Proofs of some Conjectures on Permanents

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I. Introduction. In a recent paper [1] Marcus and Minc have given an admirable summary of the theory of permanents of square matrices, especially with regard to inequalities satisfied by the permanent. Our knowledge of permanents is certainly meager compared to that for determinants, largely for the reason that while the former differs from the latter only in the replacement of minus signs by plus signs, the determinant, and not the permanent, is invariant under unitary transformations.

As far as Hermitian, positive semi-definite (h.p.s.d.) matrices are concerned, however, the few inequalities that are known for the permanent are suspiciously similar to certain special cases of classical inequalities for the determinants of such matrices—the only difference being that the *direction of the inequality is reversed*. In particular, we conjecture that the answer to a question posed in [1], referred to there as problem 2, is given by the following:

Conjecture α . *Let A be an n -square h.p.s.d. matrix and let H be a subgroup of S_n , the symmetric group on n objects. Let λ be a representation of H with characters $X^\lambda(P)$ and let $X^\lambda(I) = m$ be the degree of λ . Then*

$$(1) \quad D^\lambda \equiv \frac{1}{m} \sum_{P \in H} X^\lambda(P) \prod_{i=1}^n a_{iP(i)} \leq \text{Per } A,$$

where $\text{Per } A$ is the permanent of A .

If $\text{Per } A$ be replaced by $\text{Det } A$ in (1), and if the inequality be reversed, then we would have the classic inequality of Schur [2, 3] which is almost half a century old.

Two immediate corollaries of conjecture α would be two theorems conjectured in [1]:

Conjecture β (*M. Marcus and M. Newman*). *If A is h.p.s.d. and $1 \leq k \leq n$ then $\text{Per } A \geq \text{Per } \alpha \text{ Per } \beta$ where α is a principal k -square submatrix and β is the complementary principal $(n - k)$ -square submatrix of A . If A is positive definite then equality holds if and only if $A = \alpha \dot{+} \beta$.*

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Conjecture 9 (M. Marcus). Let A be an mk -square h.p.s.d. matrix (with $k > 1$) partitioned as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}$$

in which each A_{ij} is k -square. Let B be the m -square matrix whose (i, j) entry is $\text{Per } A_{ij}$. Then

$$\text{Per } A \geq \text{Per } B.$$

If A is positive definite then equality holds if and only if $A = \sum_{i=1}^m A_{ii}$.

If we ignore the "if and only if" statements for equality to hold, then conjecture 8 follows from conjecture α by letting H be the product of the symmetric group on k objects and the symmetric group on the remaining $n - k$ objects and by letting λ be the identity representation. Likewise, conjecture 9 would follow by letting H be the product of the m symmetric groups, each on k objects, together with those permutations that exchange *all* (and not some) of the k objects in one of the m sets with those in another set. Again λ is the identity representation.

Although we cannot prove conjecture α , we can *prove conjecture 8 as well as conjecture 9 when $m = 2$* . As partial compensation for not being able to prove conjecture 9 generally, we will prove a somewhat deeper theorem for the $m = 2$ case than is contained in conjecture 9. This theorem refers to an arbitrary partitioning of A into four blocks, rather than merely into four square blocks. It also gives the *complete* necessary and sufficient conditions for equality to hold in conjectures 8 and 9. In section III we present an analogous theorem for determinants.

II. Statement and Proof of the Theorem on Permanents.

Definition. An s -square submatrix of a matrix Γ is any matrix formed by the intersection of s rows and columns of Γ . These need not be contiguous.

Theorem. Let $p > 0$ and $r > 0$ and let A be an $n = (p + r)$ -square h.p.s.d. matrix, partitioned as follows:

$$(1) \quad A = \begin{bmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{bmatrix}$$

where α is $p \times p$, β is $r \times r$, and γ is $p \times r$. In addition A does not have a zero row. Let $A(\lambda)$ be the n -square matrix obtained from A by replacing α by the matrix $\lambda\alpha$, λ being a complex number. Then all the coefficients in the p^{th} degree polynomial, $P(\lambda) = \text{Per } A(\lambda)$, are real and non-negative. Furthermore, let s be the largest integer for which there exists an s -square submatrix of γ with non-vanishing permanent. Then the coefficients of all terms λ^t in $P(\lambda)$ with $p - s \leq t \leq p$, are strictly positive, and the remaining coefficients are zero.

Conjectures 8 and 9 ($m = 2$) follow immediately from this theorem, for $\text{Per } A = P(1)$ while $\text{Per } \alpha \text{ Per } \beta$ is the coefficient of λ^p in $P(\lambda)$. Also, if $p = r$ then $\text{Per } \gamma \text{ Per } \gamma^*$ is the constant term in $P(\lambda)$. The conditions for equality are also seen at once to be:

Addendum to Conjectures (now theorems) 8 and 9 ($m = 2$). *The necessary and sufficient conditions for equality to hold in these two theorems is that either: (1) A has an identically zero row or (2) $\gamma = 0$, which means that $A = \alpha \dagger \beta$.*

Proof of theorem. Since A is h.p.s.d. its elements can be represented as $a_{ij} = (f^i, f^j)$, where f^i is an n -dimensional vector with components f_k^i , and $(g, f) \equiv \sum_1^n g_k^* f_k$. Also, since A does not have a zero row, we can assume $a_{ii} = 1$ (all i) without loss of generality. Let $R(k_1, \dots, k_n)$ be any function defined for all $k_i = 1, \dots, n$, and define inner product by

$$(R, G) \equiv \sum_{k_1=1}^n \cdots \sum_{k_n=1}^n R^*(k_1, \dots, k_n) G(k_1, \dots, k_n).$$

Define

$$T(i_1, \dots, i_n \mid k_1, \dots, k_n) \equiv f_{k_1}^{i_1} \cdots f_{k_n}^{i_n} \equiv T(I, K),$$

where I is the n -tuple $(i_1, \dots, i_n)$ and similarly for K . Define

$$B(I, J) = a_{i_1 i_1} a_{i_2 i_2}, \dots, a_{i_n i_n},$$

whence $B(I, J) = (T(I, K), T(J, K))$, where the inner product is taken with respect to the K variables. Let P be the permutation

$$\begin{pmatrix} 1 & \cdots & n \\ \alpha_1 & & \alpha_n \end{pmatrix}$$

and for all functions R , define

$$R(PK) \equiv PR(K) \equiv R(k_{\alpha_1}, \dots, k_{\alpha_n}).$$

Then

$$\text{Per } A = \sum_{P \in S_n} B(PI, I),$$

when $I = (1, 2, \dots, n)$. But since $B(PI, J) = B(I, P^{-1}J)$, we have

$$\text{Per } A = (n!)^{-1} \sum_{P \in S_n} \sum_{Q \in S_n} B(PI, QI),$$

when $I = (1, \dots, n)$. Define $F(K) = T(I, K)$ with $I = (1, \dots, n)$ and define

$$F^S(K) = \sum_{P \in S_n} F(PK).$$

Since $T(PI, K) = T(I, P^{-1}K)$ and since $(PF, G) = (F, P^{-1}G)$, we have the important result: $\text{Per } A = (n!)^{-1}(F^S, F^S)$.

Now, let S_p be the symmetric group on the indices $1, \dots, p$ and let S_r be the symmetric group on the indices $p+1, \dots, p+r = n$. Define

$$O_p = \sum_{P \in S_p} P \quad \text{and} \quad O_r = \sum_{P \in S_r} P,$$

and let $F'(K) = O_p O_r F(K)$. Since F is a product function, F' will thus be the product of two functions $F'(K) = F_1(K_p)F_2(K_r)$, where K_p denotes the variables $k_1, \dots, k_p$, and K_r denotes the remaining r variables. Any transposition permutation which is not in S_p or S_r will be called a "cross transposition" and any product of such transpositions will be called a "cross permutation". Thus, every cross permutation has the property that if $1 \leq q \leq p$ then either the index q is left undisturbed or else q is replaced by s , where $p+1 \leq s \leq n$. Let X be the set of those cross permutations, P , having the property that all indices q_i ($1 \leq q_i \leq p$) which are "moved" by P remain in serial order after the permutation. Likewise, all indices s_i ($p+1 \leq s_i \leq n$) which are "moved" by P remain in serial order. The set X has $\binom{n}{p} - 1$ members. Define the operator O as

$$(2) \quad O = I + \sum_{P \in X} P$$

where I is the identity permutation. Then clearly

$$F^S = OF' \quad \text{and} \quad OF^S = \binom{n}{p} F^S.$$

Hence

$$(3) \quad \text{Per } A = (n!)^{-1}(OF', OF') = (p!r!)^{-1}(F', OF')$$

Let us examine the terms on the right hand side of (3). The term $(p!r!)^{-1}(F', OF')$ is evidently $\text{Per } \alpha \text{ Per } \beta$, which is the coefficient of λ^p in $P(\lambda)$: Define $X_j \subset X$ to be those cross permutations composed of exactly j transpositions, for $1 \leq j \leq \min(p, r)$, and $X_0 = I$. Hence $O = \sum_i \bar{O}_i$ where

$$\bar{O}_i = \sum_{P \in X_i} P.$$

A little reflection will show that the term $(p!r!)^{-1}(F', \bar{O}_i F')$ in (3) is the coefficient of λ^{p-i} in $P(\lambda)$.

Consider first the term $(F', F') = p!r! \text{Per } \alpha \text{ Per } \beta$. It is clearly non-negative, and to show that it is positive it is sufficient to find a function $G(K)$ such that $(G, F') \neq 0$. There always exists a vector V , with components v_k , such that $(V, f^i) \neq 0$ for all $i = 1, \dots, n$. Let

$$G(K) = \prod_{i=1}^n v_{k_i}.$$

Plainly $(G, F') \neq 0$.

As for the remaining terms in (3), (F', PF') with $P \in X_i$ and $j \leq s$, we claim that each one is real and strictly positive. Consider, for example, the permutation which interchanges the indices 1 and $p+1$. This term is

$$(4) \quad E = \sum_K F_1^*(k_1, k_2, \dots, k_p) F_2^*(k_{p+1}, k_{p+2}, \dots, k_n) \\ \cdot F_1(k_{p+1}, k_2, \dots, k_p) F_2(k_1, k_{p+2}, \dots, k_n).$$

Let $\mathbf{x}$ be the variables $(k_2, \dots, k_p)$ and $\mathbf{y}$ be the variables $(k_{p+2}, \dots, k_n)$ and define

$$U(\mathbf{x}, \mathbf{y}) = \sum_{k_1} F_1^*(k_1, \mathbf{x}) F_2(k_1, \mathbf{y}),$$

so that

$$E = \sum_{\mathbf{x}} \sum_{\mathbf{y}} |U(\mathbf{x}, \mathbf{y})|^2 \geq 0.$$

To show that E is non-zero, it is sufficient to find a function $H(\mathbf{x}, \mathbf{y})$ such that

$$\sum_{\mathbf{x}} \sum_{\mathbf{y}} H(\mathbf{x}, \mathbf{y})^* U(\mathbf{x}, \mathbf{y}) \neq 0.$$

For each of the original vectors f^i , ($i = 1, \dots, p$), let $t_i \geq 1$ be the number of times this vector appears in the set $S_- = (f^1, f^2, \dots, f^p)$. Thus, $p - t_i$ vectors in S_- are distinct from, and not parallel to f^i (recall that all f^i have unit length). Likewise, for $i = p+1, \dots, n$, let t_i be the number of times f^i appears in the set $S_+ = (f^{p+1}, \dots, f^n)$. For each $i = 1, \dots, n$ we can always find a vector v^i such that $(v^i, f^i) = 0$ and $C_{ij} = (v^i, f^j) \neq 0$ whenever $f^j \neq f^i$. For each $1 \leq i \leq p$ and $p+1 \leq j \leq n$ define the function $H^{ii}(\mathbf{x}, \mathbf{y})$ as follows:

$$(5) \quad H^{ii}(\mathbf{x}, \mathbf{y}) = \left(\prod_{\alpha=2}^{t_i} f_{k_\alpha}^i \right) \left(\prod_{\beta=t_i+1}^p v_{k_\beta}^i \right) \left(\prod_{\gamma=p+2}^{p+t_i} f_{k_\gamma}^j \right)^* \left(\prod_{\delta=p+t_i+1}^n v_{k_\delta}^j \right)^*,$$

where the first and/or the third factors are to be omitted if t_i and/or t_j is equal to one. Clearly

$$(6) \quad d_{ii}(k) = \sum_{\mathbf{x}} \sum_{\mathbf{y}} F_1^*(k, \mathbf{x}) F_2(k, \mathbf{y}) H^{ii}(\mathbf{x}, \mathbf{y})^* \\ = f_k^i f_k^j (p - t_i)! t_i! (r - t_i)! t_i! \left(\prod_{l=t_i+1}^p C_{il}^* \right) \left(\prod_{l=p+t_i+1}^n C_{il} \right).$$

Now, if $s \geq 1$, there is at least one pair (i, j) with $1 \leq i \leq p$ and $p+1 \leq j \leq n$ such that $a_{ij} = (f^i, f^j) \neq 0$. Thus,

$$\sum_{\mathbf{x}} \sum_{\mathbf{y}} H^{ij}(\mathbf{x}, \mathbf{y})^* U(\mathbf{x}, \mathbf{y}) = \sum_k d^{ij}(k) \neq 0.$$

By the same method we shall see that all terms (F', PF') are real and strictly positive if P is composed of j cross transpositions and $j \leq s$. To begin with, we note that if P and Q are any two permutations in the set X_j then $(F', PF') = (F', QF')$. This is a consequence of two facts: firstly that F_1 and F_2 are symmetric functions and hence the order in which their arguments appear is immaterial; secondly that (F', PF') is a sum, as in eqn. (4), over the "dummy" variables, K , which may conveniently be relabeled so that the variables from

the set K_p which were "moved" by P are labeled $k_1, \dots, k_i$ and the variables from K_r which were "moved" are labeled $k_{p+1}, \dots, k_{p+i}$. If we denote by E_i the common value of (F', PF') with $P \in X_i$ then

$$(7) \quad E_i = \sum_k F_1^*(k_1, k_2, \dots, k_p) F_2(k_{p+1}, k_{p+2}, \dots, k_n)^* \\ \cdot F_1(k_{p+1}, \dots, k_{p+i} k_{i+1}, \dots, k_p) F_2(k_1, \dots, k_i, k_{p+i+1}, \dots, k_n).$$

Thus,

$$(8) \quad E_i = \sum_{\mathbf{x}} \sum_{\mathbf{y}} |U(\mathbf{x}, \mathbf{y})|^2$$

where $\mathbf{x}$ is now $(k_{i+1}, \dots, k_p)$ and $\mathbf{y}$ is $(k_{p+i+1}, \dots, k_n)$ and

$$(9) \quad U(\mathbf{x}, \mathbf{y}) = \sum_{k_1} \dots \sum_{k_i} F_1^*(k_1, \dots, k_i, \mathbf{x}) F_2(k_1, \dots, k_i, \mathbf{y}).$$

By eqn. (8), E_i is clearly real and non-negative. To prove that E_i is positive for $j \leq s$, it is only necessary to construct an H function along the lines indicated in eqn. (5), with the property that

$$\sum_{\mathbf{x}} \sum_{\mathbf{y}} H(\mathbf{x}, \mathbf{y})^* U(\mathbf{x}, \mathbf{y}) \neq 0.$$

Without loss of generality (but with considerable notational simplification) we may assume that the upper left hand s -square corner of γ has a non-vanishing permanent, *i.e.* the submatrix $a_{\alpha\beta}$ with $\alpha = 1, 2, \dots, s$ and $\beta = p+1, p+2, \dots, p+s$. This submatrix must necessarily have some j -square submatrix with non-vanishing permanent, and we shall again assume it is in the upper left hand corner of γ , *i.e.*, $a_{\alpha\beta}$ with $\alpha = 1, 2, \dots, j$ and $\beta = p+1, p+2, \dots, p+j$. Let P_j denote the permanent of this submatrix.

As before, $H(\mathbf{x}, \mathbf{y})$ will be a product function, $H_1(\mathbf{x})H_2(\mathbf{y})$, and H_1 and H_2 will themselves be product functions. To define H_1 , divide the set of vectors $S_- = (f^1, \dots, f^p)$ into two sets $L_- = (f^1, \dots, f^i)$ and $L_+ = (f^{i+1}, \dots, f^p)$. Let $h \leq j$ be the number of distinct vectors in L_- and, without loss of generality, we shall assume that the vectors in the set $G = (f^1, f^2, \dots, f^h)$ are distinct. It is always possible to find vectors $g^1, g^2, \dots, g^h$ such that for all $i = 1, \dots, h$ and all k , $(g^i, f^k) = 0$ if $f^k \neq f^i$ while $(g^i, f^i) = 1$. It is also always possible to find a vector v such that $(v, f^k) = 0$ if $f^k \notin G$, while $C_k = (v, f^k) \neq 0$ whenever $f^k \in G$.

For all $i = 1, \dots, h$ let u_i and t_i be the number of times the vector f^i appears in the sets L_+ and S_- , respectively. Define $b_i = \sum_{m=1}^i u_m$ for $i > 0$, $b_0 = 0$, and define $a_i = b_{i-1} + 1$ for $i > 0$. Now, let

$$(10) \quad H_1(\mathbf{x}) = \left\{ \prod_{i=1}^h \left(\prod_{\alpha=j+a_i}^{i+b_i} g_{k_\alpha}^{u_i} \right) \right\}^* \left(\prod_{\alpha=j+a_{h+1}}^p v_{k_\alpha} \right)^*,$$

where it is understood that any product is to be omitted if its upper index is smaller than its lower. Thus,

$$\begin{aligned}
 (11) \quad \sum_{\mathbf{x}} H_1(\mathbf{x})^* F_1^*(k_1, \dots, k_i, \mathbf{x}) \\
 = J^*(k_1, \dots, k_i) \prod_{i=1}^h t_i! (p - b_h - j)! \prod_{\alpha=j+a_h+1}^p C_\alpha,
 \end{aligned}$$

where J is the symmetric function

$$(12) \quad J(k_1, \dots, k_i) = O_i[f_{k_1}^1 f_{k_2}^2 \dots f_{k_i}^i].$$

Mutatis mutandis we define the function H_2 in the same way, with the result that

$$(13) \quad \sum_{\mathbf{y}} H_2^*(\mathbf{y}) F_2(k_1, \dots, k_i, k_{p+j+1}, \dots, k_n) = \varphi J(k_1, \dots, k_i).$$

Thus,

$$(14) \quad \sum_{\mathbf{x}} \sum_{\mathbf{y}} H(\mathbf{x}, \mathbf{y})^* U(\mathbf{x}, \mathbf{y}) = \psi \sum_{k_1} \dots \sum_{k_i} |J(k_1, \dots, k_i)|^2 = \chi P_i.$$

In (13) and (14), φ , ψ , and χ are non-zero constants, thus proving that $E_i \neq 0$.

To complete the proof of the theorem we must show that d_t , the coefficient of λ^t in $P(\lambda)$, is zero whenever $t = p - j$ with $j > s$. This follows from the trivial observation that all contributions to this coefficient involve $(p - j)$ elements of the matrix α , j elements of γ , j elements of γ^* and $(r - j)$ elements of β . Every selection of j elements from γ (such that no row or column is repeated) defines a j -square submatrix of γ . Let M be any such submatrix. Plainly, the only way in which the elements of M enter into d_t is through the combination $\text{Per } M$. Thus, if $\text{Per } M = 0$ for all possible j -square submatrices of γ then $d_t = 0$. Q.E.D.

III. Theorem on determinants:

Theorem. Let $p > 0$ and $r > 0$ and let A be an $n = (p + r)$ -square h.p.s.d. matrix, partitioned as follows:

$$A = \begin{bmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{bmatrix}$$

where α is $p \times p$, β is $r \times r$, and γ is $p \times r$. In addition α and β are positive definite. Let s be the largest integer for which there exists an s -square submatrix of γ with non-vanishing determinant. Let $A(\lambda)$ be the matrix obtained from A by replacing α by the matrix $\lambda\alpha$, λ being a complex number. Then the coefficient of λ^{p-t} in the p^{th} degree polynomial $P(\lambda) = \text{Det } A(\lambda)$ is: (a) zero if $t > s$; (b) real and strictly positive if t is even and $t \leq s$; (c) real and strictly negative if t is odd and $t \leq s$. Furthermore, if α or β is merely semi-definite, then $P(\lambda)$ vanishes identically.

The proof of this theorem is almost exactly the same as for the preceding one, except that symmetrizers are replaced by anti-symmetrizers.

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