On the Kinematic Formula in Integral Geometry

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1. Introduction. The kinematic density in euclidean space was first introduced by Poincaré. In modern terminology it is the Haar measure of the group of motions which acts on the space. One of the basic problems in integral geometry is to find explicit formulas for the integrals of geometric quantities over the kinematic density in terms of known integral invariants. An important example is the kinematic formula of Blaschke, as follows [1]: Let E^3 be the euclidean three-space, and let dg be the kinematic density, so normalized that the measure of all positions about a point is $8\pi^2$. (In other words, the measure of all positions of a domain D with volume V such that D contains a fixed point is $8\pi^2 V$.) Let D_i , $i = 1, 2$, be a domain with smooth boundary, of which V_i , $M_1^{(i)}$, $M_2^{(i)}$, χ_i are respectively the volume, the area of the boundary, the integral of mean curvature of the boundary, and the Euler characteristic. Then, if D_1 is fixed and D_2 moves with kinematic density dg , we have

$$(1) \quad \int \chi(D_1 \cap g D_2) dg = 8\pi^2(4\pi\chi_1 V_2 + 4\pi\chi_2 V_1 + M_1^{(1)} M_2^{(2)} + M_2^{(1)} M_1^{(2)}),$$

where $\chi(D_1 \cap g D_2)$ is the Euler characteristic of the intersection $D_1 \cap g D_2$. Formula (1) contains as special or limiting cases many of the formulas in integral geometry in euclidean three-space. It was generalized to euclidean n -space by C. T. Yen and the present author [3] and to the non-euclidean spaces by Santalo [4].

This paper will be concerned with a pair of compact submanifolds (without boundary) M^p , M^q of arbitrary dimensions p , q in euclidean n -space E^n and with the integration with respect to dg of certain geometrical quantities of the submanifold $M^p \cap g(M^q)$. The latter depend only on the induced riemannian metric of $M^p \cap g(M^q)$ and are defined as follows:

Let X be a riemannian manifold of dimension k . In the bundle B of orthonormal frames over X we have the coframes, which consist of k linearly independent linear differential forms φ_α , such that the riemannian metric in X is

$$(2) \quad ds^2 = \sum_{\alpha} \varphi_{\alpha}^2.$$

(In this and other formulas in this section we have $1 \leq \alpha, \beta, \gamma, \delta \leq k$.) Let

$$(3) \quad \varphi_{\alpha\beta} = -\varphi_{\beta\alpha}$$

be the connection forms of the Levi-Civita connection. They are linear differential forms in B and satisfy the "structural equations"

$$(4) \quad \begin{aligned} d\varphi_\alpha &= \sum_\beta \varphi_\beta \wedge \varphi_{\beta\alpha}, \\ d\varphi_{\alpha\beta} &= \sum_\gamma \varphi_{\alpha\gamma} \wedge \varphi_{\gamma\beta} + \Phi_{\alpha\beta}, \end{aligned}$$

where

$$(5) \quad \Phi_{\alpha\beta} = \frac{1}{2} \sum_{\gamma, \delta} S_{\alpha\beta\gamma\delta} \varphi_\gamma \wedge \varphi_\delta.$$

The coefficients $S_{\alpha\beta\gamma\delta}$ are functions in B and have the symmetry properties

$$(6) \quad \begin{aligned} S_{\alpha\beta\gamma\delta} &= -S_{\alpha\beta\delta\gamma} = -S_{\beta\alpha\gamma\delta}, \\ S_{\alpha\beta\gamma\delta} &= S_{\gamma\delta\alpha\beta}, \\ S_{\alpha\beta\gamma\delta} + S_{\alpha\gamma\delta\beta} + S_{\alpha\delta\beta\gamma} &= 0. \end{aligned}$$

From these functions we construct the following scalar invariants in X :

$$(7) \quad I_e = \frac{(-1)^{e/2}(k-e)!}{2^{k/2}k!} \sum \delta \begin{pmatrix} \alpha_1 & \cdots & \alpha_e \\ \beta_1 & \cdots & \beta_e \end{pmatrix} S_{\alpha_1\alpha_2\beta_1\beta_2} \cdots S_{\alpha_{e-1}\alpha_e\beta_{e-1}\beta_e},$$

where e is an even integer satisfying $0 \leq e \leq k$,

$$\delta \begin{pmatrix} \alpha_1 & \cdots & \alpha_e \\ \beta_1 & \cdots & \beta_e \end{pmatrix}$$

is equal to $+1$ or -1 according as $\alpha_1, \dots, \alpha_e$ is an even or odd permutation of $\beta_1, \dots, \beta_e$, and is otherwise zero, and the summation is over all $\alpha_1, \dots, \alpha_e$ and $\beta_1, \dots, \beta_e$, independently from 1 to k . When X is oriented and compact, we let

$$(8) \quad \mu_e(X) = \int_X I_e dv,$$

where dv is the volume element. Thus $\mu_e(X)$ are integral invariants of X , with $\mu_0(X)$ equal to the total volume. If k is even, then the Gauss-Bonnet formula says that [2]

$$(9) \quad \mu_k(X) = \frac{(2\pi)^{k/2}}{(k-1)(k-3) \cdots 1} \chi(X),$$

where $\chi(X)$ is the Euler-Poincaré characteristic of X . The numerical coefficient before the summation in (7) is so chosen that $I_e(X) = 1$, when X is the unit k -sphere in E^{k+1} with the induced metric.

These integral invariants appear in a natural way in Weyl's formula for the

volume of a tube [8]. In fact, suppose X be imbedded in E^n and suppose T_ρ be a tube of radius ρ about X , *i.e.*, the set of all points at a distance $\leq \rho$ from X . Then, for ρ sufficiently small, Weyl proved that the volume of T_ρ is given by the formula

$$(10) \quad V(T_\rho) = O_m \sum_e \frac{(e-1)(e-3) \cdots 1}{(m+e)(m+e-2) \cdots m} \mu_e(X) \rho^{m+e},$$

$$e \text{ even}, \quad 0 \leq e \leq k, \quad m = n - k,$$

where O_m is the volume of the unit sphere (of dimension $m - 1$) in E^m , and its value is given by

$$(11) \quad O_m = 2\pi^{m/2} / \Gamma\left(\frac{m}{2}\right).$$

This formula is also valid for large values of ρ , provided that the volume is counted with multiplicities for domains where T_ρ intersects itself.

The main result of this paper is the kinematic formula

$$(12) \quad \int \mu_e(M^p \cap gM^q) dg = \sum_{\substack{0 \leq i \leq e \\ i \text{ even}}} c_i \mu_i(M^p) \mu_{e-i}(M^q),$$

$$e \text{ even}, \quad 0 \leq e \leq p + q - n,$$

where c_i , to be given below (formula (79)), are numerical constants depending on n, p, q, e .

2. Preliminaries. Let E^n be the euclidean space of dimension n , and $xe_1 \cdots e_n$ an orthonormal frame, or simply a frame, in E^n , so that $x \in E^n$ and $e_1, \dots, e_n$ are vectors through x , whose scalar products satisfy the relations

$$(13) \quad (e_A, e_B) = \delta_{AB}, \quad 1 \leq A, B, C \leq n.$$

The kinematic density is the volume element in the space of all frames, and is given by

$$(14) \quad dg = \bigwedge_{\substack{A, B, C \\ B < C}} (dx, e_A)(de_B, e_C),$$

where the right-hand side is the exterior product of the differential forms in question. By (14), dg is normalized. The factor

$$(15) \quad dg_0 = \bigwedge_{B < C} (de_B, e_C)$$

in this product is the kinematic density about a point, and we have

$$(16) \quad dg = dx dg_0,$$

where dx is the volume element in E^n .

To define the density dE^p in the space of all p -dimensional linear subspaces

E^p in E^n consider the frames $xe_1 \cdots e_n$, such that $x \in E^p$ and $e_1, \dots, e_p$ are parallel to E^p (so that $e_{p+1}, \dots, e_n$ are normal vectors of E^p). Then

$$(17) \quad dE^p = \bigwedge_{h, \lambda, \mu} (dx, e_\lambda)(de_h, e_\mu), \quad 1 \leq h \leq p, \quad p+1 \leq \lambda, \mu \leq n.$$

Similarly, the density in the space of all E^p through a fixed point O of E^n is

$$(18) \quad dE^p(O) = \bigwedge_{h, \mu} (de_h, e_\mu), \quad 1 \leq h \leq p, \quad p+1 \leq \mu \leq n,$$

where $e_1, \dots, e_p$ are vectors of the frame, which span E^p .

We will also consider the space of all (x, E^p) , such that E^p is a p -dimensional linear subspace of E^n and $x \in E^p$. Its density will be

$$(19) \quad d(x, E^p) = \bigwedge_{A, h, \mu} (dx, e_A)(de_h, e_\mu),$$

and we have

$$(20) \quad d(x, E^p) = dx(E^p) dE^p,$$

where $dx(E^p)$ is the volume element in E^p .

The total volumes

$$\int dg_0, \quad \int dE^p(O)$$

are finite, and we will give their values. Let $e_1, \dots, e_n$ be a frame through O . By mapping it to the last vector e_n , we get a fibering of the space $SO(n)$ of all frames through O , whose base space is the unit sphere in E^n and whose fiber is $SO(n-1)$. It follows by induction that

$$(21) \quad \int_{SO(n)} dg_0 = O_n O_{n-1} \cdots O_2.$$

The Grassmann manifold of all oriented E^p through O is the quotient space $SO(n)/SO(p) \times SO(n-p)$, whose total volume is $O_n \cdots O_2 / O_p \cdots O_2 O_{n-p} \cdots O_2$. It follows that the total volume of the Grassmann manifold $Gr(p)$ of all un-oriented E^p through O is

$$(22) \quad \int_{Gr(p)} dE^p(O) = O_n \cdots O_1 / O_p \cdots O_1 O_{n-p} \cdots O_1.$$

Let $E_0^p \in Gr(p)$ be a fixed linear subspace of dimension p through O . Let E_0^p and E^p be spanned by the vectors e_h^0 and e_k respectively ($1 \leq h, k \leq p$), such that

$$(23) \quad (e_h^0, e_k^0) = (e_h, e_k) = \delta_{hk}.$$

Then $\Delta(E_0^p, E^p) = |\det(e_h^0, e_k)|$ depends only on E_0^p and E^p . Since $dE^p(O)$ is invariant under the rotations about O , the integral $\int_{Gr(p)} \Delta^N(E_0^p, E^p) dE^p(O)$ is independent of E_0^p and is a numerical constant, to be denoted by $c(n, p, N)$.

We wish to show that its value is given by

$$(24) \quad c(n, p, N) = \int_{Gr(p)} \Delta^N dE^p(O) = \frac{O_{n+N} \cdots O_1}{O_{p+N} \cdots O_1} \frac{O_N \cdots O_1}{O_{n-p+N} \cdots O_1}.$$

To prove this we need a formula for $dE^p(O)$, which is also useful for other purposes. In fact, let E_0^q be a fixed q -dimensional linear subspace through O , with $q \leq p$. Let $Gr^*(p)$ be the subset of $Gr(p)$, whose E^p satisfy the condition $\dim(E^p \cap (E_0^q)^\perp) = p - q$, where $(E_0^q)^\perp$ is the $(n - q)$ -dimensional linear subspace through O orthogonal to E_0^q . $Gr^*(p)$ differs from $Gr(p)$ by a set of measure zero. We choose the frames $Oe_1 \cdots e_n$ associated to $E^p \in Gr^*(p)$, such that $e_1, \dots, e_{p-q}$ span $E^p \cap (E_0^q)^\perp$ and $e_1, \dots, e_p$ span E^p . Let $Oe_1 \cdots e_{p-q} f_{p-q+1} \cdots f_n$ be frames satisfying the condition that $f_{n-q+1}, \dots, f_n$ are constant vectors spanning E_0^q . Then e_i differ from f_i by a proper orthogonal transformation

$$(25) \quad e_i = \sum_j u_{ij} f_j, \quad p - q + 1 \leq i, j \leq n,$$

where

$$(26) \quad u_{ii} = (e_i, f_i).$$

Since $f_{n-q+1}, \dots, f_n$ are constant vectors, we have

$$(de_\alpha, f_\rho) = -(df_\rho, e_\alpha) = 0, \quad 1 \leq \alpha \leq p - q, \quad n - q + 1 \leq \rho \leq n.$$

It follows that

$$dE^p(O) = \bigwedge_{\alpha, r, \lambda, \mu} (de_\alpha, e_\lambda)(de_r, e_\mu) = D^{p-q} \bigwedge_{\alpha, r, s, \mu} (de_\alpha, f_s)(de_r, e_\mu),$$

$$p + 1 \leq \lambda, \mu \leq n; \quad p - q + 1 \leq r \leq p; \quad p - q + 1 \leq s \leq n - q,$$

where

$$(27) \quad D = \det(u_{\lambda\mu}).$$

We write the last formula as

$$(28) \quad dE^p(O) = D^{p-q} dE^{p-q}((E_0^q)^\perp) dE^p(E^{p-q}),$$

where

$$(29) \quad dE^p(E^{p-q}) = \bigwedge_{r, \mu} (de_r, e_\mu)$$

is the density in the space of all E^p about $E^{p-q} = E^p \cap (E_0^q)^\perp$, and

$$(30) \quad dE^{p-q}((E_0^q)^\perp) = \bigwedge_{\alpha, s} (de_\alpha, f_s)$$

is the density in the space of all E^{p-q} through O and in $(E_0^q)^\perp$.

We now integrate both sides of the equation (28) over $Gr^*(p)$. In integrating its right-hand side, we first hold E^{p-q} fixed and then integrate over all E^{p-q}

through O and in $(E_0^q)^\perp$. The result is

$$c(n - p + q, q, p - q) = \frac{O_n \cdots O_1 O_{p-q} \cdots O_1}{O_p \cdots O_1 O_{n-q} \cdots O_1}.$$

This reduces to (24) by a change of notation.

In the same way we can derive for dE^p a formula analogous to (28). In fact, let E_0^{n-q} be a fixed space of dimension $n - q$ in E^n , not necessarily through O , with $q \leq p$. Consider those E^p such that $\dim(E^p \cap E_0^{n-q}) = p - q$, and let $E^{p-q} = E^p \cap E_0^{n-q}$. Let $(E_0^{n-q})^\perp$ (respectively $E^{p-q}(E^p)^\perp$) be the space through O orthogonal to E_0^{n-q} (respectively the space through O orthogonal to E^{p-q} and parallel to E^p). Both are of dimension q and the invariant Δ is defined. Then we have the following density formula

$$(31) \quad dE^p = \Delta^{p-q+1} dE^{p-q}(E_0^{n-q}) dE^p(E^{p-q}),$$

where $dE^{p-q}(E_0^{n-q})$ (respectively $dE^p(E^{p-q})$) is the density in the space of all E^{p-q} in E_0^{n-q} (respectively all E^p through E^{p-q}). The difference between (28) and (31) lies in the fact that in (28) all the linear subspaces under consideration are through O , while this condition is not imposed in (31).

Let $f(E^p)$ be an integrable function which depends only on $E^{p-q} = E^p \cap E^{n-q}$. Then the integration with respect to $dE^p(E^{p-q})$ can be carried out, and we have

$$(32) \quad \begin{aligned} \int f(E^p) dE^p &= c(n - p + q, q, p - q + 1) \int f(E^{p-q}) dE^{p-q}(E_0^{n-q}) \\ &= \frac{O_{n+1} \cdots O_1 O_{p-q+1} \cdots O_1}{O_{p+1} \cdots O_1 O_{n-q+1} \cdots O_1} \int f(E^{p-q}) dE^{p-q}(E_0^{n-q}). \end{aligned}$$

3. A differential formula. Although we will need the formula only in the special case of the euclidean space, it is desirable to formulate the problem for a general homogeneous space, as follows: Let G be a Lie group, acting on a left coset space G/H by left multiplication. Let M^r, M^s be submanifolds in G/H , of dimensions r, s respectively. Consider the space

$$(33) \quad Y = \bigcup_{g \in G} (M^r \cap gM^s) \times g,$$

with an obvious topology. A point in this space can be written

$$(x = gy, g \mid x \in M^r, y \in M^s).$$

By sending this point to $(x, y) \in M^r \times M^s$, we get a mapping

$$(34) \quad \psi : Y \rightarrow M^r \times M^s.$$

By definition we have $\psi^{-1}(x, y) = (x = gH_y y, gH_y)$, where H_y is the group of isotropy at y , i.e., the subgroup of all transformations of G , which leave y fixed. Suppose that G/H has a riemannian structure invariant under G . Then $M^r \cap gM^s$ has a volume element Φ_g , and Y has the volume element $\Phi_g dg$. The

problem is to express $\Phi_d dg$ as a multiple of $dh_v \Omega_1 \Omega_2$, where Ω_1, Ω_2 are respectively the volume elements of M^r, M^s , and dh_v that of H_v .

We wish to solve this problem for a special case: By a k -field in E^n is meant a point $x \in E^n$, together with an ordered set of vectors $e_1, \dots, e_k$, satisfying

$$(e_\gamma, e_\tau) = \delta_{\gamma\tau}, \quad 1 \leq \gamma, \tau \leq k.$$

Let $F_{n,k}$ be the space of all k -fields of E^n , so that $F_{n,0} = E^n$. We can write $F_{n,k}$ as a left coset space G/H , where G is the group of all proper motions in E^n and H is isomorphic to $SO(n-k)$.

Let M^p be a submanifold of dimension p of E^n , and M^r be the submanifold of $F_{n,k}$, which consists of the k -fields $x e_1 \cdots e_k$, such that $x \in M^p$ and $e_1, \dots, e_k$ are tangent to M^p at x ($k \leq p$). Then

$$(35) \quad r = p + (p-1) + \cdots + (p-k) = \frac{1}{2}(k+1)(2p-k).$$

To define a density Θ on M^r choose the vectors $e_{k+1}, \dots, e_p$, so that $e_1, \dots, e_p$ constitute an orthonormal frame of tangent vectors to M^p at x . Then we define

$$(36) \quad \Theta = \bigwedge_{L, \gamma, \delta} (dx, e_L)(de_\gamma, e_\delta), \quad 1 \leq L \leq p, \quad 1 \leq \gamma \leq k, \quad \gamma+1 \leq \delta \leq p,$$

which is clearly independent of the choice of $e_{k+1}, \dots, e_p$.

Similarly let M^s be the manifold of all tangent k -fields of a submanifold M^q of dimension q in E^n . Since M^p and gM^q generally intersect in a submanifold of dimension $p+q-n$, our problem is meaningful, only when $k \leq p+q-n$, which we suppose from now on. We will solve our problem for the submanifolds M^r, M^s in $F_{n,k}$.

As we will be using indices over different ranges, let us list them in the following table, which will be strictly followed through the rest of this section:

$$(37) \quad \begin{aligned} 1 &\leq A, B, C \leq n; & 1 &\leq \alpha, \beta \leq p+q-n; \\ p+q-n+1 &\leq i, j \leq n; & p+1 &\leq \lambda, \mu \leq n; \\ q+1 &\leq \rho, \sigma \leq n; & p+q-n+1 &\leq a, b \leq p; \\ p+q-n+1 &\leq h, l \leq q; & k+1 &\leq \zeta \leq n; \\ 1 &\leq \gamma, \tau, \xi \leq k; & \gamma+1 &\leq \delta \leq p+q-n. \end{aligned}$$

Let Oa_A be a fixed frame and $O'a'_A$ the moving frame in E^n , i.e., the frame obtained from the former by a proper motion g . By (14) the kinematic density is

$$(38) \quad dg = \bigwedge_{A; B < C} (dO', a'_A)(da'_B, a'_C).$$

By gM^q we mean the submanifold obtained from M^q by the motion g , so that gM^q has the same relative position to $O'a'_A$ as M^q is to Oa_A .

Let xe_A be orthonormal frames, so that $x \in M^p$ and $e_1, \dots, e_p$ are tangent vectors to M^p at x . Then

$$(39) \quad (dx, e_\lambda) = 0$$

and it is well-known that

$$(40) \quad de_\lambda \equiv 0, \text{ mod } dx, e_\mu.$$

Similarly, let $x'e'_A$ be frames, such that $x' \in gM^q$ and $e'_1, \dots, e'_q$ are tangent vectors to gM^q at x' . Let d' be the differential relative to the moving frame $O'a'_A$, so that

$$(41) \quad d'O' = d'a'_A = 0.$$

Then, as above,

$$(39a) \quad (d'x', e'_\rho) = 0$$

and

$$(40a) \quad d'e'_\rho \equiv 0, \text{ mod } d'x', e'_\sigma.$$

Suppose g be generic, so that $M^p \cap gM^q$ is of dimension $p + q - n$. We restrict the above families of frames by the condition

$$(42) \quad x = x', \quad e_\alpha = e'_\alpha.$$

Geometrically the latter means that $x \in M^p \cap gM^q$ and e_α are the tangent vectors to $M^p \cap gM^q$ at x . The two submanifolds M^p and gM^q have at x a scalar invariant, which is

$$(43) \quad \Delta = |\det(e_\alpha, e'_\rho)| = |\det(e_\lambda, e'_h)|.$$

This is the invariant introduced in §2 for the spaces spanned by e_α, e'_ρ respectively in the orthogonal complement of the space spanned by e_α . For a pair of hypersurfaces ($p = q = n - 1$) it is the absolute value of the sine of the angle between their normal vectors. Using these frames we have

$$(44) \quad \Phi_\sigma = \bigwedge (dx, e_\alpha)(de_\gamma, e_\delta).$$

Let us write

$$x = x' = O' + \sum x'_A a'_A.$$

By differentiation it follows that

$$dO' \equiv dx - d'x, \quad \text{mod } da'_A.$$

Mod da'_A we have therefore

$$\begin{aligned} \bigwedge (dx, e_\alpha)(dO', a'_A) &= \bigwedge (dx, e_\alpha)(dO', e'_A) \\ &\equiv \bigwedge (dx, e_\alpha)(dx - d'x', e'_A) \\ &= \bigwedge (dx, e_\alpha)(dx - d'x', e_\alpha)(dx - d'x', e'_i) \\ &= \pm \bigwedge (dx, e_\alpha)(d'x', e_\alpha)(dx - d'x', e'_i)(dx, e'_\rho) \\ &= \pm \bigwedge (dx, e_\alpha)(dx, e'_\rho)(d'x', e_\alpha)(d'x', e'_h) \\ &= \pm \Delta \bigwedge (dx, e_\alpha)(dx, e_\alpha)(d'x', e_\alpha)(d'x', e'_h). \end{aligned}$$

In these exterior products the indices run over the ranges agreed on in (37). In the reduction the first step follows from the fact that a'_A differ from e'_A by a proper orthogonal transformation, the fourth step makes use of (39a), and the fifth step uses the fact that an exterior product is zero, if it contains more than p factors involving dx . Observe that the product in the last expression is the product of the volume elements of M^p and M^q .

Let

$$e'_A = \sum_B u'_{AB} a'_B ,$$

so that (u'_{AB}) is a proper orthogonal matrix. Then

$$(d - d')e'_A = \sum_B u'_{AB} da'_B ,$$

and we have

$$\bigwedge_{A < B} (da'_A , a'_B) = \bigwedge_{A < B} (de'_A - d'e'_A , e'_B) .$$

It follows that

$$\begin{aligned} & \bigwedge_{A < B} (de_\gamma , e_\delta)(da'_A , a'_B) \\ &= \bigwedge_{A < B} (de_\gamma , e_\delta)(de'_A - d'e'_A , e'_B) \\ &= \bigwedge_{\substack{\tau < A \\ \zeta < B}} (de_\gamma , e_\delta)(de_\tau - d'e_\tau , e'_A)(de'_\zeta - d'e'_\zeta , e'_B) \\ &= \pm \bigwedge_{\zeta < B} (de_\gamma , e_\delta)(d'e_\gamma , e_\delta)(de_\tau - d'e_\tau , e'_\tau)(de'_\zeta - d'e'_\zeta , e'_B) \\ &\equiv \pm \bigwedge_{\zeta < B} (de_\gamma , e_\delta)(d'e_\gamma , e_\delta)(de_\tau - d'e_\tau , e'_\tau)(de_\tau , e'_\tau)(de'_\zeta - d'e'_\zeta , e'_B) \\ &= \pm \Delta^k \bigwedge_{\zeta < B} (de_\gamma , e_\delta)(de_\gamma , e_\alpha)(d'e_\gamma , e_\delta)(d'e_\tau , e'_\tau)(de'_\zeta - d'e'_\zeta , e'_B) . \end{aligned}$$

Here the congruence is mod $d'x'$ and follows from (40a) in view of the relation $(d'e_\tau , e'_\rho) = -(d'e'_\rho , e_\tau)$.

The densities on M^r , M^s are by definition respectively

$$\begin{aligned} (45) \quad \Theta_1 &= \bigwedge (dx , e_\alpha)(dx , e_\alpha)(de_\gamma , e_\delta)(de_\tau , e_\alpha) , \\ \Theta_2 &= \bigwedge (d'x' , e'_\alpha)(d'x' , e'_\alpha)(d'e_\gamma , e_\delta)(d'e_\tau , e'_\tau) . \end{aligned}$$

Remembering the meaning of the operator $d - d'$, we also see that

$$(46) \quad dh_\nu = \bigwedge_{\zeta < B} (de'_\zeta - d'e'_\zeta , e'_B) .$$

Multiplying the expressions in (38), (44) and making use of the two long equations above, we get

$$(47) \quad \Phi_\sigma dg = \pm \Delta^{k+1} dh_\nu \Theta_1 \Theta_2 .$$

This is the density formula desired, the factor being Δ^{k+1} .

4. Higher sectional curvatures and mixed curvatures of a pair of riemannian manifolds. The invariants I_* discussed in §1 can be generalized to the notion of a mixed curvature of a pair of riemannian manifolds. Let M^p, M^q be two riemannian manifolds, of dimensions p, q respectively. Let B_1, B_2 be respectively the bundles of k -fields over M^p, M^q . Thus B_1 (resp. B_2) is the space of all $xe_1 \cdots e_k$ (resp. $yf_1 \cdots f_k$) such that $x \in M^p$ (resp. $y \in M^q$) and $e_1, \dots, e_k$ (resp. $f_1, \dots, f_k$) are k mutually perpendicular unit tangent vectors at x (resp. at y). Let $\tilde{B}_1$ and $\tilde{B}_2$ be the principal bundles of orthonormal frames of M^p and M^q respectively. Then we have

$$(48) \quad \tilde{B}_1 \times \tilde{B}_2 \xrightarrow{\pi} B_1 \times B_2 \xrightarrow{\psi} M^p \times M^q,$$

where the mapping π is defined by taking the first k vectors of the frames. As in §1 we regard the curvature functions

$$P_{lmrs}, Q_{\lambda\mu\rho\sigma}, 1 \leq l, m, r, s \leq p; \quad 1 \leq \lambda, \mu, \rho, \sigma \leq q$$

of M^p and M^q as scalar functions in $\tilde{B}_1 \times \tilde{B}_2$. Following the definition of I_* we set

$$(49) \quad I(f, g) = \frac{(k-f-g)!}{(-2)^{f+g/2}k!} \sum \delta \begin{pmatrix} \alpha_1 \cdots \alpha_{f+g} \\ \beta_1 \cdots \beta_{f+g} \end{pmatrix} \\ \cdot P_{\alpha_1 \alpha_2 \beta_1 \beta_2} \cdots P_{\alpha_{f-1} \alpha_f \beta_{f-1} \beta_f} Q_{\alpha_{f+1} \alpha_{f+2} \beta_{f+1} \beta_{f+2}} \cdots Q_{\alpha_{f+g-1} \alpha_{f+g} \beta_{f+g-1} \beta_{f+g}},$$

where f, g are even integers, and the α 's and β 's run from 1 to k . Clearly $I(f, g)$ is a scalar function in $B_1 \times B_2$ and is thus defined for a pair of k -fields of M^p and M^q respectively. When $M^p = M^q$ and the two k -fields are chosen to be identical, $I(f, g) = I_{f+g}$, $k = f + g$, will depend only on the k -plane spanned by the k -field. It is then the "sectional curvature for a k -plane" of a riemannian manifold, generalizing the riemannian sectional curvature for a two-plane. This high-dimensional sectional curvature has recently been studied by various authors [6], in connection with some applications of the Gauss-Bonnet formula.

The density Θ in (36), which we will now denote by Θ_1 , defines a volume element in B_1 . The product $\Theta_1 \Theta_2$, where Θ_2 is the volume element in B_2 , gives a volume element in $B_1 \times B_2$. We wish to prove the formula

$$(50) \quad \int_{B_1 \times B_2} I(f, g) \Theta_1 \Theta_2 = O_p \cdots O_{p-k+1} O_q \cdots O_{q-k+1} \mu_f(M^p) \mu_g(M^q),$$

provided that M^p and M^q are both compact.

To prove this formula we fix a point $(x, y) \in M^p \times M^q$ and integrate over the fiber $\psi^{-1}(x, y)$, i.e., over all k -fields with origins at x and y . Let e_i^0 and f_λ^0 be fixed orthonormal frames at x, y respectively and let $P_{lmrs}^0, Q_{\lambda\mu\rho\sigma}^0$ be the values of the curvature functions for these frames. For the k -fields given by

$$e_\alpha = \sum_i u_{\alpha i} e_i^0, \quad f_\alpha = \sum_\lambda v_{\alpha \lambda} f_\lambda^0,$$

we have

$$P_{\alpha\beta\gamma\delta} = \sum u_{\alpha l} u_{\beta m} u_{\gamma r} u_{\delta s} P_{lmrs}^0$$

$$Q_{\alpha\beta\gamma\delta} = \sum v_{\alpha\lambda} v_{\beta\mu} v_{\gamma\rho} v_{\delta\sigma} Q_{\lambda\mu\rho\sigma}^0.$$

Thus the integrand is a polynomial F in $u_{\alpha l}$, $v_{\beta\lambda}$, of even degrees in each of these sets of variables, and no $u_{\alpha l}$ or $v_{\beta\lambda}$ appears in a power higher than the second. If $e_1, \dots, e_k$ is a k -field at x , replacement of $u_{\alpha l}$ by $-u_{\alpha l}$, for a fixed l and all α , will give a new k -field at x . This means that the terms in F , which give a non-zero value to the integral, must contain each $u_{\alpha l}$ (and similarly each $v_{\beta\lambda}$) exactly in its square. It follows that the only terms of F to be concerned are those for which $\alpha_1, \dots, \alpha_f$ and $\alpha_{f+1}, \dots, \alpha_{f+g}$ are permutations of $\beta_1, \dots, \beta_f$ and $\beta_{f+1}, \dots, \beta_{f+g}$ respectively and hence that the integral over the fiber $\psi^{-1}(x, y)$ is a constant multiple of $I_f(x)I_g(y)$, where each factor stands for the invariant defined in (7) for the manifolds M^p and M^q respectively. This proves (50), except for the value of the constant factor. The latter is determined by taking M^p and M^q to be the unit spheres in E^{p+1} and E^{q+1} respectively. Then the integrand at the left-hand side of (50) is equal to 1 and the integral itself equal to

$$O_{p+1} \cdots O_{p-k+1} O_{q+1} \cdots O_{q-k+1}.$$

5. A Meusnier-Euler type theorem. We will pursue the situation studied in §3, where M^p and gM^q intersect in a submanifold M^{p+q-n} of dimension $p+q-n$. We will use the same notations, and in particular the ranges of indices in the table (37). The calculation of $\mu_s(M^{p+q-n})$ in the main formula (12) needs a knowledge of I_s , which in turn depends on the induced riemannian metric of M^{p+q-n} . Our first problem is therefore to express the second fundamental forms of M^{p+q-n} in terms of those of M^p and gM^q and the relative position of their tangent spaces.

For the families of frames xe_A and xe'_A introduced in §3, let

$$(51) \quad \begin{aligned} \omega_A &= (dx, e_A), & \omega'_A &= (d'x', e'_A), \\ \omega_{AB} &= (de_A, e_B), & \omega'_{AB} &= (d'e'_A, e'_B), \end{aligned}$$

so that

$$(52) \quad \omega_{AB} + \omega_{BA} = 0, \quad \omega'_{AB} + \omega'_{BA} = 0.$$

Equations (39) and (39a) can be written

$$(53) \quad \omega_\lambda = 0, \quad \omega'_\rho = 0.$$

while (40) and (40a) allow us to put, *when restricted to M^{p+q-n}* ,

$$(54) \quad \omega_{\alpha\lambda} = \sum h_{\alpha\lambda\beta} \omega_\beta, \quad \omega'_{\alpha\rho} = \sum h'_{\alpha\rho\beta} \omega_\beta,$$

where

$$(55) \quad h_{\alpha\lambda\beta} = h_{\beta\lambda\alpha} , \quad h'_{\alpha\rho\beta} = h'_{\beta\rho\alpha} .$$

The second fundamental forms of M^p and gM^q restricted to their intersection M^{p+q-n} are respectively

$$(56) \quad \begin{aligned} II_\lambda &= (e_\lambda , d^2x) = \sum_{\alpha,\beta} h_{\alpha\lambda\beta} \omega_\alpha \omega_\beta \\ II'_\rho &= (e'_\rho , d'^2x) = \sum_{\alpha,\beta} h'_{\alpha\rho\beta} \omega_\alpha \omega_\beta . \end{aligned}$$

We wish to express (e_a , d^2x) as a linear combination of II_λ and II'_ρ .

For this purpose set

$$(57) \quad e'_\rho = \sum_a u_{\rho a} e_a + \sum_\lambda u_{\rho\lambda} e_\lambda ,$$

so that

$$(58) \quad u_{\rho a} = (e'_\rho , e_a) .$$

Under our hypothesis $\Delta = |\det (u_{\rho a})| \neq 0$, let $(v_{b\sigma})$ be the inverse matrix of $(u_{\rho a})$, so that

$$(59) \quad \sum_\sigma v_{b\sigma} u_{\sigma a} = \delta_{ba} , \quad \sum_a u_{\rho a} v_{a\sigma} = \delta_{\rho\sigma} .$$

Then we have

$$(60) \quad e_a = \sum_\rho v_{a\rho} e'_\rho + \sum_\lambda v_{a\lambda} e_\lambda ,$$

where

$$(61) \quad v_{a\lambda} = - \sum_\rho v_{a\rho} u_{\rho\lambda} .$$

The condition $(e'_\rho , e'_\sigma) = \delta_{\rho\sigma}$ is expressed by

$$(62) \quad \delta_{\rho\sigma} = \sum_a u_{\rho a} u_{\sigma a} + \sum_\lambda u_{\rho\lambda} u_{\sigma\lambda} .$$

From (60) we have

$$(63) \quad II_a = (e_a , d^2x) = \sum_{\alpha,\beta} h_{\alpha a\beta} \omega_\alpha \omega_\beta = \sum_\rho v_{a\rho} II'_\rho + \sum_\lambda v_{a\lambda} II_\lambda .$$

For $n = 3$, $p = 2$, and gM^q a plane, this is essentially the classical Meusnier's theorem.

It follows from the structure equations of the euclidean space that the curvature forms of the induced riemannian metric on M^{p+q-n} and its second fundamental forms are related by

$$(64) \quad \Phi_{\alpha\beta} = \frac{1}{2} \sum_{\gamma,\delta} S_{\alpha\beta\gamma\delta} \omega_\gamma \wedge \omega_\delta = - \sum_i \omega_{\alpha i} \wedge \omega_{\beta i} ,$$

whence

$$(65) \quad S_{\alpha\beta\gamma\delta} = - \sum_i (h_{\alpha i \gamma} h_{\beta i \delta} - h_{\alpha i \delta} h_{\beta i \gamma}).$$

To express $S_{\alpha\beta\gamma\delta}$ in terms of $h_{\alpha\lambda\beta}$, $h'_{\alpha\rho\beta}$, we introduce the quantities

$$(66) \quad \begin{aligned} P_{\alpha\beta\gamma\delta} &= - \sum_{\lambda} (h_{\alpha\lambda\gamma} h_{\beta\lambda\delta} - h_{\alpha\lambda\delta} h_{\beta\lambda\gamma}), \\ Q_{\alpha\beta\gamma\delta} &= - \sum_{\rho} (h'_{\alpha\rho\gamma} h'_{\beta\rho\delta} - h'_{\alpha\rho\delta} h'_{\beta\rho\gamma}), \end{aligned}$$

which are the curvature functions of M^p , gM^q respectively. Put also

$$(67) \quad \begin{aligned} H_{\alpha\beta\gamma\delta\lambda\mu} &= h_{\alpha\lambda\gamma} h_{\beta\mu\delta} - h_{\alpha\lambda\delta} h_{\beta\mu\gamma}, \\ H'_{\alpha\beta\gamma\delta\rho\sigma} &= h'_{\alpha\sigma\gamma} h'_{\beta\rho\delta} - h'_{\alpha\sigma\delta} h'_{\beta\rho\gamma}, \\ K_{\alpha\beta\gamma\delta\rho\lambda} &= h_{\alpha\lambda\gamma} h'_{\beta\rho\delta} + h_{\beta\lambda\delta} h'_{\alpha\rho\gamma} - h_{\alpha\lambda\delta} h'_{\beta\rho\gamma} - h_{\beta\lambda\gamma} h'_{\alpha\rho\delta}, \end{aligned}$$

and

$$(68) \quad L_{\alpha\beta\gamma\delta} = - \sum_{\alpha, \lambda, \mu} v_{\alpha\lambda} v_{\alpha\mu} H_{\alpha\beta\gamma\delta\lambda\mu} + \sum_{\alpha, \rho, \sigma, \lambda} v_{\alpha\rho} v_{\alpha\lambda} v_{\sigma\lambda} H'_{\alpha\beta\gamma\delta\rho\sigma} - \sum_{\alpha, \rho, \lambda} v_{\alpha\rho} v_{\alpha\lambda} K_{\alpha\beta\gamma\delta\rho\lambda}.$$

Then we find, after a little calculation,

$$(69) \quad S_{\alpha\beta\gamma\delta} = P_{\alpha\beta\gamma\delta} + Q_{\alpha\beta\gamma\delta} + L_{\alpha\beta\gamma\delta}.$$

It is to be observed that each set of the quantities $P_{\alpha\beta\gamma\delta}$, $Q_{\alpha\beta\gamma\delta}$, $L_{\alpha\beta\gamma\delta}$ satisfies the symmetry relations (6). Formula (69) expresses the curvature functions of M^{p+q-n} in terms of those of M^p and gM^q and the "mixed curvatures" $L_{\alpha\beta\gamma\delta}$.

6. Proof of the main formula. In order to prove our formula (12) we take the differential formula (47) with $k = p + q - n$ and carry out the integrations after multiplying both sides by $I_*(M^{p+q-n})$. Integration of the left-hand side obviously gives a constant multiple of $\int \mu_*(M^p \cap gM^q) dg$. However, (47) is established under the assumption $\Delta \neq 0$, and a crucial question is that of the convergence of the integral at the right-hand side. By definition $v_{i\sigma}$ is equal to a cofactor of the matrix $(u_{\rho a})$ divided by Δ . From (68) and (61) it is seen that $L_{\alpha\beta\gamma\delta}$ is equal to $1/\Delta^2$ times a quantity which is bounded near $\Delta = 0$. The integrand is therefore equal to $\Delta^{p+q-n+1-\epsilon}$ times a bounded quantity, and is itself bounded near $\Delta = 0$, since $e \leq p + q - n$. Since the set of g for which $\Delta = 0$ is of measure zero in the group of proper motions in E^n , it suffices to evaluate the integral at the right-hand side under the assumption $\Delta \neq 0$.

To integrate the right-hand side in question (*i.e.*, the right-hand side of (47) multiplied by $I_*(M^{p+q-n})$), we first keep fixed a frame in the tangent space T_x to M^{p+q-n} at x . H_v is isomorphic to $SO(2n - p - q)$. But the integrand depends only on the linear spaces spanned by e_a and e'_ρ respectively. Thus it suffices to integrate over the open subset defined by $\Delta \neq 0$ in the Grassmann manifold of all $(n - q)$ -dimensional subspaces spanned by e'_ρ in $(T_x)^\perp$, the space of dimension $2n - p - q$ through x and orthogonal to T_x .

Actual evaluation of the integral seems to be difficult. In getting its expression we could be aided by the first main theorem on vector invariants ([7], p. 53). In fact, we observe the following:

1) The integral over H_n , to be denoted by J , is a polynomial in $h_{\alpha\lambda\beta}$, $h'_{\alpha\rho\beta}$, and is a linear combination of summands of the form

$$\sum \delta \begin{pmatrix} \alpha_1 & \cdots & \alpha_e \\ \beta_1 & \cdots & \beta_e \end{pmatrix} h_{\alpha_1\lambda_1\beta_1} \cdots h_{\alpha_t\lambda_t\beta_t} h'_{\alpha_{t+1}\rho_{t+1}\beta_{t+1}} \cdots h'_{\alpha_e\rho_e\beta_e}.$$

2) J is invariant under the transformations

$$\begin{aligned} h_{\alpha\lambda\beta} &\rightarrow \sum_{\mu} s_{\lambda\mu} h_{\alpha\mu\beta}, \\ h'_{\alpha\rho\beta} &\rightarrow \sum_{\sigma} s_{\rho\sigma} h'_{\alpha\sigma\beta}, \end{aligned}$$

where $(s_{\lambda\mu})$, $(s_{\rho\sigma})$ are arbitrary orthogonal matrices of orders $n - p$, $n - q$ respectively.

By the first main theorem on vector invariants it follows from 2) that J is equal to a polynomial in

$$\sum_{\lambda} h_{\alpha\lambda\beta} h_{\gamma\lambda\delta}, \quad \sum_{\rho} h'_{\alpha\rho\beta} h'_{\gamma\rho\delta}.$$

By 1) we see further that J must be a linear combination of the mixed curvatures $I(f, g)$, $0 \leq f, g \leq e$, $f + g = e$, introduced in (49), with numerical coefficients. Application of (50) then gives (12).

It remains to determine the constants c_i in (12). This will be achieved by carrying out the integration in a special case, i.e., the case when both M^p and M^q are spheres.

7. Determination of the constants. We wish to evaluate directly the integral

$$(70) \quad A_e = \int \mu_e(S^p \cap gS^q) dg,$$

where S^p , S^q are spheres of dimensions p , q and radii 1, a respectively. We begin by a special case:

Lemma. Let S_0^{m-1} be a fixed unit sphere in E^m , and let S^{m-1} be the sphere of center x and radius R . Let

$$(71) \quad B_e = \int \mu_e(S_0^{m-1} \cap S^{m-1}) dx,$$

where dx is the volume element in E^m . Then

$$(72) \quad B_e = \frac{2O_m O_{m-1}}{O_{m-e}} R^{m-e-1} \sum_{\substack{0 \leq 2\lambda + \mu \leq e/2 \\ 0 \leq \lambda, \mu}} \frac{\left(\frac{e}{2}\right)!}{(2\lambda)! \mu! \left(\frac{e}{2} - 2\lambda - \mu\right)!} 2^{2\lambda} \frac{O_{2\lambda+m-e+1}}{O_{2\lambda+1}} R^{2\mu+2\lambda}.$$

The right-hand side is a polynomial of degree $m - 1$ in R , and we will introduce the constants $b_{\epsilon i}$ according to the equation

$$(73) \quad B_{\epsilon} = b_{\epsilon, m-\epsilon-1} R^{m-\epsilon-1} + \cdots + b_{\epsilon, m-1} R^{m-1}.$$

To prove the lemma we suppose, without loss of generality, that $R < 1$. Let y be the distance of x from the center of S_0^{m-1} . Then the radius of the sphere $S^{m-1} \cap S_0^{m-1}$ is

$$\{4y^2 - (1 - R^2 + y^2)^2\}^{1/2}/2y,$$

and

$$\mu_{\epsilon}(S_0^{m-1} \cap S^{m-1}) = O_{m-1} \left\{ \frac{4y^2 - (1 - R^2 + y^2)^2}{4y^2} \right\}^{(m-2-\epsilon)/2}.$$

Since dx is equal to $y^{m-1} dy$ times the volume element of the unit sphere in E^m , we get

$$B_{\epsilon} = \frac{O_{m-1} O_m}{2^{m-\epsilon-2}} \int_{1-R}^{1+R} y^{\epsilon+1} \{4y^2 - (1 - R^2 + y^2)^2\}^{(m-\epsilon-2)/2} dy.$$

This integral is elementary and can be evaluated. In fact, put

$$t = y^2 - 1 - R^2.$$

We get

$$B_{\epsilon} = \frac{O_{m-1} O_m}{2^{m-\epsilon-1}} \int_{-2R}^{+2R} (t + 1 + R^2)^{\epsilon/2} (4R^2 - t^2)^{(m-\epsilon-2)/2} dt.$$

Expanding the first factor under the integral sign by the binomial theorem, we see that we will be concerned with integrals of the form

$$(74) \quad J(k, l) = \int_{-\alpha}^{\alpha} u^k (\alpha^2 - u^2)^{l/2} du.$$

Elementary calculus gives

$$(75) \quad \int_0^{\alpha} u^k (\alpha^2 - u^2)^{l/2} du = \frac{O_{k+l+3}}{O_{k+1} O_{l+2}} \alpha^{k+l+1},$$

and hence

$$(76) \quad \begin{aligned} J(k, l) &= 0, \quad \text{if } k \text{ is odd,} \\ J(k, l) &= \frac{2O_{k+l+3}}{O_{k+1} O_{l+2}} \alpha^{k+l+1}, \quad \text{if } k \text{ is even.} \end{aligned}$$

Formula (72) then follows by elementary reductions.

We now proceed to the evaluation of A_{ϵ} . Let E^{p+1} , E^{q+1} be respectively the linear spaces which contain S^p , gS^q , and let x be the center of gS^q . Then gS^q

can be identified with the pair $\{x, E^{q+1} \mid x \in E^{q+1}\}$, and we have the fiberings

$$(77) \quad G \xrightarrow{\pi} \{x, E^{q+1} \mid x \in E^{q+1}\} \xrightarrow{\psi} Gr,$$

where G is the space of all frames $xe_1 \cdots e_n$ in E^n and Gr is the Grassmann manifold of all $(q+1)$ -dimensional linear spaces in E^n ; π is defined by taking E^{q+1} to be spanned by $e_{n-q}, \dots, e_n$. The fibers of the first fibering are isomorphic to the group of isotropy of S^q and have the dimension

$$\{q(q+1) + (n-q-2)(n-q-1)\}/2$$

and total volume

$$O_{q+1} \cdots O_2 O_{n-q-1} \cdots O_2.$$

It follows that

$$A_* = 2O_{q+1} \cdots O_2 O_{n-q-1} \cdots O_2 \int \mu_*(S^p \cap gS^q) d(x, E^{q+1}),$$

where $d(x, E^{q+1})$ is the density in the homogeneous space of all $\{x, E^{q+1} \mid x \in E^{q+1}\}$. The factor 2 before the integral arises from the fact that our E^{q+1} is unoriented.

Let Gr^* be the subset of all E^{q+1} , such that

$$\dim(E^{p+1} \cap E^{q+1}) = p + q - n + 2.$$

Gr^* differs from Gr by a subset of measure zero, and it suffices to calculate the integral over $\psi^{-1}(Gr^*)$. Let $E^{p+1} \cap E^{q+1} = E^{p+q-n+2}$, and let H be the space of all $E^{p+q-n+2}$ in E^{p+1} . Then we have the natural mapping

$$\psi^{-1}(Gr^*) \rightarrow H,$$

and we calculate the integral by iteration with $E^{p+q-n+2}$ fixed. It suffices to restrict to those $E^{p+q-n+2}$, whose intersection with S^p is non-empty. The intersection $E^{p+q-n+2} \cap S^p = S^{p+q-n+1}$ is then a sphere of dimension $p+q-n+1$ and radius r , $0 \leq r \leq 1$.

The density $d(x, E^{q+1})$ can be factored as in (20), with E^p replaced by E^{q+1} . We fix E^{q+1} and integrate over the volume element $dx(E^{q+1})$ of $x \in E^{q+1}$. Let $E^{n-p-1} = (E^{p+q-n+2})^\perp$ be the orthogonal complement of $E^{p+q-n+2}$ through the center of $S^{p+q-n+1}$. A point $x \in E^{q+1}$ can be coordinatized by its orthogonal projections $x_1 \in E^{p+q-n+2}$ and $x_2 \in E^{n-p-1}$, and

$$dx(E^{q+1}) = dx_1(E^{p+q-n+2}) dx_2(E^{n-p-1}),$$

where the right-hand members are the volume elements of x_1, x_2 in their respective spaces. Let s be the distance from x to $E^{p+q-n+2}$. The intersection $S_1^{p+q-n+1} = gS^q \cap E^{p+q-n+2}$ is then a sphere of radius $\rho = (a^2 - s^2)^{1/2}$, and $dx_2(E^{n-p-1})$ is equal to $s^{n-p-2} ds$ times the volume element of the unit sphere in E^{n-p-1} . It follows that

$$A_\bullet = O_{q+1} \cdots O_1 O_{n-q-1} \cdots O_2 O_{n-p-1} \\ \cdot \int dE^{q+1} \int_0^a s^{n-p-2} ds \int \mu_\bullet(S^{p+q-n+1} \cap S_1^{p+q-n+1}) dx_1 (E^{p+q-n+2}).$$

The last integral has been evaluated in the lemma, which gives

$$\int \mu_\bullet(S^{p+q-n+1} \cap S_1^{p+q-n+1}) dx_1 (E^{p+q-n+2}) = \sum_{\substack{0 \leq i \leq e \\ i \text{ even}}} b_{\bullet, p+q-n+1-i} r^{p+q-n+1+i-e} \rho^{p+q-n+1-i}.$$

By (75) we get

$$\int_0^a s^{n-p-2} ds \int \mu_\bullet(S^{p+q-n+1} \cap S_1^{p+q-n+1}) dx_1 (E^{p+q-n+2}) \\ = \frac{1}{O_{n-p-1}} \sum_{\substack{0 \leq i \leq e \\ i \text{ even}}} b_{\bullet, p+q-n+1-i} \frac{O_{q-i+2}}{O_{p+q-n+3-i}} r^{p+q-n+1-e+i} a^{q-i}.$$

It remains to integrate with respect to dE^{q+1} . This is facilitated by the observation that the integrand depends only on $E^{p+q-n+2}$. Applying (32), we get

$$A_\bullet = \frac{O_{n+1} \cdots O_1 O_{p+q-n+3} \cdots O_1 O_{n-q-1} \cdots O_1}{O_1 O_{q+2} O_{p+2} \cdots O_1} \sum_{0 \leq i \leq e} b_{\bullet, p+q-n+1-i} \frac{O_{q-i+2}}{O_{p+q-n+3-i}} a^{q-i} \\ \cdot \int r^{p+q-n+1-e+i} dE^{p+q-n+2}(E^{p+1}).$$

To integrate with respect to $dE^{p+q-n+2}(E^{p+1})$ let u be the distance of $E^{p+q-n+2}$ from the center of S^p , so that $r = (1 - u^2)^{1/2}$. Then, by (17) and (18),

$$dE^{p+q-n+2}(E^{p+1}) = dE^{p+q-n+2}(O) u^{n-q-2} du d\omega,$$

where $dE^{p+q-n+2}(O)$ is the volume element in the space of all $E^{p+q-n+2}$ through the center of S^p and $d\omega$ is the volume element on the unit sphere in the orthogonal complement $(E^{p+q-n+2})^\perp$ of $E^{p+q-n+2}$ through the center of S^p and in E^{p+1} . Performance of the integration gives the final results

$$(78) \quad A_\bullet = \frac{O_{n+1} \cdots O_1 O_{p+q-n+3}}{O_1 O_{p+2} O_{q+2}} \sum_{0 \leq i \leq e} \frac{O_{q+2-i} O_{p+2-e+i}}{O_{p+q-n+3-i} O_{p+q-n+3-e+i}} b_{\bullet, p+q-n+1-i} a^{q-i}.$$

Comparing (12) and (78) and recalling that

$$\frac{1}{O_{p+1}} \mu_i(S^p) = 1, \quad \frac{1}{O_{q+1}} \mu_i(S^q) = u^{q-i},$$

we get

$$(79) \quad c_{\bullet-i} = \frac{O_{n+1} \cdots O_1 O_{p+q-n+3} O_{q+2-i} O_{p+2-e+i}}{O_1 O_{p+2} O_{p+1} O_{q+2} O_{q+1} O_{p+q-n+3-i} O_{p+q-n+3-e+i}} b_{\bullet, p+q-n+1-i},$$

where the b 's are given by (72) and (73), with $m = p + q - n + 2$. Thus the exact values of the constants c_i in (12) are determined.

A particular case of (12) is the formula

$$(80) \quad \int \mu_0(M^p \cap gM^q) dg = \frac{O_{n+1} \cdots O_2 O_{p+q-n+1}}{O_{p+1} O_{q+1}} \mu_0(M^p) \mu_0(M^q),$$

which was first given by Santalo [4].

8. Integration over linear space; reproductivity of the invariants. It will be natural to consider the intersections $M^p \cap E^q$ and integrate over dE^q . By considerations much less elaborate than the above, we can derive the following integral formula

$$(81) \quad \int \mu_e(M^p \cap E^q) dE^q = \frac{O_{n+1} \cdots O_{n-q+1} O_{p+q-n+2} O_{p+q-n+1} O_{p+2-e}}{O_{q+1} \cdots O_1 O_{p+2} O_{p+1} O_{p+q-n+2-e}} \mu_e(M^p),$$

where M^p is a compact submanifold of E^n and $e \leq p + q - n$. We will not go into the details here.

In conclusion we wish to remark that it would be of interest to characterize the invariants $\mu_e(M^p)$ by geometrical properties. One of them will perhaps be the "reproductive property" expressed by (81).

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