$\lim_{t \rightarrow 0^+} \int_{G'} \varphi' d(N_t \circ \pi_{G',G}^{-1}) = \varphi'(\Omega),$$

and that

$$\frac{d}{dt} \int_{G'} \varphi' d(N_t \circ \pi_{G',G}^{-1}) = \int_{G'} \Delta \varphi' d(N_t \circ \pi_{G',G}^{-1}),$$

for $t > 0$ and φ' in $C_0(G')$. Now from the alternate description of the hyperbolic geometry given in the next paragraph it follows easily that $(\Delta \varphi') \circ \pi_{G',G} = \Delta(\varphi' \circ \pi_{G',G}^{-1})$. The question, therefore, is whether (HE) and (0) are valid when φ is of the form $\varphi' \circ \pi_{G',G}$ with φ' in $C_0^\infty(G')$. The appendix gives an affirmative answer to this question.

Just as the sphere could be considered as a family of rays, so the hyperbolic geometry can be considered as the set of all rays issuing from 0 in $\mathfrak{K}$ and contained in the cone $C = \{\xi = (x_0, x) : \xi \cdot \xi > 0 \text{ and } x_0 > 0\}$. In this picture, functions on $\mathfrak{L}$ are just positively homogeneous functions of degree 0 on C . The projection π_G onto a finite-dimensional subgeometry containing Ω is just the map induced by projection of $\mathfrak{K}$ onto the linear subspace $R \oplus M$ spanned by G . Let $x_1, \dots, x_n$ be a coordinate system in M , and let x_0 be the natural coordinate on R . If a function f on G is thought of as a positively homogeneous function of degree 0 on $C \cap (R \oplus M)$, then the action of the Laplace–Beltrami operator is easily expressible: it sends f into

$$(x_0^2 - x_1^2 - \cdots - x_n^2) \left(\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} - \frac{\partial^2 f}{\partial x_0^2} \right).$$

The fact that this map does indeed give the Laplace–Beltrami operator is a consequence of the following four remarks.

- (1) The map is a second order differential operator.
- (2) The map preserves positively homogeneous functions of degree 0.
- (3) The map is invariant under the pseudo-orthogonal group that leaves the quadratic form

$$x_0^2 - x_1^2 - \cdots - x_n^2$$

invariant.

- (4) The map becomes

$$\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \quad \text{at } \Omega.$$

From this form of the Laplace–Beltrami operator the equation $(\Delta\varphi') \circ \pi_{G',G} = \Delta(\varphi' \circ \pi_{G',G})$ of the last paragraph is obvious.

The sphere with a base point. The last section suggests trying a sphere with a point mass diffused over it. Let Ω be a point on the sphere S in the Hilbert space $\mathcal{H}$. For starting random variables we note that $\mathfrak{A}(S)$ is the algebra of all functions of the form $f \circ \pi_G$ where G is a finite-dimensional great sphere on S containing Ω and f is a bounded Baire function defined almost everywhere on G . The expected value of this random variable is given by fixing a variance parameter $t > 0$ and taking the expectation to be $\int_G f dN_t$ where $\{N_t: t > 0\}$ is the one-parameter family of probability measures on G that satisfies the heat equation (HE) with initial condition (0). In these equations Δ is the Laplace–Beltrami operator (see, for example, [3], pp. 386–387) on the Riemannian manifold G , and φ varies over $C^\infty(G)$, the family of all infinitely differentiable functions on G . The existence and uniqueness of the family $\{N_t: t > 0\}$ may be established in various ways, one of which is given in the appendix. Since the random variable does not determine G uniquely, it is necessary to check that this expectation is well-defined. Let G' be a great subsphere of G containing Ω . Just as before the question is whether (HE) and (0) hold when φ is of the form $\varphi' \circ \pi_{G',G}$ with φ' in $C^\infty(G')$ where $\pi_{G',G} = \pi_{G'} \mid G$ is the natural squashing of G onto G' . Unlike the case of the hyperbolic geometry $\varphi' \circ \pi_{G',G}$ is not necessarily C^∞ at the poles $G'^\perp \cap G$ since the squashing $\pi_{G',G}$ tears there. Nevertheless, the appendix gives an affirmative answer to this question. The invariant integration theory over S developed earlier is the limit of the present integration theory as $t \rightarrow \infty$.

APPENDIX ON DIFFUSION

Spherical case. (See also [14], pp. 419–421). If G is an n -sphere and Ω is a point of G , we wish to establish first that there is a unique family $\{N_t: t > 0\}$

of probability measures on G satisfying (HE) and (O) for all φ in $C(G)$. To show uniqueness we note that such a family $\{N_t: t > 0\}$ has a Laplace transform $\{\hat{N}_\lambda: \lambda > 0\}$, given by

$$\int_G \varphi d\hat{N}_\lambda = \int_0^\infty e^{-\lambda t} dt \int_G \varphi dN_t ,$$

that satisfies the equation:

$$(LT) \quad \lambda \int_G \varphi d\hat{N}_\lambda - \int_G \Delta \varphi d\hat{N}_\lambda = \varphi(\Omega), \quad \lambda > 0,$$

for all $\varphi \in C^\infty(G)$. If ν were the difference of two such $\hat{N}_\lambda$'s, it would be a signed measure of bounded variation satisfying

$$\lambda \int_G \varphi d\nu - \int_G \Delta \varphi d\nu = 0,$$

for all φ in $C^\infty(G)$; in other words ν would be a weak solution of the elliptic equation:

$$(EE) \quad (\lambda - \Delta)f = 0, \quad \lambda > 0.$$

Now (EE) has no non-zero C^2 solutions f ; for Δf is non-positive at the maxima of f and non-negative at the minima of f . Even when ν is a weak solution of (EE), we may use the regularity theorem for solutions of an elliptic equation (see [4]) to conclude that $\nu = 0$.

The existence of $\{N_t\}$ follows easily from the Hille-Yosida theorem (see [1], section 4, or [6], section 40.2) applied to the Banach space $C(G)$ of all continuous functions on G . To use it we need to check for $\lambda > 0$ that $(\lambda - \Delta)C^\infty(G)$ is dense in $C(G)$ and that $\|(\lambda - \Delta)f\| \geq \lambda \|f\|$ for f in $C^\infty(G)$. The former assertion is equivalent to saying that no non-zero signed measure is a weak solution of (EE), and the latter assertion holds because Δf is non-positive at the maxima of f and non-negative at the minima of f . The Hille-Yosida theorem yields a strongly continuous semi-group $\{\Sigma_t: t \geq 0\}$ on $C(G)$ with infinitesimal generator containing Δ . If $(\lambda - \Delta)f \geq 0$, then $f \geq 0$, since Δf is non-negative at the minima of f ; therefore, Σ_t preserves non-negative functions. The adjoint semi-group $\{\Sigma_t^*\}$ acts on the space of signed measures on G . If we take $N_t = \Sigma_t^* \delta$ where δ is a unit mass at Ω , we get a solution of (HE) and (O) for all φ in $C^\infty(G)$.

Let G' be a great sphere on G containing Ω . We wish to verify (HE) and (O) when φ is of the form $\varphi' \circ \pi_{G', G}$ with φ' in $C(G')$. Actually, since the uniqueness of $\{N_t: t > 0\}$ was demonstrated in terms of its Laplace transform $\{\hat{N}_\lambda: \lambda > 0\}$, it is sufficient to verify equation (LT), which is the Laplace-transformed version of (HE) and (O). We write $\varphi = \varphi_1 + \varphi_2$ where φ_1 is a C^∞ function on G that coincides with φ in a neighborhood of Ω and that vanishes in a neighborhood of the poles $G'^\perp \cap G$ of G' . On $G - \{\Omega\}$, $\hat{N}_\lambda$ is a weak solution of (EE); hence by the regularity theorem [4], on $G - \{\Omega\}$, $d\hat{N}_\lambda = f dV$ where dV is the in-

variant volume element of G and f is an analytic function on $G - \{\Omega\}$ satisfying (EE). If we let G_ϵ be the part of G at distance $\geq \epsilon$ from the poles $G'^\perp \cap G$ of G' , and let F_ϵ be the boundary of G_ϵ consisting of all points at distance exactly ϵ from $G'^\perp \cap G$, we have:

$$\begin{aligned} \lambda \int_G \varphi d\hat{N}_\lambda - \int_G \Delta \varphi d\hat{N}_\lambda - \varphi(\Omega) &= \lambda \int_{G_\epsilon} \varphi_2 d\hat{N}_\lambda - \int_{G_\epsilon} \Delta \varphi_2 d\hat{N}_\lambda \\ &= \lim_{\epsilon \rightarrow 0} \int_{G_\epsilon} (\varphi_2 \Delta f - f \Delta \varphi_2) dV \\ &= \lim_{\epsilon \rightarrow 0} \int_{F_\epsilon} \left(\varphi \frac{\partial f}{\partial n} - f \frac{\partial \varphi}{\partial n} \right) dS \\ &= \lim_{\epsilon \rightarrow 0} \int_{F_\epsilon} \varphi \frac{\partial f}{\partial n} = 0. \end{aligned}$$

In this calculation $\partial/\partial n$ and dS denote, as usual, the directional derivative normal to F_ϵ and the $(n-1)$ -dimensional surface volume element on F_ϵ , respectively.

Hyperbolic case. Let G be an n -dimensional hyperbolic geometry, and let Ω be a point of G . We shall show first that there is at most one family $\{N_t: t > 0\}$ of probability measures on G satisfying (HE) and (O) for all φ in $C_0(G)$ by considering its Laplace transform $\{\hat{N}_\lambda: \lambda > 0\}$ given by:

$$(A) \quad \int \varphi d\hat{N}_\lambda = \int_0^\infty e^{-\lambda t} dt \int \varphi dN_t.$$

Then $\hat{N}_\lambda$ satisfies (LT) for all φ in $C_0^\infty(G)$, and the difference ν of two such $\hat{N}_\lambda$'s would be a signed measure of bounded variation on G that is a weak solution of (EE). We will demonstrate that 0 is the only such weak solution of (EE). We shall do this by considering an arbitrary function α in $C_0^\infty(\Gamma)$ where Γ is the group of all isometries of G , and smoothing ν via α to get a C^∞ function f on G such that

$$\int_G \varphi f dV = \int_G d\nu(x) \int_\Gamma \varphi(\sigma x) \alpha(\sigma) d\sigma,$$

where φ is an arbitrary element of $C_0^\infty(G)$, dV is the invariant n -dimensional volume element on G , and $d\sigma$ is Haar measure on Γ . The resulting function f is a bounded solution of (EE). We shall use the method of [7], p. 552 to show that $f = 0$ is the only bounded solution of (EE). Let U be the set of all points of G at distance < 1 from Ω , and let B be the set of all points of G at distance exactly 1 from Ω . According to [2], for every continuous function θ on B there is a unique continuous function f on $B \cup U$ that agrees with θ on B and that satisfies (EE) on U . If $0 \leq \theta \leq M$, then $0 \leq f(\Omega) \leq M$ because Δf is non-positive at any maximum of f in U and non-negative at any minimum of f in U . By the Riesz–Markoff theorem, there exists a positive measure m on

B such that

$$f(\Omega) = \int_B f \, dm.$$

Furthermore, $m(B) < 1$, since $f(\Omega) < 1$ when $\theta = 1$ because Δf is non-positive at any maximum of f in U . If f satisfies (EE) on all of G , then for any point p in G

$$f(p) = \int_B f(\sigma x) \, dm(x),$$

where σ is an element of Γ that sends Ω into p . If f is bounded, then $\|f\| \leq m(B)\|f\|$ and so $f = 0$.

Next, we shall show the existence of a family $\{N_t: t > 0\}$ of probability measures on G satisfying (HE) and (O) not only for all φ in $C_0(G)$ but also for all φ of the form $\varphi' \circ \pi_{G',G}$ where φ' is in $C_0^\infty(G')$ and G' is a subgeometry containing Ω . We shall accomplish this by embedding G as an open ball in a Euclidean n -space, letting $\tilde{G}$ be the corresponding closed ball, establishing (HE) and (O) for all φ in $C^\infty(\tilde{G})$, and noting that all functions of the required form are in $C^\infty(\tilde{G})$. In the section on the hyperbolic geometry we realized G as $\mathcal{L} \cap [R \oplus M]$, or equivalently as the set of all rays in the cone $C \cap [R \oplus M]$. Each of these rays cuts the n -dimensional affine space $M + \Omega$ exactly once, and the set of all these intersections is the open ball in $M + \Omega$ with center Ω and radius 1. Thus G has been identified with this ball, and we take $\tilde{G}$ to be the closed ball in $M + \Omega$ with center Ω and radius 1. In this picture of G , the squashing $\pi_{G',G}$ of G onto G' is the appropriate restriction of projection of $R \oplus M$ onto $R \oplus M'$ the linear space spanned by G' as a hyperbolic sheet. Since this projection maps $\tilde{G}$ onto $\tilde{G}'$, if φ' is in $C^\infty(\tilde{G}')$ we may consider $\varphi' \circ \pi_{G',G}$ as a C^∞ function on $\tilde{G}$. We now establish the existence of a family $\{N_t: t > 0\}$ of probability measures on $\tilde{G}$ satisfying (HE) and (O) for all φ in $C^\infty(\tilde{G})$ where the integration in each case is here understood *to extend over all of* $\tilde{G}$. The existence of such a family $\{N_t\}$ follows easily from the Hille–Yosida theorem (see [1], section 4, or [6], section 40.2) applied to the Banach space $C(\tilde{G})$ of all continuous functions on $\tilde{G}$. To use it we need to check for $\lambda > 0$ that $(\lambda - \Delta)C^\infty(\tilde{G})$ is dense in $C(\tilde{G})$ and that $\|(\lambda - \Delta)f\| \geq \lambda \|f\|$ for all f in $C^\infty(\tilde{G})$. The former assertion is equivalent to saying that no non-zero signed measure on $\tilde{G}$ is a weak solution to (EE) on *all of* $\tilde{G}$. Such a signed measure ν would have to vanish on G by the argument of the previous paragraph; and, therefore, $\int \varphi \, d\nu = 0$ for all φ in $C^\infty(\tilde{G})$, since $\Delta\varphi$ is automatically zero on $\tilde{G} - G$. The latter assertion holds because Δf is non-positive at the maxima of f and non-negative at the minima of f even on $\tilde{G} - G$, since Δf is automatically zero there. The Hille–Yosida theorem yields a strongly continuous semi-group $\{\Sigma_t: t \geq 0\}$ on $C(\tilde{G})$ with infinitesimal generator containing Δ . If $(\lambda - \Delta)f \geq 0$, then $f \geq 0$, since Δf is non-negative at the minima of f ; therefore, Σ_t preserves non-negative functions. The adjoint semi-group $\{\Sigma_t^*\}$ acts on the space of signed measures on $\tilde{G}$. If we take $N_t =$

$\Sigma^*_t \delta$ where δ is a unit mass at Ω , we get a solution to (HE) and (0) for all φ in $C^\infty(\bar{G})$, the integrations being understood *to extend over all of $\bar{G}$* . It remains to show that N_t is confined to G ; or, equivalently, that $\hat{N}_\lambda(\bar{G} - G) = 0$ where $\{\hat{N}_\lambda: \lambda > 0\}$ is the Laplace transform of $\{N_t: t > 0\}$ given by (A). If $\varphi(\Omega) = 0$, then

$$\lambda \int_{\bar{G}} \varphi d\hat{N}_\lambda = \int_{\bar{G}} \Delta\varphi d\hat{N}_\lambda .$$

Let φ depend only on r , the distance from Ω , as follows: $\varphi = 0$, for $r \leq N$, and $\varphi = \exp[-1/(r - N)]$ for $r > N$. Then

$$\Delta\varphi = (\sinh r)^{-n} \frac{\partial}{\partial r} \left(\sinh^n r \frac{\partial\varphi}{\partial r} \right)$$

is uniformly bounded as $N \rightarrow \infty$. Since as $N \rightarrow \infty$, $\Delta\varphi \rightarrow 0$ pointwise, the Lebesgue dominated convergence theorem gives:

$$\lambda \int_{\bar{G}-G} d\hat{N}_\lambda = 0.$$

Presumably, of course, N_t has an analytic density with respect to the invariant volume element on G . Explicit formulas for N_t in dimensions 2 and 3 occur in [5].

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