Unitary Operators Induced by Measure Preserving Transformations

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1. Let $(X, \mathbf{M}, m)$ be a totally finite, separable measure space, and T an invertible, measure preserving transformation on this space. We shall only be concerned with measure preserving transformations modulo sets of measure zero, that is, effectively, with measure preserving automorphisms of the measure algebra of $(X, \mathbf{M}, m)$. Since X splits into an atomic and a non-atomic part, which are necessarily invariant under T , we can consider separately the cases when X is atomic and when it is non-atomic. If X is atomic, with n atoms say, T is just a permutation, and the set of all invertible, measure preserving transformations of X is a subgroup of the symmetric group S_n ($n \leq \aleph_0$). We therefore assume that X is non-atomic. There is then no loss of generality in assuming that X is the unit interval, $\mathbf{M}$ the class of Lebesgue measurable sets and m Lebesgue measure (*cf.* Halmos [3, p. 171–174], [4, p. 42–44]). Further since every automorphism of the measure algebra of the unit interval is induced by a measure preserving point transformation of the unit interval [10, p. 582–584], we shall speak of “transformations”, when in fact we mean automorphisms.

The invertible transformation (automorphism) T induces a unitary operator U on $L^2(X, \mathbf{M}, m)$ defined by $(Uf)(x) = f(Tx)$ for $f \in L^2(X, \mathbf{M}, m)$. It is known, von Neumann [9, p. 618–619] that every unitary operator on L^2 , satisfying $U(fg) = (Uf)(Ug)$ whenever f, g and fg belong to L^2 , can be induced by a measure preserving transformation, (see §4 for a proof). Some spectral properties of U are also known. It is easy to show that the whole spectrum, as well as its discrete part, are symmetric about the real axis. In fact more is known, the spectrum of the unitary operator of an aperiodic transformation (*i.e.* one which does not satisfy $T^n = I$ for any integer n) is the whole circle. See [4, p. 70–72 and 74–75] from which the result may be easily deduced. For transformations with discrete spectrum there are more detailed results, see [9], [4] and [6]. There are also some results for ergodic and mixing transformations Halmos [4, p. 34 and 39], Jacobs [7, §8.8, §9.3 and §9.8] and Sinai [11], and also some for groups of operators, Foias [2]. However, the general question still remains open as to what unitary

operators on a Hilbert space can be induced by (necessarily invertible, see §4) measure preserving transformations. More precisely:

- Given a unitary operator U on an abstract, separable Hilbert space $\mathcal{H}$ under*
 (★) *what conditions will there exist a representation of $\mathcal{H}$ as $L^2[0, 1]$ and an invertible (Lebesgue) measure preserving transformation T on $[0, 1]$ such that U is the unitary operator induced by T ?*

By what we have said above if U is induced by a measure preserving transformation on some totally finite, non-atomic, separable measure space, then it is induced by a transformation on $[0, 1]$. Further it can easily be shown that the property of being inducible is a spectral property, *i.e.* if U is inducible by a measure preserving transformation, and P is *any* unitary operator, $P^{-1}UP$ is also inducible. We do not give this proof as the result follows at once (Theorem 6) from our main theorems. We give here two closely related sets of necessary and sufficient conditions for an operator on an abstract Hilbert space $\mathcal{H}$ to be induced by a measure preserving transformation. Both depend on characterizing those sets in $\mathcal{H}$ which can serve as the family of characteristic functions of measurable sets in a representation of $\mathcal{H}$ as $L^2[0, 1]$. The first set of conditions are in terms of an auxiliary spectral measure. We have unfortunately been unable to give a characterization purely in terms of the spectral measure of U . Nor have we been able to use our results to settle the problem of existence of a strongly mixing transformation of finite type [4, p. 97 problem 5]. Girsanov [15] has in fact announced (in 1958) the existence of such a transformation, and even of a transformation of type one. His approach is based on spectral properties of stationary Gaussian processes. As far as we are aware, no proofs have yet appeared.

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2. Our first theorem characterises in terms of a spectral measure those subsets of $\mathcal{H}$ which can serve as the set of characteristic functions of measurable sets in a representation of $\mathcal{H}$ as $L^2[0, 1]$. For all definitions and properties of spectral measures, see Halmos [5] or Stone [12].

Theorem 1. *Given a subset $\mathcal{C}$ of an abstract separable Hilbert space $\mathcal{H}$, a necessary and sufficient condition that there exists a representation of $\mathcal{H}$ as $L^2[0, 1]$, in which $\mathcal{C}$ corresponds to the set $\tilde{\mathcal{C}}$ of characteristic functions of Lebesgue measurable sets modulo null functions, is that there exists a simple, purely continuous, spectral measure $E(\cdot)$, and an element $i \in \mathcal{H}$, $\|i\| = 1$ with $\{E(\cdot)i\}$ total in $\mathcal{H}$, such that $\mathcal{C} = \{E(\cdot)i\}$.*

Proof. Necessity. Consider the spectral measure given by $E(M)f = \chi_M f$, where $f \in L^2[0, 1]$ and χ_M is the characteristic function of the Lebesgue measurable set M in $[0, 1]$. Then it is well-known that $E(M)$ is simple and purely continuous; if $\mathbf{1}$ denotes the constant function with value 1, then $\{E(\cdot)\mathbf{1}\}$ is total in $L^2[0, 1]$ and $\tilde{\mathcal{C}} = \{E(\cdot)\mathbf{1}\}$.

Sufficiency. We give one proof in detail and sketch an alternative proof.

First proof of sufficiency. Let the spectral measure $E(\cdot)$ be defined on the measurable space $(X, \mathbf{S})$. We introduce a lattice structure on $\mathfrak{C}$ as follows. We note that if $E(M_1)i = E(N_1)i$ and $E(M_2)i = E(N_2)i$, then

$$\begin{aligned} E(N_1 \cap N_2)i &= E(N_1)E(N_2)i \\ &= E(N_1)E(M_2)i = E(M_2)E(N_1)i = E(M_2)E(M_1)i = E(M_1 \cap M_2)i; \end{aligned}$$

and

$$\begin{aligned} E(N_1 \cup N_2)i &= E(N_1)i + E(N_2)i - E(N_1 \cap N_2)i \\ &= E(M_1)i + E(M_2)i - E(M_1 \cap M_2)i = E(M_1 \cup M_2)i. \end{aligned}$$

Hence if we write

$$E(M_1)i \wedge E(M_2)i = E(M_1 \cap M_2)i$$

and

$$E(M_1)i \vee E(M_2)i = E(M_1 \cup M_2)i$$

then $\wedge$ and $\vee$ are well-defined and independent of the particular sets M_1 and M_2 . Further, $E(M)i \wedge E(M)i = E(M)i \vee E(M)i = E(M)i$. $\wedge$ and $\vee$ are both commutative and associative operations. Finally

$$\begin{aligned} E(M)i \wedge (E(M)i \vee E(N)i) \\ = E(M)i \wedge E(M \cup N)i = E(M \cap (M \cup N))i = E(M)i, \end{aligned}$$

and

$$\begin{aligned} E(M)i \vee (E(M)i \wedge E(N)i) \\ = E(M)i \vee E(M \cap N)i = E(M \cup (M \cap N))i = E(M)i. \end{aligned}$$

So $\mathfrak{C}$ is a lattice under $\wedge$ and $\vee$ with order relation given by $E(M)i \leq E(N)i$ if $E(M \cap N)i = E(M)i$ or equivalently $E(M \cup N)i = E(N)i$. See [1, p. 18].

The lattice $\mathfrak{C}$ has a unique minimal element $0 = E(\emptyset)i$ and a unique maximal element $i = E(X)i$. For any $E(M)i \in \mathfrak{C}$, $i - E(M)i$ is the uniquely defined complement in $\mathfrak{C}$ since

$$E(M)i \wedge (i - E(M)i) = 0, E(M)i \vee (i - E(M)i) = i;$$

and if $E(N)i$ is such that $E(M)i \wedge E(N)i = 0$, $E(M)i \vee E(N)i = i$, then $E(M)i + E(N)i = i$. Moreover $\mathfrak{C}$ is a distributive lattice [1, p. 133] since

$$E(M)i \wedge (E(N_1)i \vee E(N_2)i) = (E(M)i \wedge E(N_1)i) \vee (E(M)i \wedge E(N_2)i).$$

Thus $\mathfrak{C}$ is a Boolean algebra [1, p. 152]. Note also that

$$E(M)i \vee E(N)i + E(M)i \wedge E(N)i = E(M)i + E(N)i.$$

Now let $E(M_n)i$ be an increasing sequence in $\mathfrak{C}$. Put $N_n = \bigcup_{r=1}^n M_r$. Then $E(N_n)i = E(M_n)i$ for all n , and the projections $E(N_n)$ form an increasing sequence. Hence $E(N_n)i$ converge in norm to

$$\left(\bigvee_{n=1}^{\infty} E(N_n) \right) i = E \left(\bigcup_{n=1}^{\infty} N_n \right) i = E \left(\bigcup_{n=1}^{\infty} M_n \right) i.$$

This is the least upper bound of the sequence $E(N_n)i$. It is clearly an upper bound; if $E(N)i$ is any other, $E(N \cap N_n)$ is also an increasing sequence of projections and

$$E(N)i \geq E \left(N \cap \bigcup_{n=1}^{\infty} N_n \right) i = \lim_{n \rightarrow \infty} E(N \cap N_n)i = \lim_{n \rightarrow \infty} E(N_n)i = E \left(\bigcup_{n=1}^{\infty} N_n \right) i.$$

Thus $\mathfrak{C}$ is a Boolean σ -algebra. [The whole proof could be simplified with the use of a little more lattice theory. The projections $\{E(\cdot)\}$ form a Boolean σ -algebra. The set $\mathfrak{C} = \{E(\cdot)i\}$ is then isomorphic to the quotient Boolean σ -algebra of $\{E(\cdot)\}$ modulo the σ -ideal $\{E(\cdot)i = 0\}$. The lattice structure is that described above.]

Next we note that

$$\begin{aligned} (E(N)i, E(M)i) &= (E(M)i, E(N)i) = (i, E(M)E(N)i) = (i, E(M \cap N)i) \\ &= (i, (E(M \cap N))^2 i) = (E(M \cap N)i, E(M \cap N)i) \\ &= \|E(M \cap N)i\|^2 = \|E(M)i \wedge E(N)i\|^2. \end{aligned}$$

Now we define a measure on $\mathfrak{C}$ by

$$m(E(M)i) = \|E(M)i\|^2.$$

$m(E(M)i)$ is strictly positive; moreover m is finitely additive since $E(M)i \wedge E(N)i = 0$ implies $(E(M)i, E(N)i) = \|E(M)i \wedge E(N)i\|^2 = 0$ and so

$$\begin{aligned} \|E(M)i \vee E(N)i\|^2 &= \|E(M)i + E(N)i - (E(M)i \wedge E(N)i)\|^2 \\ &= \|E(M)i + E(N)i\|^2 = \|E(M)i\|^2 + \|E(N)i\|^2. \end{aligned}$$

Further if $E(M_n)i$ increase to $E(M)i$, then $E(M_n)i \rightarrow E(M)i$ in norm and $\|E(M_n)i\| \uparrow \|E(M)i\|$, so m is continuous from below, and so countably additive. Thus $(\mathfrak{C}, m)$ is a measure algebra, m is obviously totally finite and normalized, since $\|i\| = 1$. Since the spectral measure $E(\cdot)$ is continuous, and since $\{E(\cdot)i\}$ is total in $\mathfrak{H}$ it follows that the measure m is non-atomic. [Proof. Let $E(M)i$ have no proper smaller element. There are projections smaller than $E(M)$, since $E(\cdot)$ is continuous. Let $E(N)$ be one and let $x \in \mathfrak{H}$ such that $x \in \text{Range}(E(N))$ and $x \neq \lambda E(M)i$. Then, since $\{E(\cdot)i\}$ is total in $\mathfrak{H}$, $x = \sum \lambda_r E(N_r)i$ for some N_r , so

$$x = E(M)x = \sum \lambda_r E(M \cap N_r)i.$$

At least one of $E(M \cap N_r)i$ is not zero and is not equal to $E(M)i$, else x is a constant multiple of $E(M)i$. This $E(M \cap N_r)i < E(M)i$.

Since $\mathcal{H}$ is separable in the norm metric, so is $\mathcal{C}$, and this metric is equivalent to that induced by the measure m , for

$$\begin{aligned}
 m(E(M)i \Delta E(N)i) &= m(E(M \cup N)i - E(M \cap N)i) \\
 &= m(E(M \cup N)i) - m(E(M \cap N)i) \\
 &= m(E(M)i) + m(E(N)i) - 2m(E(M \cap N)i) \\
 &= ||E(M)i||^2 + ||E(N)i||^2 - 2 ||E(M \cap N)i||^2 \\
 &= ||E(M)i||^2 + ||E(N)i||^2 - 2(E(M)i, E(N)i) \\
 &= ||E(M)i - E(N)i||^2.
 \end{aligned}$$

Hence $(\mathcal{C}, m)$ is separable as a measure algebra. It follows by the result mentioned in the introduction [3, p. 171-174] that $\mathcal{C}$ is isomorphic to the measure algebra of $[0, 1]$ with Lebesgue measure, and hence (by a norm preserving isomorphism) to the Boolean σ -algebra $\tilde{\mathcal{C}}$ of Lebesgue measurable characteristic functions in $[0, 1]$ modulo null functions. Thus we have a norm preserving Boolean isomorphism between $\mathcal{C}$, which is a subset of $\mathcal{H}$, and $\tilde{\mathcal{C}}$ which is a subset of $L^2[0, 1]$. Let α be the isomorphism $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$. Note that if $E(M)i + E(N)i \in \mathcal{C}$ (which happens if and only if $E(M)i \wedge E(N)i = 0$) then $\alpha(E(M)i + E(N)i) \in \tilde{\mathcal{C}}$ and $\alpha(E(M)i + E(N)i) = \alpha(E(M)i) + \alpha(E(N)i)$ since $\alpha(E(M)i) \wedge \alpha(E(N)i) = 0$. We now extend α to the linear manifolds generated by $\mathcal{C}$ and $\tilde{\mathcal{C}}$ respectively, by writing

$$\alpha\left(\sum_{r=1}^n \lambda_r E(M_r)i\right) = \sum_{r=1}^n \lambda_r \alpha(E(M_r)i).$$

Since

$$\begin{aligned}
 \left\| \sum_{r=1}^n \lambda_r E(M_r)i \right\|^2 &= \sum_{r=1}^n \sum_{s=1}^n \lambda_r \bar{\lambda}_s (E(M_r)i, E(M_s)i) \\
 &= \sum_{r=1}^n \sum_{s=1}^n \lambda_r \bar{\lambda}_s ||E(M_r)i \wedge E(M_s)i||^2 \\
 &= \sum_{r=1}^n \sum_{s=1}^n \lambda_r \bar{\lambda}_s ||\alpha(E(M_r)i) \wedge \alpha(E(M_s)i)||^2 \\
 &= \sum_{r=1}^n \sum_{s=1}^n \lambda_r \bar{\lambda}_s ||\alpha(E(M_r)i) \wedge \alpha(E(M_s)i)||^2 \\
 &= \sum_{r=1}^n \sum_{s=1}^n \lambda_r \bar{\lambda}_s (\alpha(E(M_r)i), \alpha(E(M_s)i)) \\
 &= \left\| \sum_{r=1}^n \lambda_r \alpha(E(M_r)i) \right\|^2;
 \end{aligned}$$

it follows that this mapping is norm preserving, and that

$$\sum_{r=1}^n \lambda_r E(M_r) i = \sum_{s=1}^m \mu_s E(N_s) i$$

if and only if

$$\sum_{r=1}^n \lambda_r \alpha(E(M_r) i) = \sum_{s=1}^m \mu_s \alpha(E(N_s) i).$$

The mapping α is thus a norm-preserving linear isomorphism of the linear manifolds generated by $\mathcal{C}$ and $\tilde{\mathcal{C}}$ respectively and may therefore be extended to the respective closures of these, namely $\mathcal{H}$ and $L^2[0, 1]$. This gives the required representation of $\mathcal{H}$ as $L^2[0, 1]$.

Sketch of second proof of sufficiency. Let $E(\cdot)$ be defined on the measurable space $(X, \mathbf{S})$. Put

$$m(M) = \|E(M)i\|^2$$

for all $M \in \mathbf{S}$. Then m is a measure on $\mathbf{S}$, and is totally finite since $m(X) = 1$. Further there is a (1 - 1) correspondence, modulo m -null functions, of characteristic functions of sets in $\mathbf{S}$ and elements in $\mathcal{C}$, χ_M corresponding to $E(M)i$. Further, as in the first proof, given any M , there exists N such that

$$\|E(M \cap N)i\| < \|E(M)i\|,$$

so the measure m is non-atomic. If $\mathcal{H} = \{M : m(M) = 0\}$, then $(\mathbf{S}/\mathcal{H}, m)$ is a totally finite normalised measure algebra and since, as in the first proof,

$$m(M \triangle N) = \|E(M)i - E(N)i\|^2,$$

it is separable. It now follows that it is isomorphic to the measure algebra of the unit interval. Further, since

$$(E(M)i, E(N)i) = \|E(M \cap N)i\|^2,$$

one may obtain, as in the first proof, an isomorphism of $\mathcal{H}$ onto $L^2(X, \mathbf{S}, m)$ in which $\mathcal{C}$ is mapped onto the set of characteristic functions of sets in $\mathbf{S}$ modulo m -null functions. Since $(\mathbf{S}/\mathcal{H}, m)$ is isomorphic to the measure algebra of $[0, 1]$, we obtain an isomorphism of $\mathcal{H}$ onto $L^2[0, 1]$ taking $\mathcal{C}$ onto $\tilde{\mathcal{C}}$.

Remark. In the definition of a spectral measure, there is no real need to assume an underlying measurable space $(X, \mathbf{S})$ [5, p. 88-89]. A spectral measure can be regarded as just a Boolean σ -algebra of commuting projections. A simple spectral measure $\mathcal{E}$ is then one in which the projections of some $i \in \mathcal{H}$ are total in $\mathcal{H}$. A continuous spectral measure $\mathcal{E}$ is one in which for every $E \in \mathcal{E}$, there exists $F \in \mathcal{E}$ with $F \leq E$ and $F \neq E$, i.e. a non-atomic Boolean σ -algebra. The proof of Theorem 1 would need only slight re-wording if we take this to be the definition of spectral measure. Of course standard theorems on the representation of Boolean σ -algebras as σ -algebras of sets, ([1, p. 167], [3, p. 171]) would

then produce an underlying measurable space. Notice that we do not use multiplicity theory at all in the above definitions of simple and continuous spectral measures.

3. In this section we give another characterisation of those subsets of $\mathcal{H}$ which can serve as the set of characteristic functions in a representation of $\mathcal{H}$ as $L^2[0, 1]$. One of the two sets of necessary and sufficient conditions for (*) which we have given (Theorem 4) is independent of the results of this section. We begin with a

Definition. Let $\mathcal{H}$ be an abstract Hilbert space. A subset $\mathcal{C}$ of $\mathcal{H}$ is called a **C-family** if the following nine conditions are satisfied.

- (i) $\mathcal{C}$ is closed in the norm topology.
- (ii) $\mathcal{C}$ is total in $\mathcal{H}$.
- (iii) there exists a unique element $i \in \mathcal{C}$ such that $\|\phi\| \leq \|i\|$ for all $\phi \in \mathcal{C}$; we assume that $\|i\| = 1$.
- (iv) For all $\phi \in \mathcal{C}$, we have $i - \phi \in \mathcal{C}$ and $(\phi, i - \phi) = 0$; in particular $0 = i - i \in \mathcal{C}$.
- (v) If ϕ, ψ and $\chi \in \mathcal{C}$ and $(\psi, i - \phi) = (\chi, i - \psi) = 0$, then $(\chi, i - \phi) = 0$.
- (vi) If $\phi, \psi \in \mathcal{C}$, then $\phi + \psi \in \mathcal{C} \Leftrightarrow (\phi, \psi) = 0$.
- (vii) If $\phi \in \mathcal{C}$, then there exists $\psi \in \mathcal{C}$ such that $\psi \neq i - \phi$ and $(\phi, \psi) = 0$.
- (viii) If $\phi, \psi \in \mathcal{C}$, $[(\phi, \psi) \text{ is real}]$ and $(\phi, \psi) \geq 0$.
- (ix) If $\phi, \psi \in \mathcal{C}$, there exists a unique element $\chi \in \mathcal{C}$ such that $(\chi, i - \phi) = (\chi, i - \psi) = 0$ and $\|\chi\|^2 = (\phi, \psi)$; [that $\|\chi\|^2 \leq (\phi, \psi)$ follows from (i), ... (viii)].

We now state some simple consequences of the definition, giving proofs for those which are not absolutely trivial. In all that follows, $\mathcal{C}$ denotes a C-family in a Hilbert space $\mathcal{H}$.

- I. If $\phi \in \mathcal{C}$, then $\|\phi\|^2 = (\phi, i)$.
- II. If $\phi, \psi \in \mathcal{C}$, $(\psi, i - \phi) = 0 \Leftrightarrow \|\psi\|^2 = (\phi, \psi) \Leftrightarrow \phi - \psi \in \mathcal{C}$.

Proof. That $(\psi, i - \phi) = 0 \Leftrightarrow \|\psi\|^2 = (\phi, \psi)$ is trivial. Also

$$(\psi, i - \phi) = 0 \Leftrightarrow i - \phi + \psi \in \mathcal{C} \Leftrightarrow i - (i - \phi + \psi) = \phi - \psi \in \mathcal{C}.$$

- III. If $\phi \in \mathcal{C}$, $\phi \neq 0$, then $\lambda\phi \in \mathcal{C} \Rightarrow \lambda = 0$ or 1 .

Proof. By (iv) and I

$$|\lambda|^2 \|\phi\|^2 = \|\lambda\phi\|^2 = (\lambda\phi, i) = \lambda(\phi, i) = \lambda \|\phi\|^2.$$

So $|\lambda|^2 = \lambda$, i.e. $\lambda = 0$ or 1 .

- IV. If $\phi, \psi \in \mathcal{C}$ then $(\psi, i - \phi) = 0 \Rightarrow (\psi, \chi) \leq (\phi, \chi)$ for any $\chi \in \mathcal{C}$; in particular $\|\psi\| \leq \|\phi\|$.

Proof. By II, $\phi - \psi \in \mathcal{C}$ so that

$$(\phi, \chi) = (\psi, \chi) + (\phi - \psi, \chi) \geq (\psi, \chi)$$

by (viii). Also by II,

$$||\phi||^2 = (\phi, \phi) \geq (\phi, \psi) = ||\psi||^2.$$

V. If for $\phi, \psi \in \mathcal{C}$ we write $\psi \leq \phi$ whenever $(\psi, i - \phi) = 0$, then the relation so defined is an order relation.

Proof. $\phi \leq \phi$ by (iv).

$$\psi \leq \phi \quad \text{and} \quad \chi \leq \psi \Rightarrow \chi \leq \phi \quad \text{by (v).}$$

If $\phi \leq \psi$ and $\psi \leq \phi$, then by IV, $||\phi|| = ||\psi||$. So since $(\psi, i - \phi) = 0$, $\psi + i - \phi \in \mathcal{C}$ and

$$||\psi + i - \phi||^2 = ||\psi||^2 + ||i - \phi||^2 = ||\phi||^2 + ||i - \phi||^2 = ||i||^2 = 1;$$

so by (iii), $\psi + i - \phi = i$, and $\psi = \phi$.

Definition. For all $\phi, \psi \in \mathcal{C}$, we write $\phi\psi$ for the element χ defined by (ix). Obviously $\phi\psi = \psi\phi$.

VI. $\phi\psi$ is the greatest element χ such that $\chi \leq \phi$ and $\chi \leq \psi$; i.e., $\phi\psi \leq \phi$, $\phi\psi \leq \psi$, and $\chi \leq \phi$, $\chi \leq \psi$, implies $\chi \leq \phi\psi$.

Proof. That $\phi\psi \leq \phi$, $\phi\psi \leq \psi$ follows from (ix). By II and IV it follows that if $\chi \leq \phi$, and $\chi \leq \psi$, then

$$||\chi||^2 = (\chi, \phi) \leq (\phi, \psi).$$

If $\chi \not\leq \phi\psi$, then let $\xi = \chi(i - \phi\psi)$, so $||\xi||^2 = (\chi, i - \phi\psi) > 0$. Now $(\xi, \phi\psi) = 0$ so $\xi + \phi\psi \in \mathcal{C}$ and

$$||\xi + \phi\psi||^2 = ||\xi||^2 + ||\phi\psi||^2 > ||\phi\psi||^2 = (\phi, \psi).$$

But

$$(\xi + \phi\psi, i - \phi) = (\xi, i - \phi) + (\phi\psi, i - \phi) = 0$$

since $\xi \leq \chi \leq \phi$. So $\xi + \phi\psi \leq \phi$. Similarly $\xi + \phi\psi \leq \psi$ and so by IV,

$$||\xi + \phi\psi||^2 = (\xi + \phi\psi, \psi) \leq (\phi, \psi).$$

Contradiction.

Corollary 1. If $\phi \in \mathcal{C}$, $i\phi = \phi\phi = \phi$. If $\psi \in \mathcal{C}$, $\psi \leq \phi$ then $\phi\psi = \psi$.

Corollary 2. If $\phi, \psi, \chi \in \mathcal{C}$, $\phi(\psi\chi) = (\phi\psi)\chi$.

Proof. $\phi(\psi\chi) \leq \phi$ and $\leq \psi\chi$, so $\phi(\psi\chi) \leq \psi$ and χ , so $\phi(\psi\chi) \leq \phi\psi$ and so $\phi(\psi\chi) \leq (\phi\psi)\chi$. Similarly $(\phi\psi)\chi \leq \phi(\psi\chi)$.

VII. (a) $(\phi, \psi) = 0 \Rightarrow (\chi\phi, \chi\psi) = 0$.

(b) $\psi \leq \phi, \chi \leq \phi$ and $(\psi, \chi) = 0 \Rightarrow \psi + \chi \leq \phi$.

(c) $(\chi, \phi) = 0 \Rightarrow \phi \leq \chi + \phi$.

Proof. (a) $(\phi, \psi) = 0 \Rightarrow \phi \leq i - \psi \Rightarrow \chi\phi \leq \phi \leq i - \psi$, i.e. $(\chi\phi, \psi) = 0$. Repeating this argument with ψ in place of ϕ , $\chi\phi$ in place of ψ , we have $(\chi\phi, \chi\psi) = 0$.

(b) is immediate from the definition of $\leq$.

(c) $(\phi, i - \phi - \chi) = (\phi, i - \phi) - (\phi, \chi) = 0$, so $\phi \leq \phi + \chi$.

VIII. If $\phi_n \in \mathcal{C}$ and $\phi_n \leq \phi_{n+1}$ (i.e. $\phi_n \uparrow$) then there exists $\phi \in \mathcal{C}$ such that $\phi_n \rightarrow \phi$ (in norm). Further $\phi_n \leq \phi$, for all n ; and if $\psi \in \mathcal{C}$ is such that $\phi_n \leq \psi$ for all n , then $\phi \leq \psi$.

Proof.

$$\begin{aligned} \|\phi_{n_1} - \phi_{n_2}\|^2 &= (\phi_{n_1} - \phi_{n_2}, \phi_{n_1} - \phi_{n_2}) \\ &= \|\phi_{n_1}\|^2 + \|\phi_{n_2}\|^2 - 2(\phi_{n_1}, \phi_{n_2}), \quad (\text{by viii}), \\ &= \|\phi_{n_1}\|^2 - \|\phi_{n_2}\|^2, \quad \text{if } n_1 > n_2 \quad (\text{by II}), \\ &\rightarrow 0, \end{aligned}$$

since $\|\phi_n\| \uparrow$ and $\|\phi_n\| \leq 1$. So $\phi_n \rightarrow \phi$ say, in norm and $\phi \in \mathcal{C}$ by (i). Also, if $n_1 > n$, $(\phi_n, i - \phi_{n_1}) = 0$, $i - \phi_{n_1} \rightarrow i - \phi$ as $n_1 \rightarrow \infty$, so $(\phi_n, i - \phi) = 0$, i.e. $\phi_n \leq \phi$. Further, if $\phi_n \leq \psi$ for all n , i.e. $(\phi_n, i - \psi) = 0$, then $(\phi, i - \psi) = 0$, i.e. $\phi \leq \psi$.

IX. If ϕ, ψ and $\chi \in \mathcal{C}$, and $(\phi, \psi) = 0$, then

$$\chi(\phi + \psi) = \chi\phi + \chi\psi.$$

Proof. By (ix) and VII (c)

$$\chi\phi \leq \phi + \psi, \quad \chi\psi \leq \phi + \psi;$$

by VII (a), $(\chi\phi, \chi\psi) = 0$; and so by VII (b),

$$\chi\phi + \chi\psi \leq \phi + \psi.$$

Also, by (ix) $\chi\phi \leq \chi$, $\chi\psi \leq \chi$ and so by VII (a) and (b),

$$\chi\phi + \chi\psi \leq \chi.$$

So by VI

$$(\chi\phi + \chi\psi) \leq \chi(\phi + \psi).$$

Since $(\chi\phi, \chi\psi) = 0$,

$$\begin{aligned} ||\chi\phi + \chi\psi||^2 &= ||\chi\phi||^2 + ||\chi\psi||^2 \\ &= (\chi, \phi) + (\chi, \psi) \\ &= (\chi, \phi + \psi) \\ &= ||\chi(\phi + \psi)||^2, \end{aligned}$$

whence, by the uniqueness part of (ix), we have

$$\chi(\phi + \psi) = \chi\phi + \chi\psi.$$

Corollary. If $\phi \leq \psi$, then $\chi(\psi - \phi) = \chi\psi - \chi\phi$.

Proof. By II and IX,

$$\chi\psi = \chi(\psi - \phi + \phi) = \chi(\psi - \phi) + \chi\phi.$$

We are now in a position to introduce a lattice structure on $\mathcal{C}$.

Definition. For all $\phi, \psi \in \mathcal{C}$ we write

$$\begin{aligned} \phi \wedge \psi &= \phi\psi, \\ \phi \vee \psi &= i - (i - \phi)(i - \psi). \end{aligned}$$

X. If $\phi, \psi \in \mathcal{C}$, then $\phi + \psi - \phi\psi \in \mathcal{C}$ and $\phi + \psi - \phi\psi = \phi \vee \psi$.

Proof. By IX Corollary and VI Corollary 1,

$$\begin{aligned} (i - \phi)(i - \psi) &= i(i - \psi) - \phi(i - \psi) \\ &= i - \psi - (\phi - \phi\psi) \\ &= i - \psi - \phi + \phi\psi. \end{aligned}$$

So $\phi \vee \psi = i - (i - \phi)(i - \psi) = \phi + \psi - \phi\psi$.

XI. If $\phi, \psi \in \mathcal{C}$, $\phi \leq \phi \vee \psi$. If $\chi \in \mathcal{C}$, $\phi \leq \chi$ and $\psi \leq \chi$, then $\phi \vee \psi \leq \chi$, i.e. $\phi \vee \psi$ is the least element greater than both ϕ and ψ .

Proof. By IX Corollary, I and II,

$$\begin{aligned} (\phi, i - (\phi \vee \psi)) &= (\phi, (i - \phi)(i - \psi)) \\ &= (\phi, i - \phi - \psi + \phi\psi) \\ &= (\phi, i) - (\phi, \phi) - (\phi, \psi) + (\phi, \phi\psi) \\ &= ||\phi||^2 - ||\phi||^2 - ||\phi\psi||^2 + ||\phi\psi||^2 \\ &= 0. \end{aligned}$$

Hence $\phi \leq \phi \vee \psi$. If $\phi \leq \chi$ and $\psi \leq \chi$, then $\phi\psi \leq \chi$. So

$$(\phi + \psi - \phi\psi, i - \chi) = (\phi, i - \chi) + (\psi, i - \chi) - (\phi\psi, i - \chi) = 0.$$

Hence $\phi \vee \psi \leq \chi$.

XI and VI together show that $\mathcal{C}$ is a lattice; we next show that it is distributive.

XII. *If $\phi, \psi, \chi \in \mathcal{C}$ then*

$$\phi \wedge (\psi \vee \chi) = (\phi \wedge \psi) \vee (\phi \wedge \chi),$$

$$\phi \vee (\psi \wedge \chi) = (\phi \vee \psi) \wedge (\phi \vee \chi).$$

Proof. We prove the first of these. The other can be proved in the same way; however it is well known [1, p. 133], that it follows automatically from the first in a lattice. By IX Corollary and VI Corollary 2

$$\begin{aligned} \phi \wedge (\psi \vee \chi) &= \phi(\psi \vee \chi) \\ &= \phi(i - (i - \psi)(i - \chi)) \\ &= \phi - \phi((i - \psi)(i - \chi)) \\ &= \phi - (\phi(i - \psi))(i - \chi) \\ &= \phi - (\phi - \phi\psi)(i - \chi) \\ &= \phi - \phi + \phi\psi + (\phi - \phi\psi)\chi \\ &= \phi\psi + \phi\chi - \phi\psi\chi \\ &= \phi\psi + \phi\chi - (\phi\psi)(\phi\chi) \\ &= (\phi\psi) \vee (\phi\chi) \\ &= (\phi \wedge \psi) \vee (\phi \wedge \chi). \end{aligned}$$

We now come to our next theorem which asserts that every $\mathbf{C}$ -family in an abstract separable Hilbert space is representable as the family of characteristic functions in $[0, 1]$.

Theorem 2. *Let $\mathcal{C}$ be a $\mathbf{C}$ -family in an abstract Hilbert space $\mathcal{H}$. We have*

(a) *$\mathcal{C}$ is a Boolean σ -algebra under the operations $\vee, \wedge$, complementation being defined as difference with respect to i .*

(b) *The function $m(\phi) = \|\phi\|^2$, $\phi \in \mathcal{C}$, defines a strictly positive measure on $\mathcal{C}$ under which $\mathcal{C}$ becomes a non-atomic, totally finite, normalized measure algebra.*

(c) *If further, $\mathcal{H}$ is separable, then $\mathcal{C}$ is a separable measure algebra, and so is isomorphic to the measure algebra of $[0, 1]$, and to the lattice $\tilde{\mathcal{C}}$ of Lebesgue measurable characteristic functions of $[0, 1]$ modulo null functions. This isomorphism can be extended to an isomorphism of $\mathcal{H}$ and $L^2[0, 1]$. The multiplication in $\mathcal{C}$ induces the ordinary multiplication in $L^2[0, 1]$.*

Proof. (a) It follows from (iv) and (ix) of the definition, and VI, X, XI and XII that $\mathcal{C}$ is a Boolean algebra with respect to the operations mentioned. [(iv), (ix) and X imply that if $\phi \in \mathcal{C}$, then $\phi \wedge (i - \phi) = 0$, $\phi \vee (i - \phi) = i$]. It follows from VIII that $\mathcal{C}$ is a Boolean σ -algebra.

(b) The finite additivity of $m(\phi)$ follows since

$$||\phi + \psi||^2 = ||\phi||^2 + ||\psi||^2, \text{ if } (\phi, \psi) = 0.$$

Countable additivity follows from VIII. Strict positivity follows since $||\phi|| = 0 \Leftrightarrow \phi = 0$. Non-atomicity follows from (viii) of the definition. The measure is totally finite and normalized, since $\phi \leq i$ for all $\phi \in \mathcal{C}$ and $m(i) = 1$.

(c) The proof of (c) is essentially the same as the proof of sufficiency in Theorem 1. If $\mathcal{H}$ is separable in the norm metric, so is $\mathcal{C}$, and this metric is equivalent to that induced by the measure since

$$\begin{aligned} m(\phi \triangle \psi) &= m(\phi \vee \psi - \phi \wedge \psi) \\ &= m(\phi) + m(\psi) - 2m(\phi \wedge \psi) \\ &= ||\phi||^2 + ||\psi||^2 - 2(\phi, \psi) \\ &= ||\phi - \psi||^2. \end{aligned}$$

So $\mathcal{C}$ is separable as a measure algebra, and as in the proof of Theorem 1 is isomorphic to the measure algebra of the unit interval and to the Boolean algebra $\tilde{\mathcal{C}}$ of characteristic functions of $[0, 1]$ modulo null functions. This isomorphism $\alpha : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ is norm-preserving. Also if $\phi, \psi \in \mathcal{C}$, $\phi + \psi \in \mathcal{C} \Leftrightarrow \phi \wedge \psi = 0 \Leftrightarrow \alpha(\phi) \wedge \alpha(\psi) = 0 \Leftrightarrow \alpha(\phi) + \alpha(\psi) \in \tilde{\mathcal{C}}$, and then $\alpha(\phi + \psi) = \alpha(\phi) + \alpha(\psi)$. We can again, as in Theorem 1, extend α to the linear manifolds generated by $\mathcal{C}$ and $\tilde{\mathcal{C}}$ in $\mathcal{H}$ and $L^2[0, 1]$ respectively. α is then norm-preserving, the argument being exactly as in Theorem 1, using the fact that $||\phi \wedge \psi||^2 = (\phi, \psi)$ in both $\mathcal{C}$ and $\tilde{\mathcal{C}}$. We can thus extend α to the closure of the linear manifold spanned by $\mathcal{C}$, and get an isometry of $\mathcal{H}$ onto $L^2[0, 1]$.

Since for $\phi, \psi \in \mathcal{C}$, $\phi\psi = \phi \wedge \psi$, $\alpha(\phi)\alpha(\psi) = \alpha(\phi) \wedge \alpha(\psi)$, the multiplication in $\mathcal{C}$ induces the ordinary multiplication in $\tilde{\mathcal{C}}$, which extends to the ordinary multiplication in $L^2[0, 1]$. This completes the proof.

Theorem 3. *Given a subset $\mathcal{C}$ of an abstract, separable Hilbert space $\mathcal{H}$, a necessary and sufficient condition that there exists a representation of $\mathcal{H}$ as $L^2[0, 1]$, in which $\mathcal{C}$ corresponds to the lattice $\tilde{\mathcal{C}}$ of characteristic functions of Lebesgue measurable sets modulo null functions, is that $\mathcal{C}$ be a **C-family**.*

Proof. Necessity follows because the characteristic functions do form a **C-family** with the constant function **1** as i . Sufficiency is proved in Theorem 2(c).

Remark. It follows from Theorems 1, 2, and 3 that every **C-family** can be obtained from a simple, continuous spectral measure; this can, of course, be proved directly, but we do not give the proof here. Given any x in $\mathcal{H}$ of norm 1, one can construct a **C-family** with this element as maximal element. This is done by repeated orthogonal projection—essentially, one constructs a spectral measure. Also, one could reduce the proof of sufficiency of Theorem 1 to that of Theorem 2(c) by verifying that the set $\mathcal{C} = \{E(\cdot)i\}$ forms a **C-family**. This is not difficult, and suggests that the complicated conditions defining a **C-family**

may, in practice, be easier to verify than the condition referring to a spectral measure.

4. In this section we give our two main theorems, both giving answers to (*), one in terms of spectral measures, the other in terms of $\mathbf{C}$ -families.

We first give a proof of von Neumann's Theorem [9, p. 618–619].

Every unitary operator U on $L^2[0, 1]$ satisfying $U(f)U(g) \in L^2$ and $U(fg) = U(f)U(g)$ whenever f, g and $fg \in L^2$, can be induced by an invertible measure preserving transformation on $[0, 1]$.

von Neumann's proof is incomplete, and later authors *e.g.* [4, p. 45, 7, p. 213 Theorem 2] have always strengthened the hypotheses of the theorem by assuming that the multiplicativity condition also holds for U^{-1} . This is unnecessary. As we need the strong form of the theorem and as no proof of this appears in the literature to our knowledge, we give the proof here.

Proof of von Neumann's Theorem. Let χ_E be the characteristic function of E . Then $(\chi_E)^2 = \chi_E \in L^2$, so $(U(\chi_E))^2 = U(\chi_E^2) = U(\chi_E)$, *i.e.* $U(\chi_E)$ is also a characteristic function. Further given χ_E, χ_F ,

$$\begin{aligned} U(\chi_E \chi_F) &= U(\chi_E)U(\chi_F); \\ U(\chi_E \vee \chi_F) &= U(\chi_E + \chi_F - \chi_E \chi_F) \\ &= U(\chi_E) + U(\chi_F) - U(\chi_E)U(\chi_F) \\ &= U(\chi_E) \vee U(\chi_F). \end{aligned}$$

If $\chi_E \leq \chi_F$, then $U(\chi_F - \chi_E) = U(\chi_F) - U(\chi_E)$. So U is a Boolean isomorphism of the set $\tilde{\mathcal{C}}$ of characteristic functions into itself. Further if $\chi_{E_n} \uparrow \chi_E$,

$$U(\chi_E) = U(\lim_{n \rightarrow \infty} \chi_{E_n}) = \lim_{n \rightarrow \infty} U(\chi_{E_n}),$$

(the limits being in norm) so U is a σ -isomorphism, and is, of course, norm preserving. Thus the unitary operator U defines an isomorphism $\bar{T}$ of the measure algebra of $[0, 1]$ into itself. The mapping $f \rightarrow f\bar{T}$, which is an isometry of L^2 into itself coincides with U on $\tilde{\mathcal{C}}$, and so, since $\tilde{\mathcal{C}}$ is total in L^2 , it coincides with U on the whole of L^2 . Hence U takes bounded functions into bounded functions, unbounded functions into unbounded functions; and if f is bounded then

$$\text{ess sup } |Uf| = \text{ess sup } |f|.$$

So if f is bounded so is $U^{-1}f$. (Note that $L^\infty[0, 1] \subseteq L^2[0, 1]$). Further if f and g are bounded so are $fg, Uf, Ug, U(fg), (Uf)(Ug)$ and $U(fg) = (Uf)(Ug)$. Let f, g be bounded; $\bar{f} = U^{-1}f, \bar{g} = U^{-1}g$ are then also bounded and hence so is $\bar{f}\bar{g}$ and

$$U(\bar{f}\bar{g}) = (U\bar{f})(U\bar{g}) = fg;$$

so

$$U^{-1}(fg) = (U^{-1}f)(U^{-1}g).$$

In particular if $\chi_E \in \tilde{\mathcal{C}}$, $U^{-1}\chi_E \in \tilde{\mathcal{C}}$, so U is a Boolean σ -automorphism of $\tilde{\mathcal{C}}$ onto itself and the isomorphism $\bar{T}$, which induces U , is also an automorphism. Hence by [10, p. 582–584] $\bar{T}$ arises from an invertible measure preserving transformation T on $[0, 1]$, which, of course, induces U . This completes the proof.

Corollary 1. *It is enough to assume that*

$$U(fg) = (Uf)(Ug)$$

whenever f and g are characteristic functions.

Corollary 2. *If a measure preserving transformation induces a unitary operator, it is invertible.*

We now come to our main theorems.

Theorem 4. *Given a unitary operator U on an abstract separable Hilbert space $\mathfrak{H}$, the following two conditions (a) and (b) are together necessary and sufficient for the existence of a representation of $\mathfrak{H}$ as $L^2[0, 1]$ in which U is induced by an invertible measure preserving transformation T on $[0, 1]$.*

(a) *U has an eigenvector i with eigenvalue 1;*

(b) *there exists a simple, purely continuous spectral measure $E(\cdot)$, such that the set $\{E(\cdot)i\}$ is total in $\mathfrak{H}$, and such that the set of projections $\{U^{-1}E(\cdot)U\}$ either contains, or is contained in the set of projections $\{E(\cdot)\}$ [i.e. the range of the spectral measure $U^{-1}E(\cdot)U$ either contains, or is contained in the range of $E(\cdot)$].*

Proof. Necessity. If U is induced by an invertible measure preserving transformation T , then the constant function $\mathbf{1}$ is an eigenvector with eigenvalue 1. Consider the spectral measure on $([0, 1], \mathfrak{L})$, $\mathfrak{L}$ being the class of Lebesgue measurable sets, given by $E(M)f = \chi_M f$ where $f \in L^2[0, 1]$ and χ_M is the characteristic function of M . This is simple and purely continuous, $E(M)\mathbf{1} = \chi_M$, so $\{E(\cdot)\mathbf{1}\}$ is total in $L^2[0, 1]$. Finally, for any $f \in L^2[0, 1]$,

$$\begin{aligned} U^{-1}E(M)Uf(x) &= U^{-1}E(M)f(Tx) \\ &= U^{-1}\chi_M(x)f(Tx) \\ &= \chi_M(T^{-1}x)f(x) \\ &= E(TM)f(x). \end{aligned}$$

So $U^{-1}E(M)U = E(TM)$, and the sets $\{U^{-1}E(\cdot)U\}$ and $\{E(\cdot)\}$ of projections are identical.

Sufficiency. Suppose there exists an element $i \in \mathfrak{H}$ such that $Ui = i$, and a spectral measure $E(\cdot)$, such that $\{E(\cdot)i\}$ is total in $\mathfrak{H}$ and $\{U^{-1}E(\cdot)U\} \supseteq \{E(\cdot)\}$. [If $\{U^{-1}E(\cdot)U\} \subseteq \{E(\cdot)\}$, we note that $\{UE(\cdot)U^{-1}\} \supseteq \{E(\cdot)\}$ and

work with U^{-1} in place of U . If U^{-1} is induced by an invertible measure preserving transformation clearly so is U . We may assume that $\|i\| = 1$. Let $\mathfrak{C} = \{E(\cdot)\}i = \{E(\cdot)i\}$. By Theorem 1, there exists a representation of $\mathfrak{C}$ as $L^2[0, 1]$ in which $\mathfrak{C}$ corresponds to $\tilde{\mathfrak{C}}$, the set of characteristic functions of Lebesgue measurable sets modulo null functions, the correspondence being moreover a (norm preserving) Boolean σ -isomorphism. We let $\mathbf{1}$ denote the constant function with value 1. Now since

$$\{U^{-1}E(\cdot)U\} \supseteq \{E(\cdot)\},$$

we have

$$U^{-1}\{E(\cdot)\}i = U^{-1}\{E(\cdot)\}Ui \supseteq \{E(\cdot)\}i,$$

[where $\{E(\cdot)\}x$ is the set $\{E(\cdot)x\}$]. So $U^{-1}\{E(\cdot)\}i \supseteq \{E(\cdot)\}i$ and so

$$U\{E(\cdot)\}i \subseteq UU^{-1}\{E(\cdot)\}i = \{E(\cdot)\}i.$$

i.e. $U\mathfrak{C} \subseteq \mathfrak{C}$. So if $\tilde{U}$ is the operator corresponding to U in $L^2[0, 1]$, we have $\tilde{U}\tilde{\mathfrak{C}} \subseteq \tilde{\mathfrak{C}}$. We show that $\tilde{U}$ is multiplicative in $\tilde{\mathfrak{C}}$, *i.e.* if $\chi_E, \chi_F \in \tilde{\mathfrak{C}}$ then

$$\tilde{U}(\chi_E \cdot \chi_F) = (\tilde{U}\chi_E) \cdot (\tilde{U}\chi_F).$$

We note the following properties of characteristic functions.

- (i) If $\chi_E, \chi_F \in \tilde{\mathfrak{C}}$, then $\chi_{E \cap F} = \chi_E \cdot \chi_F$ and $\|\chi_E \cdot \chi_F\|^2 = (\chi_E, \chi_F)$.
- (ii) $\chi_E \leq \chi_F$ if and only if $(\chi_E, \mathbf{1} - \chi_F) = (\chi_E, \chi_{\{0,1\}-F}) = 0$.
- (iii) If $\chi_G \leq \chi_E$ and χ_F , then $\chi_G \leq \chi_E \cdot \chi_F$.
- (iv) If $\chi_E \leq \chi_F$ and $\|\chi_E\| = \|\chi_F\|$, then $\chi_E = \chi_F$.

Now since $\tilde{U}$ is unitary $\|\tilde{U}\mathbf{1}\| = \|\mathbf{1}\| = 1$, and since $\mathbf{1}$ is the only element in $\tilde{\mathfrak{C}}$ with norm 1, it follows that $\tilde{U}\mathbf{1} = \mathbf{1}$. If $\chi_E, \chi_F \in \tilde{\mathfrak{C}}$, then $\chi_E \cdot \chi_F = \chi_{E \cap F} \leq \chi_E$ and χ_F so by (ii)

$$(\chi_E \cdot \chi_F, \mathbf{1} - \chi_E) = (\chi_E \cdot \chi_F, \mathbf{1} - \chi_F) = 0,$$

and so

$$(\tilde{U}(\chi_E \cdot \chi_F), \mathbf{1} - \tilde{U}\chi_E) = (\tilde{U}(\chi_E \cdot \chi_F), \mathbf{1} - \tilde{U}\chi_F) = 0.$$

So since $\tilde{U}(\chi_E \cdot \chi_F), (\tilde{U}\chi_E), (\tilde{U}\chi_F)$, all belong to $\tilde{\mathfrak{C}}$, by (iii)

$$\tilde{U}(\chi_E \cdot \chi_F) \leq (\tilde{U}\chi_E) \cdot (\tilde{U}\chi_F).$$

Also, by (i)

$$\|\tilde{U}(\chi_E \cdot \chi_F)\|^2 = \|\chi_E \cdot \chi_F\|^2 = (\chi_E, \chi_F) = (\tilde{U}\chi_E, \tilde{U}\chi_F) = \|\tilde{U}\chi_E \cdot \tilde{U}\chi_F\|^2$$

i.e.

$$\|\tilde{U}(\chi_E \cdot \chi_F)\| = \|\tilde{U}\chi_E \cdot \tilde{U}\chi_F\|.$$

Hence by (iv)

$$\tilde{U}(\chi_E \cdot \chi_F) = \tilde{U}\chi_E \cdot \tilde{U}\chi_F.$$

Now by Corollary 1 to the theorem of von Neumann proved above, it follows that $\tilde{U}$ is induced by an invertible measure preserving transformation (automorphism) T of $[0, 1]$. This completes the proof.

Note. It follows incidentally that $\{U^{-1}E(\cdot)U\} = \{E(\cdot)\}$. If we had assumed this apparently stronger condition, our proof would have been somewhat simpler. For then we would have had $\tilde{U}\tilde{\mathcal{C}} = \tilde{\mathcal{C}} = \tilde{U}^{-1}\tilde{\mathcal{C}}$.

Theorem 5. *Given a unitary operator U on an abstract, separable Hilbert space $\mathfrak{H}$, a necessary and sufficient condition that there exist a representation of $\mathfrak{H}$ as $L^2[0, 1]$ in which U is induced by an invertible measure preserving transformation T is that there exists a $\mathbf{C}$ -family $\mathcal{C}$ in $\mathfrak{H}$ such that either $U\mathcal{C} \subseteq \mathcal{C}$ or $U\mathcal{C} \supseteq \mathcal{C}$.*

Proof. Necessity. Suppose there exists a representation of $\mathfrak{H}$ as $L^2[0, 1]$ in which U is induced by a measure preserving transformation T . Let $\tilde{\mathcal{C}}$ be the set of characteristic functions of Lebesgue measurable sets modulo null functions. $\tilde{\mathcal{C}}$ is a $\mathbf{C}$ -family, and $U\tilde{\mathcal{C}} = \tilde{\mathcal{C}}$.

Sufficiency. Suppose there exists a $\mathbf{C}$ -family $\mathcal{C}$ in $\mathfrak{H}$ such that $U\mathcal{C} \subseteq \mathcal{C}$. [If $U\mathcal{C} \supseteq \mathcal{C}$, then $U^{-1}\mathcal{C} \subseteq \mathcal{C}$, and we argue with U^{-1} in place of U]. By Theorem 3 there exists a representation of $\mathfrak{H}$ as $L^2[0, 1]$ in which $\mathcal{C}$ corresponds to the lattice $\tilde{\mathcal{C}}$ of characteristic functions of Lebesgue measurable sets modulo null functions, the correspondence being also a norm-preserving Boolean σ -isomorphism. Let $\tilde{U}$ denote the operator corresponding to U in $L^2[0, 1]$. We have $\tilde{U}\tilde{\mathcal{C}} \subseteq \tilde{\mathcal{C}}$. The remainder of the proof is identical with that in Theorem 4. We could, however, as an alternative, have proved U was multiplicative in $\mathcal{C}$, i.e. if $\phi, \psi \in \mathcal{C}$ then $U(\phi\psi) = U(\phi)U(\psi)$.

We write $\mathcal{S}$ for the set of all unitary operators in $\mathfrak{H}$ which are inducible, (not necessarily under the same representation of $\mathfrak{H}$ as $L^2[0, 1]$) by a measure preserving transformation. It follows at once from Theorem 4 or 5 that

Theorem 6. *For any unitary operator P on $\mathfrak{H}$, $P^{-1}\mathcal{S}P = \mathcal{S}$ (i.e. $(*)$ is a spectral property).*

5. Remarks on Theorems 4, 5 and 6.

(a) There are essentially only two important steps in our results. We notice that if a unitary operator U on $L^2[0, 1]$ maps the set of characteristic functions into itself, then U is multiplicative, and so by von Neumann's result, is induced by an invertible measure preserving transformation. We also notice that sets which can act as the set of characteristic functions can be characterized in purely spectral terms. Our results follow.

(b) Ideally, a result of this nature would give conditions on the spectral measure of U itself, in terms of multiplicity functions *etc.*, and not on an auxillary spectral measure. It would seem, however, that such a set of conditions must be very complicated, whereas the conditions on the auxillary spectral measure are very simple and do not really require multiplicity theory.

(c) If we replace $[0, 1]$ by the circle $\{|z| = 1\}$, then the integral $\int z \cdot dE_*$ of the spectral measure E_* given by the characteristic functions is itself a unitary operator on $L^2\{|z| = 1\}$, namely the operator $f(z) \rightarrow zf(z)$. It would be interesting to have conditions for (*) relating U to such an operator, instead of to its spectral measure.

(d) If we drop the assumption of separability of $\mathcal{H}$, we still get conditions for unitary operators to be induced by measure preserving transformations on *some* measure space; but in general this measure space will depend on the $\mathbf{C}$ -family or spectral measure concerned. In the separable case, it follows from Theorem 1, (respectively Theorem 3) that any two simple continuous spectral measures (respectively $\mathbf{C}$ -families) are isomorphic under an isomorphism of $\mathcal{H}$. In the non-separable case this is no longer true. The reason is simple. All separable non-atomic measure algebras are *homogeneous*, this is no longer true in the non-separable case (see Maharam [8]).

(e) There exist measure preserving transformations which are spectrally equivalent, but not conjugate as measure preserving transformations. See Halmos [4] for the definition, Jacobs [7, §10] for the most comprehensive and up to date account available at the moment. Thus a unitary operator U may leave invariant two different $\mathbf{C}$ -families $\mathcal{C}_1$ and $\mathcal{C}_2$, but may treat them both very differently. Our results have some slight bearing on the conjugacy problem. We say that two unitary operators U and V in $\mathcal{H}$ form a *conjugate pair*, if there exists a representation of $\mathcal{H}$ as $L^2[0, 1]$ in which U and V are induced by conjugate measure preserving transformations T and S . If U and V form a conjugate pair they are, of course, equivalent, but not conversely. We have however the following easily proved result. *Two equivalent operators U and $P^{-1}UP$ form a conjugate pair if and only if there exists a $\mathbf{C}$ -family, $\mathcal{C}$, such that $U\mathcal{C} = P\mathcal{C} = \mathcal{C}$.* (It follows at once that $P^{-1}UP\mathcal{C} = \mathcal{C}$, but this is *not* enough for U and $P^{-1}UP$ to form a conjugate pair, as the examples of equivalent, non-conjugate measure preserving transformations show. See [4 p. 57], [13] and [14].)

(f) Let $\mathcal{J}$ be as in Theorem 6, and let $\mathcal{J}_p$ denote the subset of $\mathcal{J}$ consisting of periodic operators, *i.e.* those operators, U , for which $U^n = I$ for some positive integer n . The minimum integer n for which $U^n = I$ is called the period of U . Let U be unitary and periodic with period n . Then U has discrete spectrum and all eigenvalues of U are roots of unity. It is easily shown that $U \in \mathcal{J}_p$ if and only if $e^{2\pi ir/n}$ is an eigenvalue for each $r = 0, 1, 2, \dots, n-1$, and the multiplicity of $e^{2\pi ir/n}$ for every r is infinite. It follows from the uniform approximation theorem for invertible measure preserving transformations [4, p. 75] that $\mathcal{J}$ is a subset of the closure of $\mathcal{J}_p$ in the uniform (norm) topology. One might believe that $\mathcal{J}$ is in fact equal to the closure of $\mathcal{J}_p$ in the uniform topology, which follows if and only if $\mathcal{J}$ is closed in this topology. This seems, however, unlikely for the following reason. If $U_n \in \mathcal{J}_p$ then each U_n has an eigenvector i_n with eigenvalue 1, and the i_n may be all distinct. If U_n converges in the norm topology to U say, it is unlikely that U will necessarily have an eigenvector with eigenvalue 1. (*Added in proof.* It is false that $\mathcal{J}$ is the closure of $\mathcal{J}_p$ in the uniform topology.

This follows from Theorem 1 of R. V. Chacon & N. Friedman, Approximation and invariant measures, *Z. Wahrscheinlichkeitstheorie*, **3**(1965) 286–295).

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