*The Singular, Second Order Oblique Derivative Problem**

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1. Introduction. In recent years there has been considerable interest in generalizing the Schauder estimates proven by Agmon–Douglis–Nirenberg [1] and Browder [5] for the linear elliptic boundary problem

$$(1) \quad \begin{aligned} P(x, D)u &= f \quad \text{in a domain } G, \\ B_t(x, D)u &= \varphi_t, \quad t = 1, 2, \dots, q \quad \text{on } \Gamma = \partial G, \end{aligned}$$

with scalar differentiable polynomials $P(x, D)$ and $B_t(x, D)$. In [1], the Schauder estimates for (1) take the form

$$(2) \quad \|u\|_m \leq C \{ \|P(x, D)u\|_0 + \sum \|B_t(x, D)u\|_{r_t} + \|u\|_0 \}$$

with suitable norms $\|u\|_m$, $\|u\|_{r_t}$ and $\|u\|_0$. Under appropriate smoothness assumptions on problem (1) it was shown in [1] that the estimates (2) hold if and only if the problem satisfies a condition which has been called by various authors regular, elliptic or Shapiro–Lopatinski.

In the case of a second order operator and a single real boundary condition

$$B(x, D)u = \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i}, \quad \text{with } n \geq 3,$$

it is well known that the problem (1), the so called oblique derivative problem, is regular in the above sense if and only if the vector field $(b_1(x), \dots, b_n(x))$ is never tangent to Γ . In this paper we shall be concerned with deriving estimates for second order, elliptic operators with a boundary condition whose associated vector field $(b_1, \dots, b_n)$ is tangent to Γ in a prescribed way; in view of the above remark, such problems are not regular. Problems of this type have been investigated recently by Bitsadze [3], and in a very general context by Hörmander [10], and others. To describe our results more precisely it will be convenient to introduce the following terminology: Let $\partial G = \Gamma$ be an $n - 1$ dimen-

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sional manifold, $n \geq 3$ and $\mathbf{V}$ the vector field on Γ associated with the above oblique derivative condition. Let $\mathbf{V}$ be tangent to Γ along an $n - 2$ dimensional submanifold $\mathcal{C}$; then $\mathcal{C}$ is said to be a submergent or emergent cycle for $\mathbf{V}$ accordingly as the field $\mathbf{V}$ submerges into or emerges out of the region G when following $\mathbf{V}$ across $\mathcal{C}$ in the direction induced by the tangential component of $\mathbf{V}$ on Γ . It is easy to see by means of examples (see Section 4.1) that the estimate (2) in general cannot hold when the boundary condition gives rise to emergent cycles; however, it is interesting to note that such an estimate can be derived for the problem $P(x, D)u = f$, $B(x, D)u = 0$ if the boundary condition gives rise only to submergent cycles of a prescribed type. More generally, an estimate of the type (2) can be shown to hold for the problem $P(x, D)u = f$, $(1/\varphi)B(x, D)u = \psi$ provided $B(x, D)$ generates only submergent cycles of a prescribed kind where φ must vanish sufficiently strongly on the cycles of the tangency of B and be non-zero otherwise. In this paper we shall derive local estimates which imply the global estimates described immediately above. The details of obtaining the global estimates have been treated by the author but are not included here; they will appear subsequently in another paper.

In order to derive a local estimate for the oblique derivative problem with submergent cycles, it suffices to restrict our attention to the semi-space $H = \{y : y \in E_n, y_n > 0\}$ with boundary E and the problem

$$(3) \quad \begin{aligned} &(\nabla^2 - 1)u = f \text{ in } H, \\ &\frac{1}{x_{n-1}} \left\{ x_{n-1} \frac{\partial u}{\partial x_n} + b \frac{\partial u}{\partial x_{n-1}} \right\} = \varphi \text{ on } E. \end{aligned}$$

Observe that the manifold $\mathcal{C} = \{x : x \in E_n, x_n = x_{n-1} = 0\}$ is a submergent or emergent cycle for the boundary condition in (3) according as $b > 0$ or $b < 0$. We shall prove in this paper that an estimate of type (2) holds for solutions of the problem (3) provided that $b > 0$.

Recently, H. O. Cordes proposed a treatment of the theory of singular elliptic boundary problems from the point of view of the so-called generalized Fredholm operators in suitable Hilbert spaces. As elaborated in Cordes [7] the natural background for such an investigation is the theory of C^* -algebras of operators with certain proper, closed, two-sided $*$ -ideals. Using some results in [7] it can be shown that the *a priori* estimates derived in this paper imply that the operator is left $\mathfrak{E}^{\mathcal{C}}$ -Fredholm, where $\mathfrak{E}^{\mathcal{C}}$ is a suitable ideal generated by the submergent cycle $\mathcal{C}$.

Sections 2 and 3 of this paper serve to introduce fundamental definitions and give a brief summary of well-known results which we find useful for our purposes. An exception is Section 3.1 where we prove an integral estimate which plays a central role in the derivation of local estimates for problem (3). Our principal results are contained in Section 4. In the final section of this paper we attempt to derive a local estimate in the presence of emergent cycles using the same methods which were successful in the submergent case; however the results there do not quite take a satisfactory form.

This paper is a portion of the author's PhD Thesis, [2] at the University of California in Berkeley and was published as a UC Technical Report in May 1963. There was no necessity to give wider publication to these results at that time; however, there recently appears to be some interest in the oblique derivative problem (see Hörmander [10], and Bitsadze [3]), and hence publication seemed desirable at this time.

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2. Basic definitions. Since we are concerned with only a local estimate we will be dealing with the semi-space $H = \{y : y \in E_n, \text{ and } y_n \geq 0\}$ and its boundary E . All functions considered will be complex valued and Lebesgue measurable on H or E . The results which appear below are well-known and are stated only to fix the concepts and notations which are useful for our purposes.

It will be convenient to introduce the following notations and conventions for norms in the spaces H and E . Let us put

$$||u||^2 = \int_H |u|^2, \quad ||\varphi||^2 = \int_E |\varphi|^2,$$

with integration taken in the Lebesgue sense, and we shall use the multi-index notation to denote differentiation as follows: Let

$$D_j = -i \frac{\partial}{\partial x_j}, \quad j = 1, 2, \dots, n,$$

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \quad |\alpha| = \sum_{i=1}^n \alpha_i,$$

where the α_i are non-negative integers and put $D^{(\alpha)} = D_1^{\alpha_1} \dots D_n^{\alpha_n}$. For $u \in C_0^\infty(H)$ and $\varphi \in C_0^\infty(E)$ we define the norms

$$||u||_k^2 = \sum_{|\alpha| \leq k} ||D^{(\alpha)}u||^2, \quad ||\varphi||_k^2 = \sum_{\substack{|\alpha| \leq k \\ \alpha_n = 0}} ||D^{(\alpha)}\varphi||^2.$$

We denote by $\mathfrak{S}_k$ and $\mathfrak{S}_k^*$ the completion of the spaces $C_0^\infty(H)$ and $C_0^\infty(E)$ under the norms $|| \cdot ||_k$ and $|| \cdot ||_k^*$ respectively and retain the same norm symbols for general elements in $\mathfrak{S}_k$ and $\mathfrak{S}_k^*$. Clearly $\mathfrak{S}_i \supset \mathfrak{S}_k$, $\mathfrak{S}_i^* \supset \mathfrak{S}_k^*$ whenever $k \geq i$, and $\mathfrak{S}_0$ and $\mathfrak{S}_0^*$ correspond to the usual spaces of square integrable functions over H and E respectively whose norms were defined earlier without the zero subscript.

Let $u \in \mathfrak{S}_k$ and $|\alpha| \leq k$. We say that $v \in \mathfrak{S}_0$ is the α -th strong derivative of u if and only if there exists a sequence $u^n \in C_0^\infty(H)$ such that $||u - u^n||_k \rightarrow 0$ and $||v - D^{(\alpha)}u^n|| \rightarrow 0$, and we write $v = D^{(\alpha)}u$. We have a similar definition for

strong derivatives of functions $\varphi \in \mathfrak{S}_k^*$, $|\alpha| \leq k$, $\alpha_n = 0$. Observe that if $u \in \mathfrak{S}_k$, $k \geq 1$ and α is an index such that $|\alpha| \leq k$, then $D^{(\alpha)}u \in \mathfrak{S}_{k-|\alpha|}$.

Next we define on $\mathfrak{S}_k^*$ a norm equivalent to $\|\cdot\|_k$ by the use of Fourier Transforms. Let $\hat{x} = (x_1, \dots, x_{n-1})$ and $\hat{\xi} = (\xi_1, \dots, \xi_{n-1})$ denote general points in E and put $\hat{x} \cdot \hat{\xi} = \sum_{j=1}^{n-1} x_j \xi_j$. For $\varphi \in C_0^\infty(E)$ we define the Fourier Transform of φ , denoted by $\mathcal{F}(\varphi)$, as follows:

$$\mathcal{F}(\varphi) = \psi(\hat{\xi}) = (2\pi)^{-(n-1)/2} \int_E \varphi(\hat{x}) e^{-i\hat{x} \cdot \hat{\xi}} d\hat{x}.$$

For $\varphi \in C_0^\infty(E)$ the following statements are true:

1. $\varphi \in \mathfrak{S}_0^*$ and $\|\varphi\| = \|\psi\|$; $\varphi(\hat{x}) = (2\pi)^{-(n-1)/2} \int_E \psi(\hat{\xi}) e^{i\hat{x} \cdot \hat{\xi}} d\hat{\xi}$.
2. $\psi(\hat{\xi})$ is analytic for all $\hat{\xi} \in E$.
3. Let P and Q be any polynomials in the variables $s_1, s_2, \dots, s_{n-1}$ and for any index α such that $\alpha_n = 0$ we put

$$\xi^{(\alpha)} = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_{n-1}^{\alpha_{n-1}},$$

then we have that $\mathcal{F}[Q(D)(P(\hat{x})\varphi)] = Q(\hat{\xi})[P(-D)\mathcal{F}(\varphi)]$.

4. Let the index α be such that $\alpha_n = 0$, then $|\xi^{(\alpha)} D^{(\beta)} \psi| < C_\alpha$, with C_α independent of $\hat{\xi}$, for all $\hat{\xi} \in E$ and all β such that $\beta_n = 0$.

We say that $\varphi \in \mathfrak{S}_0^*$ is the Fourier Transform of an element $\varphi \in \mathfrak{S}_0^*$ if and only if there exists a sequence $\varphi^n \in C_0^\infty(E)$ such that $\|\varphi - \varphi^n\| \rightarrow 0$ and $\|\psi - \mathcal{F}(\varphi^n)\| \rightarrow 0$. Statements 1.-4. above remain true for general elements $\varphi \in \mathfrak{S}_0^*$ if $\mathcal{F}(\varphi)$ is understood in the sense of the above definition and $D^{(\alpha)}\varphi$ and $D^{(\beta)}\psi$ are understood as strong derivatives.

Let $\sigma^2 = 1 + \xi_1^2 + \dots + \xi_{n-1}^2$, then for $\varphi \in C_0^\infty(E)$ we define the norm

$$[\varphi]_s^2 = \int_E \sigma^{2s} |\mathcal{F}(\varphi)|^2 d\hat{\xi} = \|\sigma^s \mathcal{F}(\varphi)\|^2,$$

for any real number $s \geq 0$. There exist positive constants C_1 and C_2 such that

$$C_1[\varphi]_k \leq \|\varphi\|_k \leq C_2[\varphi]_k,$$

for all $\varphi \in C_0^\infty(E)$, where k is a non-negative integer. Since $C_0^\infty(E)$ is dense in $\mathfrak{S}_k^*$ it is clear that $[\varphi]_k$ is equivalent to the norm $\|\varphi\|_k$ on $\mathfrak{S}_k^*$.

For $u \in \mathfrak{S}_k$, we denote by $r_0(u)$ that function on E which is the restriction of u to E . Let $\varphi \in C_0^\infty(E)$, then we define the norm, for $k \geq 1$

$$\|\varphi\|_k = \inf \{ \|u\|_k : u \in C_0^\infty(H), r_0(u) = \varphi \}.$$

For $k \geq 1$, $\varphi \in C_0^\infty(E)$, there exists a positive constant K such that

$$K^{-1} \|\varphi\|_k \leq [\varphi]_{k-1/2} \leq K \|\varphi\|_k.$$

For $s \geq 0$ we denote by $\mathfrak{S}_s^*$ the completion of the space $C_0^\infty(E)$ under the norm $[\varphi]_s$.

There exist positive constants C_1 and C_2 such that

$$C_1 \sum_{i=0}^k \int_0^\infty dx_n [D_n^i v]_{k-i}^2 \leq \|u\|_k^2 \leq C_2 \sum_{i=0}^k \int_0^\infty dx_n [D_n^i v]_{k-i}^2,$$

for all $u \in \mathfrak{S}_k$, where $v = \mathfrak{F}(u)$. Hence the norms

$$\|u\|_k^2 \quad \text{and} \quad \sum_{i=0}^k \int_0^\infty dx_n [D_n^i v]_{k-i}^2$$

are equivalent for $\mathfrak{S}_k$. A well-known result of Soboleff states that

$$\mathfrak{S}_{k+[n/2]+1} \subset C^k(H).$$

A bounded region G in E_n with boundary Γ is said to be of Class C^* if for every point $x \in \Gamma$ there exists a neighborhood N of x and a C^* mapping, F , on $N \cap \bar{G}$ and into the semi-space $\bar{H}$ such that 1) F is one-one on $N \cap \bar{G}$, and F^{-1} is C^* , 2) $F(x) = 0$, 3) $F(N \cap \Gamma) \subset E$, and 4) F has non-vanishing Jacobian.

3. Some preparatory estimates.

3.1. An integral estimate. In this paper we will consistently use the following notations: Let

$$\sigma^2(t) = t^2 + \zeta^2 = t^2 + \xi_1^2 + \cdots + \xi_{n-2}^2, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_{n-2}),$$

$$\tilde{\xi} = (\xi_1, \xi_2, \cdots, \xi_{n-1}) = (\xi, \xi_{n-1}), \quad p = e^{-b\sigma} \sigma^3 t^{-2}, \quad \rho = \sigma e^{2b\sigma}$$

where b is a constant, $b \neq 0$.

Theorem 3.1.1. *There exists a positive constant K independent of ξ such that*

$$\int_0^\infty (-pu_t)_t dt \geq K \int_0^\infty \rho u^2 dt,$$

for all $u \in C_0^2(0, \infty)$.

To prove the above theorem we need the following lemmas:

Lemma 3.1.1. *If there exists a function $h \in C^2(0, \infty)$ such that $h \neq 0$ in $(0, \infty)$ and $-(ph_t)_t > \eta ph$, where η is a positive constant independent of ξ , then the estimate of Theorem 3.1.1 holds with $K = \eta$.*

Proof. Let $u = hu^*$, then

$$\int_0^\infty -(pu_t)_t dt = \int_0^\infty [-ph^2 u_{tt}^* u^* - (ph^2)_t u_t^* u^* - h(ph_t)_t u^{*2}] dt.$$

Integrating the first term by parts we obtain

$$\begin{aligned} \int_0^\infty -(pu_t)_t dt &= \int_0^\infty [ph^2 u_t^{*2} - h(ph_t)_t u^{*2}] dt \\ &\geq \int_0^\infty \rho h^2 \left[-\frac{1}{\rho h} (ph_t)_t \right] u^{*2} dt \\ &\geq \eta \int_0^\infty \rho h^2 u^{*2} dt = \eta \int_0^\infty \rho u^2 dt, \end{aligned}$$

and the lemma is proven.

Lemma 3.1.2. *There exists a function $h = h(t)$ satisfying the hypothesis of lemma 3.1.1 if and only if there exists a function $h^* = h^*(y)$ such that $h^*(y) \neq 0$ for $y \in (0, \infty)$, $h^* \in C^2(0, \infty)$ and such that*

$$-h_{vv}^*/h^* + Q \geq \eta, \quad y \in (0, \infty),$$

where η is a positive constant independent of ξ , and

$$Q = (p\rho)_{vv}^{1/4}/(p\rho)^{1/4},$$

where t and y are related by the equation

$$y = \int_0^t \left(\frac{\rho}{p}\right)^{1/2} ds.$$

Proof. Let

$$h^* = (p\rho)^{1/4}h = gh, \quad y = \int_0^t \left(\frac{\rho}{p}\right)^{1/2} ds,$$

then

$$-\frac{1}{\rho h} (ph)_t = -\frac{g}{\rho h^*} (gh_{vv}^* - h^*g_{vv}) \left(\frac{\rho}{p}\right)^{1/2} = -h_{vv}^*/h^* + g_{vv}/g$$

and the lemma is proven.

Expanding Q we obtain

$$Q = \frac{2b}{\sigma} - \frac{b\sigma}{t^2} + b^2 - \frac{3}{2} \frac{1}{t^2} + \frac{5}{4} \frac{\sigma^2}{t^4}$$

and we prove theorem 3.1.1 in the

Case I. $b > 0$. Put $h^*(y) = (y)^{1/2}$, then $-h_{vv}^*/h^* = 1/4y^2$ and since

$$Q = b^2 + \frac{b[(y + \xi)^2 - 2\xi^2]}{y(y + \xi)(y + 2\xi)} + \frac{-y^2 - 2y\xi + 5\xi^2}{4y^2(y + 2\xi)^2},$$

hence,

$$Q + \frac{1}{4y^2} \geq b^2 \text{ if } y > \xi.$$

But for all $y \in (0, \infty)$ we have

$$\begin{aligned} Q + \frac{1}{4y^2} &\geq \frac{b[y^2 + 2y\xi - \xi^2]}{y(y + \xi)(y + 2\xi)} + b^2 + \frac{\xi}{2y^2(y + 2\xi)} \\ &\geq \frac{\xi^2(by - 1)^2 + 3b^2\xi^2y^2}{2y^2(y + \xi)(y + 2\xi)} \geq \frac{3}{2} b^2 \frac{\xi^2}{(y + \xi)(y + 2\xi)}. \end{aligned}$$

Hence, for $y \leq \zeta$ we have

$$Q \geq \frac{3}{2} b^2 \frac{\zeta^2}{(2\zeta)(3\zeta)} = \frac{b^2}{4},$$

and so we can take $\eta = b^2/4$.

Case II. $b < 0$. Put $h^*(y) = y^{1/2} e^{-\alpha y}$, $\alpha > 0$ then $-h_{yy}^*/h^* = 1/(4y^2) + \alpha/y - \alpha^2$. Clearly, we have

$$Q - h_{yy}^*/h^* \geq b^2 - \alpha^2 \quad \text{for } y \leq \frac{\zeta}{4}.$$

In any case, however, if we put $\alpha = -\gamma b$ and choose γ to be such that $16/19 \leq \gamma < 1$, then

$$Q - h_{yy}^*/h^* \geq \frac{4(b + \alpha)y^2 + 4(b^2 - \alpha^2)y^3 + \alpha y}{4(y + \zeta)(y + 2\zeta)^2}.$$

Since

$$4(b + \alpha)y^2 + \alpha y \geq 0 \quad \text{whenever } y \leq \frac{\gamma}{4(1 - \gamma)},$$

and since

$$(b^2 - \alpha^2)y^3 + 4(b + \alpha)y^2 \geq 0 \quad \text{whenever } y \geq \frac{4}{|b|(1 + \gamma)},$$

hence if we choose γ such that $16/19 \leq \gamma < 1$ and such that

$$4 \frac{\gamma}{(1 - \gamma)} \geq \frac{4}{|b|(1 + \gamma)},$$

then we have that

$$Q - h_{yy}^*/h^* \geq \frac{3}{4}(b^2 - \alpha^2) \left\{ \frac{y^3}{(y + \zeta)(y + 2\zeta)^2} \right\},$$

for all $y \in (0, \infty)$. Hence, for $y \geq \zeta/4$ we have

$$Q - h_{yy}^*/h^* \geq \frac{b^2 - \alpha^2}{540}$$

and the theorem is proven.

3.2. Regular elliptic boundary problems. We give a brief summary of some well-known results for elliptic boundary problems and at the same time develop some concepts which are useful for our purposes. Let G be a bounded region in E_n and Γ its boundary, and let $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^{(\alpha)}$ where the a_α are assumed continuous on $\bar{G}$.

The principal part of P , denoted by $\hat{P}$, is defined as follows: $\hat{P}(x, D) = \sum_{|\alpha| = m} a_\alpha(x) D^\alpha$. We say that $P(x, D)$ is elliptic at a point $x^0 \in G$ if $\hat{P}(x^0, \xi) = 0$

for real ξ implies that $\xi = 0$. $P(x, D)$ is said to be elliptic in $\bar{G}$ if $P(x, D)$ is elliptic for every $x \in \bar{G}$. Two early results in this direction are given by the following theorems.

Theorem 3.2.1. (Gårding). *There exists a positive constant K such that the estimate $\|u\|_m \leq K(\|P(x, D)u\| + \|u\|)$ holds for all $u \in C_0^m(G)$ if and only if $P(x, D)$ is elliptic in $\bar{G}$.*

Theorem 3.2.2. (Aronszajn). *Let G be of class C^{m+1} . There exists a positive constant K such that the estimate*

$$\|u\|_m \leq K(\|P(x, D)u\| + \|u\|)$$

holds for all $u \in C^m(G)$ if and only if $P(x, D)$ is elliptic in $\bar{G}$ and for no $x^0 \in \Gamma$ does there exist a ξ such that $P(x^0, \xi) = 0$ and $\text{Im } \xi$ is perpendicular to Γ at x^0 .

Since the latter condition can never be satisfied it follows that the desired estimate cannot hold for all $u \in C^m(G)$. Hence it is clear that either other estimates must be found which are satisfied by all $u \in C^m(G)$, or suitable restrictions of $C^m(G)$ (or extensions of $C_0^m(G)$) must be found such that the above estimate holds. The former problem was treated by Agmon–Douglis–Nirenberg in [1] and by F. Browder in [5]; a brief description of their result follows. From this point on we assume that G is of class C^{m+1} , and that $P(x, D)$ is elliptic in $\bar{G}$. Let φ be on Γ and assume that $\varphi \in C^s(\Gamma)$, then there exist extensions v of φ into G such that $v \in C^s(\bar{G})$, and define the norm $\|\varphi\|_{s(G)}$ as follows: Let

$$\|\varphi\|_{s(G)} = \inf \{ \|v\|_{s(G)} : v \in C^s(G), v = \varphi \text{ on } \Gamma \},$$

where $\|\cdot\|_{s(G)}$ is the usual $\mathfrak{S}_s$ norm over G .

Consider the boundary operators

$$Q_t(x, D) = \sum_{|\alpha| \leq r_t} b_{t\alpha}(x) D^{(\alpha)}, \quad t = 1, 2, \dots, q,$$

where r_t is the order of Q_t and the $b_{t\alpha} \in C^{m-r_t}(\Gamma)$. For $p \in \Gamma$, let R_p be a rotation which carries the positive x_n direction into the direction of the outer normal to Γ at p . Applying R_p to $P(p, D)$ and the $Q_t(p, D)$ we obtain the expressions $P^*(p, D)$ and $Q_t^*(p, D)$. Let $\hat{Q}_t^*(p, D)$ and $\hat{P}^*(p, D)$ denote the principal parts of the $Q_t^*(p, D)$ and $P^*(p, D)$ respectively. Let $\hat{\xi} = (\xi_1, \xi_2, \dots, \xi_{n-1})$ and $\hat{\eta} = (\hat{\xi}, \eta)$ and put

$$\hat{P}^*(p, \hat{\xi}) = \hat{P}^*(p, \hat{\xi}, \eta) = \hat{P}^*(\eta),$$

$$\hat{Q}_t^*(p, \hat{\xi}) = \hat{Q}_t^*(p, \hat{\xi}, \eta) = \hat{Q}_t^*(\eta), \quad t = 1, 2, \dots, q.$$

Since P was assumed elliptic in $\bar{G}$ it follows that if $n \geq 3$ then m is even, and this conclusion remains true when $n = 2$ only if the coefficients of P are real-valued. Moreover, the equation $\hat{P}^*(\eta) = 0$ has precisely $m/2$ roots with positive imaginary parts and $m/2$ roots with negative imaginary parts for each choice of $p \in \Gamma$ and $\hat{\xi} \neq 0$. Denote the roots above by $\eta_1^+, \eta_2^+, \dots, \eta_{m/2}^+$ and $\eta_1^-, \eta_2^-, \dots, \eta_{m/2}^-$,

$\dots, \eta_{m/2}^-$ respectively. Let us define the polynomials

$$\hat{P}_+^*(\eta) = \prod_{i=1}^{m/2} (\eta - \eta_i^+), \quad \hat{P}_-^*(\eta) = \prod_{i=1}^{m/2} (\eta - \eta_i^-),$$

then clearly $\hat{P}^*(\eta) = \hat{P}_+^*(\eta) \hat{P}_-^*(\eta)$.

The Q_i are said to cover P at $p \in \Gamma$ if for each $\xi \neq 0$ there exist $m/2$ polynomials from the set $\{\hat{Q}_t^*(\eta) : t = 1, 2, \dots, q\}$ which are linearly independent modulo $\hat{P}_+^*(\eta)$, i.e., no non-trivial linear combination of them is a multiple of $\hat{P}_+^*(\eta)$. It is clear that the Q_i cannot cover P if $q < m/2$. Furthermore, it can be shown that if the Q_i cover P at $p \in \Gamma$, then for each choice of $\xi \neq 0$ there exist $m/2$ polynomials from the set $\{\hat{Q}_t^*(\eta) : t = 1, 2, \dots, q\}$ which are linearly independent modulo $\hat{P}_-^*(\eta)$. If the Q_i cover P for every $p \in \Gamma$, then the Q_i are said to cover P .

The elliptic boundary problem

$$P(x, D)u = f \text{ in } G,$$

$$Q_i(x, D)u = \varphi_i \text{ on } \Gamma, \quad i = 1, 2, \dots, q,$$

is said to be regular if and only if the Q_i cover P .

Theorem 3.2.1. (Agmon–Douglis–Nirenberg). *There exists a positive constant K such that the estimate*

$$\|u\|_{m(G)} \leq K \left\{ \|P(x, D)u\|_{0(G)} + \sum_{i=1}^q \|Q_i(x, D)u\|_{m-r_i(G)} + \|u\|_{0(G)} \right\}$$

holds for all $u \in C^m(G)$ if and only if P is elliptic in $\bar{G}$ and the Q_i cover P .

4. Singular oblique derivative problems.

4.1. Introduction. Let G be a bounded region of E_n of class C^3 and let Γ denote the boundary of G . A vector field $\mathbf{V}$ defined on Γ will be called *principal* if and only if the following conditions are satisfied: 1) $\mathbf{V}$ is non-vanishing and continuous on Γ ; 2) if $\mathbf{V}$ is tangent to Γ at all then this can only be the case along finitely many closed cycles $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_r$ of dimension $n - 2$, i.e., each $\mathcal{C}_i$ has local representations of the form $x_i = x_i(\eta_1, \eta_2, \dots, \eta_{n-2})$, $j = 1, 2, \dots, n$, where the Jacobian matrix $\partial(x)/\partial(\eta)$ has rank $n - 2$; 3) $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$, $i, j = 1, 2, \dots, r$, and 4) $\mathbf{V}(x)$, $x \in \mathcal{C}_i$ is never tangent to $\mathcal{C}_i$, $i = 1, 2, \dots, r$. Let V be a continuous and non-vanishing vector field on Γ , then $\mathbf{V}$ is said to be *regular* if either $n = 2$, or, $n \geq 3$ and $\mathbf{V}$ is never tangent to Γ .

Let $\mathbf{V}$ be principal and $\mathbf{v}(x)$ be the unit outer normal to Γ at x . For each $x \in \Gamma$ we have the unique representation $\mathbf{V}(x) = \mathbf{V}_T(x) + \mathbf{V}_N(x)$ where $\mathbf{V}_T(x)$ is tangent to Γ and $\mathbf{V}_N(x)$ is orthogonal to Γ at x . Let $\mathcal{C}_i$ be a cycle of tangency for $\mathbf{V}$ then $\mathbf{V}_T$ is non-vanishing on $\mathcal{C}_i$, and there exists a neighborhood $\mathfrak{N}_i$ on Γ about $\mathcal{C}_i$ such that $\mathfrak{N}_i \cap \mathcal{C}_j = \emptyset$, $j \neq i$, $\|\mathbf{V}_T(x)\| > 0$ for $x \in \mathfrak{N}_i$ and the vector lines on Γ induced by $\mathbf{V}_T$ in $\mathfrak{N}_i$ all intersect $\mathcal{C}_i$ once and only once. Let each vector line ℓ in $\mathfrak{N}_i$ be parameterized by arc length s and let the orientation on ℓ

be that induced by $\mathbf{V}_T$ in $\mathfrak{N}_i$. Further, let the point on ℓ corresponding to $s = 0$ be the point of intersection of ℓ and the cycle $\mathfrak{C}_i$. We put

$$v(x) = \langle \mathbf{V}_N(x), \mathbf{v}(x) \rangle, \quad x \in \Gamma,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in E_n . Let v' be the restriction of v to the vector line ℓ in $\mathfrak{N}_i$. The cycle $\mathfrak{C}_i$ is called an *emergent cycle* for $\mathbf{V}$ if $v'(s) < 0$ for $s < 0$ and $v'(s) > 0$ for $s > 0$, for every vector line ℓ in $\mathfrak{N}_i$. The cycle $\mathfrak{C}_i$ is called a *submergent cycle* for $\mathbf{V}$ if $v'(s) > 0$, for $s < 0$ and $v'(s) < 0$ for $s > 0$, for every vector line ℓ in $\mathfrak{N}_i$.

Now let us consider the so-called oblique derivative problem:

$$(I) \quad \begin{aligned} P(x, D)u &= \sum_{|\alpha| \leq 2} A_\alpha(x) D^{(\alpha)}u = f \text{ in } G \subset E_n, \\ B(x, D)u &= \sum_{i=1}^n a_i(x) \frac{\partial u}{\partial x_i} = \varphi \text{ on } \Gamma, \end{aligned}$$

where we assume that G is a bounded region of E_n , $n \geq 3$, of Class C^3 , and Γ its boundary. Moreover we assume that P is elliptic in $\bar{G}$, and that the vector field $\sum_{i=1}^n a_i(x) \mathbf{e}_i$ is continuous and non-vanishing for $x \in \Gamma$, where the $\mathbf{e}_i$ are unit vectors in the positive x_i directions. The invariance with respect to smooth, non-singular coordinate transformations of the emergent and submergent cycles for problem (I) can be shown by direct computation. The invariance of regular vector fields over Γ under such transformations is well known. Moreover, it is well-known that the elliptic boundary problem (I) is regular in the sense of Section 3.2 if and only if the vector field $\sum_{i=1}^n a_i(x) \mathbf{e}_i$ is regular on Γ .

As an example let $n \geq 3$, $G = \{x : |x| < 1\}$, $\Gamma = \{x : |x| = 1\}$, and the boundary condition be given by

$$B_0 u = \left\{ -x_n(1 + bR^2) \sum_{i=1}^{n-1} x_i \frac{\partial u}{\partial x_i} + (bR^4 - x_n^2) \frac{\partial u}{\partial x_n} \right\} \Big|_{\Gamma},$$

where $R^2 = \sum_{i=1}^{n-1} x_i^2$. We have $\mathbf{V}_0(x) = -x_n \mathbf{e}_\rho + bR^2 \mathbf{e}_\varphi$, $x \in \Gamma$ with

$$\mathbf{e}_\rho = \sum_{i=1}^n x_i \mathbf{e}_i, \quad \mathbf{e}_\varphi = -x_n \sum_{i=1}^{n-1} x_i \mathbf{e}_i + R^2 \mathbf{e}_n.$$

Since $\langle \mathbf{e}_\rho(x), \mathbf{e}_\varphi(x) \rangle = 0$ and $\langle \mathbf{e}_\rho(x), \mathbf{V}_0(x) \rangle = -x_n$ on Γ it is easily verified that the cycle $\mathfrak{C} = \{x : x_n = 0, R = 1\}$ is emergent for $b < 0$ and submergent for $b > 0$, and $\mathbf{V}_0(x)$ is tangent to Γ only for $x \in \mathfrak{C}$. Hence for $b \neq 0$, $\mathbf{V}_0$ is principal and non-regular. Therefore the Agmon–Douglis–Nirenberg–Browder estimates do not hold for the boundary problem (I) with $B = B_0$. It was shown in Borrelli [4] that with some suitable restrictions on the domain of functions u , estimates similar to the Agmon–Douglis–Nirenberg–Browder estimates hold for the boundary problem (I) with B replaced by the condition $B_1(x, D) = (1/x_n)B_0(x, D)$ if $b > 0$, i.e., if the cycle $\mathfrak{C}$ is submergent. More generally, it can be shown that if the boundary condition $B_0(x, D)$ in problem (I) gives rise to a vector field $\mathbf{V}$ on Γ which is principal and has only submergent cycles, then, with suitable

restrictions on the functions u estimates of the Agmon–Douglis–Nirenberg–Browder type hold for the boundary problem (I) with B replaced by $B_1(x, D) = (1/\varphi(x))B_0(x, D)$ provided φ is sufficiently smooth on Γ and vanished sufficiently strongly on the submergent cycles and is non-zero elsewhere on Γ . That these estimates are not generally valid when the condition B_0 gives rise to emergent cycles is shown by the following example. Consider problem (I) with $P(x, D)$, $B_1(x, D)$, and G defined as follows:

$$\begin{aligned} G &= \{x : x \in E_n, n \geq 3, |x| < 1\}, \quad \Gamma = \partial G, \\ P(x, D) &= \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \cdots \frac{\partial^2}{\partial x_n^2}, \\ B_1(x, D) &= \frac{1}{\varphi(x)} \frac{\partial}{\partial x_n}. \end{aligned}$$

Since $\mathbf{V} = \mathbf{e}_n$ it is easy to see that the cycle $\mathcal{C} = \{x : x \in \Gamma, x_n = 0\}$ is an emergent cycle for $\mathbf{V}$. Let $u^m = u^m(x_1, x_2)$ be a sequence of polynomials in x_1 and x_2 such that

1. u^m is homogeneous of degree m , $m = 1, 2, \dots$,
2. $\nabla^2 u^m = 0$, $m = 1, 2, \dots$,
3. $\|u^m\|^2 = \int_G |u^m|^2 dx = 1$, $m = 1, 2, \dots$.

Clearly $B_1 u^m = 0$ on Γ , $m = 1, 2, \dots$, and

$$\begin{aligned} \|u^m\|_2^2 &\geq \sum_{i=1}^n \left\| \frac{\partial u^m}{\partial x_i} \right\|^2 \\ &\geq K \left\| \sum_{i=1}^n x_i \frac{\partial u^m}{\partial x_i} \right\|^2, \quad (K \text{ independent of } m), \\ &= K \|mu\|^2 = Km^2, \quad m = 1, 2, 3, \dots \end{aligned}$$

Hence no estimate of the Agmon–Douglis–Nirenberg–Browder type is possible for this problem, no matter how the function φ is defined. H. O. Cordes has conjectured that a somewhat generalized ADNB estimate can be proven for problem (I) when the condition $B_1(x, D) = (1/\varphi(x))B_0(x, D)$ is such that the vector field $\mathbf{V}$ generated by B_0 is principal and gives rise to an emergent cycle $\mathcal{C}$, φ is suitably defined, the additional terms appearing in the generalized estimate presumably involving data, associated with the function being estimated, along the emergent cycles only. In the final section of this paper we attempt to derive a local estimate in the presence of emergent cycles using the same method which was successful in the submergent case; however the results there do not quite take a satisfactory form.

The remainder of this paper will be devoted to the details in the derivation of local estimates for singular oblique boundary problems as set forth in the above paragraphs.

4.2. Some preliminaries for the local estimate. We proceed to extend the domain of applicability of an integral estimate derived earlier in section 3.1. In this connection we use the same notations as in 3.1 and assume explicitly that $b \neq 0$. Theorem 3.1.1 can trivially be generalized to give the

Lemma 4.2.1. *There exists a constant $K > 0$ independent of ξ such that the estimate*

$$\int_0^\infty p |u_t|^2 dt \geq K \int_0^\infty \rho |u|^2 dt$$

holds for all $u \in C_0^1[(0, \infty)]$ with u'' piecewise continuous.

Now we assert that Theorem 4.2.1 holds for functions $u = u(t)$, where

1. $u \in C^2[[0, \infty)]$, $u(0) = u'(0) = 0$,
2. $\int_0^\infty |u|^2 \rho dt < \infty$, $\int_0^\infty p |u_t|^2 dt < \infty$.

This fact is essentially implied by

Lemma 4.2.2. *Let $u \in C^2[[0, \infty)]$, $u(0) = 0$, $u'(0) = 0$ and*

$$\int_0^\infty |u|^2 \rho dt < \infty, \quad \int_0^\infty |u'|^2 p dt < \infty.$$

Then there exists a sequence $\{u_n\}$ such that $u_n \in C_0^1[(0, \infty)]$, u_n'' piecewise continuous and,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty |u_n|^2 \rho dt &= \int_0^\infty |u|^2 \rho dt, \\ \lim_{n \rightarrow \infty} \int_0^\infty |u_n'|^2 p dt &= \int_0^\infty |u'|^2 p dt. \end{aligned}$$

Proof. We may as well restrict ourselves to real functions u , otherwise the following constructions can be carried out separately on the real and imaginary parts of u . Let

$$\eta_n(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{n}, \\ \sin^2 (nt - 1), & \frac{1}{n} \leq t \leq \frac{1}{n} + \frac{\pi}{2n}, \\ 1, & \frac{1}{n} + \frac{\pi}{2n} \leq t \leq n, \\ \cos^2 (t - n), & n \leq t \leq n + \frac{\pi}{2}, \\ 0, & t \geq n + \frac{\pi}{2}, \end{cases}$$

and define $u_n = \eta_n(t)u(t)$, then certainly $u_n \in C_0^1(0, \infty)$, u_n'' piecewise continuous. It follows that

$$\begin{aligned} \int_0^\infty u^2 \rho \, dt - \int_0^\infty u_n^2 \rho \, dt &= \int_0^\infty (1 - \eta_n^2) u^2 \rho \, dt \\ &\leq \int_0^{1/n + \pi/2n} u^2 \rho \, dt + \int_n^\infty u^2 \rho \, dt \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned} \int_1^\infty u^2 \eta_n'^2 e^{2b\sigma(t)} \frac{\sigma^3(t)}{t^2} \, dt &\leq C \int_1^\infty u^2 \eta_n'^2 e^{2b\sigma} \sigma \, dt \\ &\leq C' \int_n^{n + \pi/2} u^2 e^{2b\sigma} \sigma \, dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Likewise,

$$\int_0^1 u^2 \eta_n'^2 e^{2b\sigma} \frac{\sigma^3}{t^2} \, dt \leq C \int_0^1 u^2 \eta_n'^2 \frac{1}{t^2} \, dt = A.$$

But since $u(0) = u'(0) = 0$ we have by Taylor's theorem,

$$u(t) = \frac{t^2}{2} u''(\theta t), \quad 0 \leq \theta \leq 1,$$

and

$$|u''(\theta t)| \leq C \quad \text{for } 0 \leq t \leq 1.$$

Hence,

$$A \leq C' \int_{1/n}^{1/n + \pi/2n} t^2 n^2 \, dt = \frac{C''}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and hence

$$\int_0^\infty u^2 \eta_n'^2 \rho \, dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Finally, we have,

$$\begin{aligned} - \int_0^\infty u'^2 \eta_n^2 \rho \, dt + \int_0^\infty u'^2 \rho \, dt &= \int_0^\infty (1 - \eta_n^2) u'^2 \rho \, dt \\ &\leq C' \int_{1/n}^{1/n + \pi/2n} u'^2 \rho \, dt + C'' \int_n^{n + \pi/2} u'^2 \rho \, dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Letting

$$\|y\|_p^2 = \int_0^\infty y^2 \rho \, dt$$

we have by the triangle inequality:

$$||u'\eta_n||_p + ||u\eta'_n||_p \geq ||u'\eta_n + u\eta'_n||_p \geq ||u'\eta_n||_p - ||u\eta'_n||_p,$$

hence

$$|||u'_n||_p - ||u'\eta_n||_p| \leq ||u\eta'_n||_p.$$

But since $||u\eta'_n||_p \rightarrow 0$ and $||u'\eta_n||_p \rightarrow ||u'||_p$ as $n \rightarrow \infty$ we have that $||u'_n||_p \rightarrow ||u'||_p$ as $n \rightarrow \infty$ as was to be shown. Now we generalize the estimates to a still more general class of functions:

Lemma 4.2.3. *Let $z(t) \in C^2[[0, \infty)]$ and let*

$$\int_0^\infty |z|^2 \rho \, dt < \infty, \quad \int_0^\infty |z'|^2 p \, dt < \infty.$$

Then,

$$\int_0^\infty |z|^2 \rho \, dt \leq C \int_0^\infty |z'|^2 p \, dt + C' \zeta^{3/2} e^{2b\zeta} |z(0)|^2,$$

where the constants are independent of z and also independent of ζ .

Proof. Let $\eta(t)$ be such that

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1, \\ 0 & \text{for } 2 \leq t < \infty, \end{cases}$$

$$\eta \in C^\infty[[0, \infty)],$$

$$0 \leq \eta(t) \leq 1, \quad 0 \leq t < \infty,$$

and put $\lambda(t) = \eta(t/\zeta^{1/2})$. Noting that $z'(0) = 0$ is implied by the hypotheses we observe that the function $u(t) = z(t) - z(0)\lambda(t)$ satisfies the hypothesis of lemma 4.2.2. Letting

$$||y||_\rho^2 = \int_0^\infty y^2 \rho \, dt$$

we have that there exists a positive constant C such that

$$||z(t) - z(0)\lambda(t)||_\rho \leq C ||(z(t) - z(0)\lambda(t))'||_\rho.$$

But now from the hypothesis we have

$$||z||_\rho - ||z(0)\lambda(t)||_\rho \leq ||z(t) - z(0)\lambda(t)||_\rho,$$

and

$$C ||(z - z(0)\lambda(t))'||_\rho \leq C ||z'||_\rho + C ||z(0)\lambda'(t)||_\rho.$$

Hence,

$$||z||_\rho \leq C ||z'||_\rho + |z(0)| \{ ||\lambda(t)||_\rho + C ||\lambda'(t)||_\rho \}.$$

Now the support of $\lambda(t) \subset \{t : 0 \leq t \leq 2(\xi)^{1/2}\}$ and the support of $\lambda'(t) \subset \{t : (\xi)^{1/2} \leq t \leq 2(\xi)^{1/2}\}$. Hence, there exists a positive constant $C_2 > 0$ such that

$$\|\lambda(t)\|_p^2 = \int_0^{2\xi^{1/2}} \lambda^2 e^{2b\sigma} \sigma dt \leq C_2 \xi^{3/2} e^{2b\xi}$$

and such that

$$\|\lambda'(t)\|_p^2 = \int_{(\xi)^{1/2}}^{2(\xi)^{1/2}} \frac{|\eta(t/(\xi)^{1/2})|^2}{\xi} e^{2b\sigma} \frac{\sigma^3}{t^2} dt \leq C_2 e^{4b} \xi^{3/2} e^{2b\xi}.$$

Hence the desired estimate follows.

Lemma 4.2.4. *Let $b > 0$, and let the function $\psi(t) \in C^1[[0, \infty)]$ be such that*

$$\int_0^\infty |\psi|^2 \sigma dt < \infty \text{ for all } \xi$$

and such that the function

$$\omega(t) = -\frac{e^{b\sigma(t)}}{\sigma(t)} \int_t^\infty \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau$$

satisfies the condition

$$\int_0^\infty |\omega|^2 \sigma^3(t) dt < \infty \text{ for all } \xi.$$

Then we have the estimate

$$\int_0^\infty |\omega(t)|^2 \sigma^3(t) dt \leq C \int_0^\infty |\psi|^2 \sigma(t) dt$$

where the positive constant C is independent of ψ (under above conditions) and ξ .

Proof. Put $z(t) = \sigma(t) e^{-b\sigma(t)} \omega(t)$ and observe that the hypothesis of lemma 4.2.3 is satisfied for $z = z(t)$. Hence lemma 4.2.3 implies that there exist positive constants C and C' such that

$$\int_0^\infty |z|^2 e^{2b\sigma} \sigma dt \leq C \int_0^\infty |z'|^2 e^{2b\sigma} \frac{\sigma^3}{t^2} dt + C' \xi^{3/2} e^{2b\xi} |z(0)|^2,$$

or,

$$\int_0^\infty |\omega|^2 \sigma^3 dt \leq C \int_0^\infty |\psi|^2 \sigma dt + C' \xi^{3/2} e^{2b\xi} \left| \int_0^\infty \frac{t}{\sigma(t)} e^{-b\sigma(t)} \psi(t) dt \right|^2.$$

However, we have that

$$\left| \int_0^\infty \frac{t}{\sigma(t)} e^{-b\sigma} \psi(t) dt \right|^2 \leq \int_0^\infty \frac{t^2}{\sigma^3} e^{-2b\sigma} dt \int_0^\infty \sigma |\psi|^2 dt.$$

If we let $y = \sigma(t) - \zeta$ it follows that

$$A = \int_0^\infty \frac{t^2}{\sigma^3(t)} e^{-2b\sigma} dt = e^{-2b\zeta} \int_0^\infty \frac{(y + 2\zeta)^{1/2}}{(y + \zeta)^2} (y)^{1/2} e^{-2by} dy$$

and we observe that there exist positive constants C_1 and C_2 such that

$$C_2 \frac{1}{\zeta^{3/2}} \leq \frac{(y + 2\zeta)^{1/2}}{(y + \zeta)^2} \leq C_1 \frac{1}{\zeta^{3/2}}.$$

We have the following estimate for A :

$$A \leq C_1 e^{-2b\zeta} \frac{1}{\zeta^{3/2}} \int_0^\infty (y)^{1/2} e^{-2by} dy = C_3 \zeta^{-3/2} e^{-2b\zeta},$$

where C_3 is independent of ζ . Combining the above results the desired estimate follows.

Lemma 4.2.5. *Let $b > 0$, and let $\psi(t) \in C^1 [(-\infty, 0]]$ be such that*

$$\int_{-\infty}^0 |\psi|^2 \sigma(t) dt < \infty \text{ for all } \xi,$$

and also such that if

$$\omega(t) = \frac{e^{b\sigma(t)}}{\sigma(t)} \int_{-\infty}^t \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau,$$

then

$$\int_{-\infty}^0 |\omega|^2 \sigma^3(t) dt < \infty \text{ for all } \xi.$$

Then there exists a positive constant C such that

$$\int_{-\infty}^0 |\omega(t)|^2 \sigma^3(t) dt \leq C \int_{-\infty}^0 |\psi|^2 \sigma(t) dt,$$

where the constant C is independent of ζ .

Proof. Extend $\psi(t)$ into $0 \leq t < \infty$ as an even function, calling this extended function again $\psi(t)$. For $t > 0$ now we define as in lemma 4.2.4 the function

$$\alpha(t) = -\frac{e^{b\sigma(t)}}{\sigma(t)} \int_t^\infty \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau,$$

and since the integrand is odd we have

$$\begin{aligned} \alpha(t) &= -\frac{e^{b\sigma(-t)}}{\sigma(-t)} \left\{ -\int_{-\infty}^{-t} \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau \right\} \\ &= \omega(-t). \end{aligned}$$

Hence, $\alpha(t)$ is the result of extending ω into $0 \leq t < \infty$ as an even function.

Thus, it is clear that $\alpha(t)$ satisfies the hypothesis of lemma 4.2.4. Hence, we have by lemma 4.2.4 that

$$\int_0^\infty |\alpha|^2 \sigma^3 dt \leq C \int_0^\infty |\psi|^2 \sigma dt,$$

and because the integrands are even we also have that

$$\int_{-\infty}^0 |\omega|^2 \sigma^3 dt \leq C \int_{-\infty}^0 |\psi|^2 \sigma dt,$$

where C is independent of ζ as was to be shown.

Lemma 4.2.6. *Let $b > 0$, and let $\psi(t) \in C^1[(-\infty, \infty)]$ be such that*

$$\int_{-\infty}^\infty |\psi|^2 \sigma dt < \infty \text{ for all } \xi,$$

and such that if

$$\omega(t) = \frac{e^{b\sigma(t)}}{\sigma(t)} \int_{-\infty}^t \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau$$

satisfies the condition

$$\int_{-\infty}^\infty |\omega|^2 \sigma^3 dt < \infty \text{ for all } \xi.$$

Then we have the estimate

$$\int_{-\infty}^\infty |\omega|^2 \sigma^3 dt \leq C \int_{-\infty}^\infty |\psi|^2 \sigma dt,$$

where C is independent of ζ .

Proof. We first of all note that

$$\int_{-\infty}^\infty \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau < \infty,$$

hence, since also

$$\int_{-\infty}^\infty |\omega|^2 \sigma^3 dt < \infty,$$

it follows that

$$\int_{-\infty}^\infty \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau = 0 \text{ for all } \xi.$$

But this implies that

$$\omega(t) = -\frac{e^{b\sigma(t)}}{\sigma(t)} \int_t^\infty \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau, \quad t \geq 0,$$

and that

$$\omega(t) = \frac{e^{b\sigma(t)}}{\sigma(t)} \int_{-\infty}^t \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau, \quad t \leq 0.$$

Furthermore it is clear that the hypotheses of lemmas 4.2.4 and 4.2.5 are satisfied, hence combining the estimates given by these lemmas we obtain the estimate for the present lemma.

Lemma 4.2.7. *Let $b > 0$, and let the function ψ satisfy the hypotheses of lemma 4.2.6. If we put*

$$v_0(t) = -\frac{\psi(t)}{\sigma(t)} - \frac{be^{b\sigma(t)}}{\sigma(t)} \int_{-\infty}^t \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau,$$

then we have the estimate

$$\int_{-\infty}^{\infty} \sigma^3 |v_0|^2 dt \leq C \int_{-\infty}^{\infty} \sigma |\psi|^2 dt,$$

where the constant C is independent of ζ .

Proof. We have that

$$\int_{-\infty}^{\infty} |v_0|^2 \sigma^3 dt \leq 2 \int_{-\infty}^{\infty} |\psi|^2 \sigma dt + 2b^2 \int_{-\infty}^{\infty} |\omega|^2 \sigma^3 dt < \infty.$$

However, from Lemma 4.2.6 it follows that

$$\int_{-\infty}^{\infty} |\omega|^2 \sigma^3 dt \leq C \int_{-\infty}^{\infty} |\psi|^2 \sigma dt,$$

where C is independent of ζ . Hence the desired estimate follows.

4.3. The local estimate for submergent cycles. It will be convenient in this and the next section to denote a general element of E by the symbol $\hat{x}$, i.e., $\hat{x} = (x_1, x_2, \dots, x_{n-1})$.

Let us consider the problem

$$(I) \quad \begin{cases} (\nabla^2 - 1)w = f \in C_0^\infty(H), \\ w(\hat{x}, 0) = 0, \end{cases}$$

and let the support of f be contained in a sphere about the origin of radius $= \delta$.

Lemma 4.3.1. *There exists a solution w to problem (I) such that*

1. $w \in C^\infty(H)$,
2. $\left\| \frac{\partial w}{\partial x_n}(\hat{x}, 0) \right\|_1 < \infty, \quad \left\| \frac{\partial w}{\partial x_n}(\hat{x}, 0) \right\|_1^* < \infty,$

3. We have the estimate

$$\left[\frac{\partial w}{\partial x_n}(\hat{x}, 0) \right]_{1/2} \leq K \|f\|_0$$

where K is a positive constant depending only on δ ,

4. If $v(\hat{\xi}, x_n) = \mathfrak{F}(w)$, then v is infinitely differentiable in the variables $\xi_1, \xi_2, \dots, \xi_{n-1}$ and x_n .

Proof. Let $g(\hat{\xi}, x_n) = \mathfrak{F}(f)$, then g is infinitely differentiable in the variables $\xi_1, \dots, \xi_{n-1}$ and x_n . Moreover, g as a function of x_n has its support in the interval $0 \leq x_n \leq \delta$ uniformly in $\hat{\xi}$. If we put $v(\hat{\xi}, x_n) = \mathfrak{F}(w)$, then problem (I) becomes

$$(II) \quad \begin{cases} v_{x_n x_n} - \sigma^2 v = g, \\ v(\hat{\xi}, 0) = 0. \end{cases}$$

For simplicity we put $x_n = x$. The Green's function, $G(x, y)$, for problem (II) is given by

$$G(x, y) = \begin{cases} \frac{e^{-y\sigma} \sinh \sigma x}{-\sigma}, & x \leq y, \\ \frac{e^{-x\sigma} \sinh \sigma y}{-\sigma}, & y \leq x, \end{cases}$$

and the function

$$v(\hat{\xi}, x) = \int_0^\infty G(x, y) g(y) dy$$

is a solution of problem (II) which is infinitely differentiable in the variables $\xi_1, \dots, \xi_{n-1}$ and x . Furthermore, we have

$$|v|^2 \leq \frac{2}{\sigma^2} \sinh^2 \sigma x \left| \int_x^\infty e^{-\sigma y} g(y) dy \right|^2 + \frac{2}{\sigma^2} e^{-2\sigma x} \left| \int_0^x \sinh \sigma y g(y) dy \right|^2,$$

and hence

$$|v|^2 \leq \begin{cases} \frac{1}{4\sigma^3} \int_0^\infty |g|^2 dy, & \text{for } x \leq \delta, \\ \frac{1}{4\sigma^3} e^{-2\sigma(x-\delta)} \int_0^\infty |g|^2 dy, & \text{for } x \geq \delta, \end{cases}$$

thus

$$\int_0^\infty |v|^2 dx \leq K \frac{\delta}{\sigma^3} \int_0^\infty |g|^2 dy,$$

where K is a positive constant independent of σ and δ . Hence $\|\sigma^2 v\| < \infty$. Similarly it can be shown that $\|\sigma v_x\| < \infty$. Letting $w = \mathfrak{F}^{-1}(v)$ we observe that

$$\|w\|_2^2 \sim \|v_{xx}\|^2 + 2 \|\sigma v_x\|^2 + \|\sigma v\|^2 \leq 3 \|\sigma^2 v\|^2 + 2 \|\sigma v_x\|^2 + 2 \|g\|^2 < \infty,$$

hence $w \in \mathfrak{S}_2$. In an exactly analogous manner it can be shown that $w \in \mathfrak{S}_k$, $k = 3, 4, \dots$. Hence by the Sobelov Lemma we conclude that $w \in C^\infty(H)$. Further we have that

$$\frac{\partial v}{\partial x}(\xi, 0) = - \int_0^\infty e^{-\sigma y} g(y) dy,$$

and thus

$$\left| \frac{\partial v}{\partial x}(\xi, 0) \right|^2 \leq \frac{1}{\sigma} \int_0^\infty |g|^2 dy.$$

Hence, it follows that

$$\left\| \frac{\partial w}{\partial x_n}(\hat{x}, 0) \right\|_1 < \infty \quad \text{and that} \quad \left[\frac{\partial w}{\partial x_n}(\hat{x}, 0) \right]_{1/2} \leq K \|f\|.$$

Finally, we see from lemma 2.5 that

$$\left\| \frac{\partial w}{\partial x_n}(\hat{x}, 0) \right\|_1 < \infty$$

and the proof of the lemma is completed.

Lemma 4.3.2. *Let w be the solution of problem (I) given in the above lemma, then we have the estimate $\|w\|_2 \leq \|f\|$.*

Proof. We have

$$\begin{aligned} \|f\|^2 &= \langle (\nabla^2 - 1)w, (\nabla^2 - 1)w \rangle \\ &= \sum_{k=1}^n \int_H \left| \frac{\partial^2 w}{\partial x_k^2} \right|^2 + 2 \sum_{\substack{i,k=1 \\ i \neq k}}^n \int_H \frac{\partial^2 w}{\partial x_i^2} \frac{\partial^2 w}{\partial x_k^2} \\ &\quad - 2\Re \sum_{k=1}^n \int_H \frac{\partial^2 w}{\partial x_k^2} \bar{w} + \|w\|^2 \end{aligned}$$

the above integrals all existing since $w \in \mathfrak{S}_k$, $k = 0, 1, 2, \dots$. Now we claim that the boundary terms which arise when integrating the integrals appearing in the second and third terms by parts drop out. Indeed, direct computation shows that

$$\left\| x_k \frac{\partial w}{\partial x_k} \right\| < \infty, \quad \left\| x_k \frac{\partial^2 w}{\partial x_k^2} \right\| < \infty, \quad k = 1, 2, \dots, n.$$

Hence for almost all $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ we have that

$$\int_{0,-\infty}^\infty dx_k \left| \frac{\partial^2 w}{\partial x_k^2} \right|^2 = \left| \frac{\partial^2 w}{\partial x_k^2} \right| x_k \Big|_{0,-\infty}^\infty - 2 \int_{0,-\infty}^\infty x_k \Re \left(\frac{\partial^2 w}{\partial x_k^2} \frac{\partial^3 w}{\partial x_k^3} \right) dx_k,$$

where the lower limit is zero when $k = n$ and $-\infty$ otherwise. Since both integrals

exist it follows that for almost all $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ we have that

$$\frac{\partial^2 w}{\partial x_k^2} \rightarrow 0 \quad \text{as } x_k \rightarrow \pm \infty,$$

for $k = 1, 2, \dots, n-1$, and that

$$\frac{\partial^2 w}{\partial x_n^2} \rightarrow 0 \quad \text{as } x_n \rightarrow +\infty,$$

for almost all $(x_1, \dots, x_{n-1})$. Similar statements are true about the derivatives $(\partial w / \partial x_k)$, $k = 1, 2, \dots, n$. Hence integration by parts gives us that

$$||f||^2 = \sum_{k=1}^n \int_H \left| \frac{\partial^2 w}{\partial x_k^2} \right|^2 + 2 \sum_{\substack{i,k=1 \\ i \neq k}}^n \int_H \left| \frac{\partial^2 w}{\partial x_i \partial x_k} \right|^2 + 2 \sum_{k=1}^n \int_H \left| \frac{\partial w}{\partial x_k} \right|^2 + ||w||^2 \geq ||w||_2$$

and the lemma follows.

Lemma 4.3.3. *Let w be as in the above lemma then we have that*

$$x_{n-1} \left(\frac{\partial w}{\partial x_n} \right) (\hat{x}, 0) \in \mathcal{HC}_0.$$

Proof. We have that

$$\begin{aligned} \left\| x_{n-1} \frac{\partial w}{\partial x_n} (\hat{x}, 0) \right\|^2 &= \left\| \frac{\partial^2 v}{\partial \xi_{n-1} \partial x_n} (\xi, 0) \right\|^2 \\ &= \int_E \left| \int_0^\infty e^{-\sigma y} \left(-y \frac{\xi_{n-1}}{\sigma} \right) g(y) dy + \int_0^\infty e^{-\sigma y} g_{\xi_{n-1}}(y) dy \right|^2 d\xi \\ &\leq 2 \int_E \left| \int_0^\infty e^{-\sigma y} y g(y) dy \right|^2 d\xi + 2 \int_E \left| \int_0^\infty e^{-\sigma y} g_{\xi_{n-1}} dy \right|^2 d\xi \\ &\leq 2 \int_E d\xi \int_0^\infty e^{-2\sigma y} y^2 dy \int_0^\infty |g|^2 dy + 2 \int_E d\xi \int_0^\infty e^{-2\sigma y} dy \int_0^\infty |g_{\xi_{n-1}}|^2 dy \\ &\leq C_1 \int_E d\xi \int_0^\infty |g|^2 dy + C_2 \int_E d\xi |g_{\xi_{n-1}}|^2 dy \\ &= C_1 ||f||_0^2 + C_2 ||x_{n-1} f||_0^2 < \infty. \end{aligned}$$

Now we consider the problem

$$(III) \quad \begin{cases} (\nabla^2 - 1)u = f \in C_0^\infty(H), \\ \frac{1}{x_{n-1}} \left(x_{n-1} \frac{\partial u}{\partial x_n} + b \frac{\partial u}{\partial x_{n-1}} \right) (\hat{x}, 0) = \varphi \in C_0^\infty(E), \end{cases}$$

where b is some positive constant. Assume that (III) has a solution $u \in C_0^\infty(H)$, and let w be the solution to problem (I) described in lemma 4.3.1 which corresponds to the function f in problem (III). Then clearly the function $z = u - w$

is a solution to the problem

$$(IV) \quad \begin{cases} (\nabla^2 - 1)z = 0, \\ \frac{1}{x_{n-1}} \left(x_{n-1} \frac{\partial z}{\partial x_n} + b \frac{\partial z}{\partial x_{n-1}} \right) (\hat{x}, 0) = \varphi - \frac{\partial w}{\partial x_n} (\hat{x}, 0) = \eta. \end{cases}$$

Letting $v(\xi, x_n) = \mathfrak{F}(z)$ we see that v has the form $v(\xi, x_n) = v_0(\xi)e^{-\sigma x_n}$ and since

$$\|z\|_2^2 \sim \|\sigma^2 v\|^2 + 2 \|\sigma v_{x_n}\|^2 + \|v_{x_n x_n}\|^2 = 4 \|\sigma^{3/2} v_0\|^2,$$

hence

$$\int_E \sigma^3 |v_0|^2 d\xi < \infty.$$

Moreover, letting $\psi(\xi) = \mathfrak{F}(\eta)$ we have in view of lemma 4.3.1 that $\|\sigma^{1/2}\psi\|^2 < \infty$. Furthermore we see from lemma 4.3.1 that $\psi(\xi)$ and $v_0(\xi)$ are infinitely differentiable in the variables $\xi_1, \dots, \xi_{n-1}$.

Transforming the boundary condition in problem (IV) we obtain

$$\frac{\partial}{\partial \xi_{n-1}} (\sigma v_0) - b \frac{\xi_{n-1}}{\sigma} (\sigma v_0) = -\psi_{\xi_{n-1}}.$$

But the left-hand side can be rewritten as

$$e^{b\sigma} (e^{-b\sigma} \sigma v_0)_{\xi_{n-1}} = -\psi_{\xi_{n-1}}.$$

Since $b > 0$ we can integrate both sides of the above equation from $-\infty$ to ξ_{n-1} with respect to ξ_{n-1} to obtain

$$e^{-b\sigma} \sigma v_0 = - \int_{-\infty}^{\xi_{n-1}} e^{-b\sigma(\tau)} \psi_\tau d\tau.$$

Integrating by parts we have

$$v_0(\xi, t) = -\frac{\psi(\xi, t)}{\sigma(t)} - b \frac{e^{b\sigma(t)}}{\sigma(t)} \int_{-\infty}^t \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \psi(\tau) d\tau,$$

where we have written t for the variable ξ_{n-1} . Now we observe that since

$$\int d\xi \int_{-\infty}^{\infty} dt \sigma^3(t) |v_0(\xi, t)|^2 < \infty,$$

and

$$\int d\xi \int_{-\infty}^{\infty} dt \sigma(t) |\psi(\xi, t)|^2 < \infty,$$

we have by Fubini's theorem that

$$\int_{-\infty}^{\infty} dt \sigma(t) |\psi(\xi, t)|^2 < \infty, \quad \int_{-\infty}^{\infty} dt \sigma^3(t) |v_0(\xi, t)|^2 < \infty,$$

for almost all $\check{\xi}$. Hence the hypothesis of lemma 4.2.7 is satisfied for the functions v_0 , ψ and ω , thus we have the estimate

$$\int_{-\infty}^{\infty} \sigma^3(t) |v_0|^2 dt \leq C \int_{-\infty}^{\infty} \sigma(t) |\psi|^2 dt,$$

for almost all $\check{\xi}$ and the constant C is independent of $\check{\xi}$. Hence it follows that

$$\int d\check{\xi} \int_{-\infty}^{\infty} \sigma^3(t) |v_0|^2 dt \leq C \int d\check{\xi} \int_{-\infty}^{\infty} \sigma(t) |\psi|^2 dt,$$

which is equivalent to the statement that

$$||z||_2 < C[\eta]_{1/2}.$$

Lemmas 4.3.1 and 2.6 imply that

$$||z||_2 \leq C \left\{ [\varphi]_{1/2} + \left[\frac{\partial w}{\partial x_n}(\hat{x}, 0) \right]_{1/2} \right\} \leq C \{ \|\varphi\|_1 + \|f\|_0 \}.$$

On the other hand we have

$$||z||_2 = ||u - w||_2 \geq ||u||_2 - ||w||_2 \geq ||u||_2 - ||f||_0.$$

Hence we have the

Theorem 4.3.1. *Let $\delta > 0$ and let u be any solution of problem (III) with the support of u contained in a sphere about the origin of radius δ , then we have the estimate*

$$||u||_2 \leq C \{ \|\varphi\|_1 + \|f\|_0 \},$$

where C is a positive constant depending only on δ .

4.4. The local estimate for emergent cycles. In this section we will be concerned with the derivation of an *a priori* estimate for problem (III), section 4.3 in the semi-space H in the case $b < 0$ for which, it will be recalled,

$$\mathcal{C} = \{y \mid y \in E_n, y_n = 0, y_{n-1} = 0\}$$

become an emergent cycle. Put $\check{x} = (x_1, x_2, \dots, x_{n-2})$ and let $\hat{\mathcal{F}}$ and $\check{\mathcal{F}}$ denote Fourier Transforms in the space E and $\mathcal{C}$ respectively. We define the following norms for $\lambda \in C_0^\infty(\mathcal{C})$:

$$||\lambda||_{\cdot}^2 = \sum_{\substack{|\alpha| \leq k \\ \alpha_n = \alpha_{n-1} = 0}} \int_{\mathcal{C}} |D^\alpha \lambda|^2, \quad k = 0, 1, 2, \dots,$$

and

$$[[\lambda]]_s^2 = \int_{\mathcal{C}} \zeta^{2s} |\check{\mathcal{F}}(\lambda)|^2, \quad s \geq 0.$$

We denote the completion of the space $C_0^\infty(\mathcal{C})$ under the norms $|| \cdot ||_k$ and $[[\cdot]]$, by $\mathfrak{S}_k$ and $\mathfrak{S}_s$ respectively and observe that these norms are equivalent when

$s = k$. And finally, for $\lambda \in C_0^\infty(\mathcal{C})$ we define the norms

$$\begin{aligned} \|\lambda\|_s &= \inf_{\substack{v \in C_0^\infty(\mathcal{B}) \\ v = \lambda \circ \mathfrak{F}}} [v]_s, \quad s \geq 0, \\ \|\lambda\|_k &= \inf_{\substack{v \in C_0^\infty(\mathcal{B}) \\ v = \lambda \circ \mathfrak{F}}} \|v\|_k, \quad k = 1, 2, 3, \dots, \end{aligned}$$

and denote the completion of $C_0^\infty(\mathcal{C})$ under the above norms by $\mathfrak{S}_s$ and $\mathfrak{S}_k$ respectively. Let $u \in C_0^\infty(H)$ and put $\lambda(\tilde{x}) = u(\tilde{x}, 0, 0)$, $\mu(\tilde{\xi}) = \mathfrak{F}(\lambda)$, $v(\tilde{\xi}, x_n) = \mathfrak{F}(u)$, $v_0(\tilde{\xi}) = v(\tilde{\xi}, 0)$, $y(t) = v_0(\tilde{\xi}, t)$ and we observe that $v_0(\tilde{\xi}) = \mathfrak{F}(u(\tilde{x}, 0))$.

Lemma 4.4.1. *We have that*

$$\mu(\tilde{\xi}) = (2\pi)^{1/2} \int_{-\infty}^{\infty} y(t) dt.$$

Proof.

$$\begin{aligned} \lambda(\tilde{x}) &= u(\tilde{x}, 0, 0) = (2\pi)^{1/2} (2\pi)^{(2-n)/2} \int_{-\infty}^{\infty} dt \int_{\mathcal{C}} \exp i(\tilde{x} \cdot \tilde{\xi}) v_0(\tilde{\xi}, t) d\tilde{\xi} \\ &= (2\pi)^{1/2} (2\pi)^{(2-n)/2} \int_{\mathcal{C}} d\tilde{\xi} \exp i(\tilde{x} \cdot \tilde{\xi}) \int_{-\infty}^{\infty} dt y(t). \end{aligned}$$

Hence:

$$\lambda(\tilde{x}) = (2\pi)^{1/2} \mathfrak{F}^{-1} \left(\int_{-\infty}^{\infty} y(t) dt \right),$$

thus

$$\mu(\tilde{\xi}) = (2\pi)^{1/2} \int_{-\infty}^{\infty} y(t) dt$$

as was to be shown.

Lemma 4.4.2. *There exist positive constants K and K' such that the estimates*

$$\begin{aligned} [\|\lambda\|]_{\alpha-1/2} &\leq K \|\lambda\|_\alpha, \quad \alpha > \tfrac{1}{2}, \\ \|\lambda\|_{k-1/2} &\leq K' \|\lambda\|_k, \quad k = 1, 2, \dots, \end{aligned}$$

hold for all $\lambda \in C_0^\infty(\mathcal{C})$.

Proof. We have from lemma 4.4.1 that

$$\begin{aligned} |\mu(\tilde{\xi})|^2 &= 2\pi \left| \int_{-\infty}^{\infty} \frac{1}{\sigma^\alpha} \sigma^\alpha y dt \right|^2 \\ &\leq 2\pi \int_{-\infty}^{\infty} \frac{dt}{\sigma^{2\alpha}} \int_{-\infty}^{\infty} \sigma^{2\alpha} |y|^2 dt \leq 2\pi \frac{1}{\zeta^{2\alpha-1}} \int_{-\infty}^{\infty} \sigma^{2\alpha} |y|^2 dt, \end{aligned}$$

hence it follows that

$$\int_e d\xi \xi^{2\alpha-1} |\mu|^2 \leq 2\pi \int_e d\xi \int_{-\infty}^{\infty} \sigma^{2\alpha} |y|^2 dt,$$

and hence the first inequality follows with $K = (2\pi)^{1/2}$. Furthermore, since from lemma 2.5 there exists a constant K' such that

$$[v]_{k-1/2} \leq K' \|v\|_k$$

for all $v \in C_0^\infty(E)$ it follows that

$$\|\lambda\|_{k-1/2} \leq K' \|\lambda\|_k.$$

To obtain an *a priori* estimate in the case $b < 0$ for problem (III) section 4.3 we proceed in the following manner: It will be recalled that after transforming the boundary condition we obtained the differential equation

$$e^{b\sigma(t)}(e^{-b\sigma(t)}\sigma(t)y(t))_t = -\theta_t,$$

where

$$\psi(\xi) = \mathfrak{F}(\varphi) \quad \text{and} \quad \psi(\xi, t) = \theta(t).$$

Writing

$$(e^{-b\sigma(t)}\sigma(t)y(t))_t = -e^{-b\sigma(t)}\theta_t$$

we integrate from some finite point " a " to t obtaining:

$$e^{-b\sigma(t)}\sigma(t)y(t)\Big|_a^t = -\int_a^t e^{-b\sigma(\tau)}\theta_\tau d\tau,$$

and integrating the right hand side by parts we obtain

$$y(t) = e^{-b\sigma(a)} \frac{e^{b\sigma(t)}}{\sigma(t)} \{\sigma(a)y(a) + \theta(a)\} - \frac{\theta(t)}{\sigma(t)} - \frac{e^{b\sigma(t)}}{\sigma(t)} \int_a^t b \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \theta(\tau) d\tau.$$

Multiplying both sides of the above equality by $e^{b\sigma(a)}\sigma^{-1}(a)$ and integrating over $-\infty < a < +\infty$ we obtain

$$\begin{aligned} y(t) \int_{-\infty}^{\infty} \frac{e^{b\sigma(a)}}{\sigma(a)} da &= \frac{e^{-b\sigma(t)}}{\sigma(t)} \int_{-\infty}^{\infty} y(a) da + \frac{e^{b\sigma(t)}}{\sigma(t)} \int_{-\infty}^{\infty} \frac{\theta(a)}{\sigma(a)} da - \frac{\theta(t)}{\sigma(t)} \int_{-\infty}^{\infty} \frac{e^{b\sigma(a)}}{\sigma(a)} da \\ &\quad - \frac{be^{b\sigma(t)}}{\sigma(t)} \int_{-\infty}^{\infty} \frac{e^{b\sigma(a)}}{\sigma(a)} \left\{ \int_a^t \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \theta(\tau) d\tau \right\} da. \end{aligned}$$

The integrals above exist since by assumption:

$$\|\varphi\|_0^2 \sim \int |\psi|^2 d\xi < \infty,$$

$$\|\varphi\|_1^2 \sim \int \sigma |\psi|^2 d\xi < \infty,$$

$$\|u\|_2^2 \sim \int \sigma^3 |v_0|^2 d\xi < \infty.$$

Moreover, since

$$\left| \frac{e^{b\sigma(a)}}{\sigma(a)} \int_a^t \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \theta(\tau) d\tau \right|^2 \leq \frac{e^{2b\sigma(a)}}{\sigma^2(a)} \int_a^t \frac{\tau^2}{\sigma^3(\tau)} e^{-2b\sigma(\tau)} d\tau \cdot \int_a^t \sigma(\tau) |\theta|^2 d\tau$$

and since

$$\frac{\tau}{\sigma^2(\tau)} \leq \frac{\tau}{\sigma(\tau)} \leq 1$$

it follows that

$$\int_a^t \frac{\tau^2}{\sigma^3(\tau)} e^{-2b\sigma(\tau)} d\tau \leq \int_a^t \frac{\tau}{\sigma(\tau)} e^{-2b\sigma(\tau)} d\tau = \frac{e^{-2b\sigma(\tau)}}{-2b} \Big|_a^t \leq \frac{e^{-2b\zeta}}{-2b}.$$

Hence Fubini's theorem holds for the last integral in the above equation, moreover the convergence is uniform for all ζ . Applying Fubini's theorem to the last integral we obtain after setting

$$\int_{-\infty}^{\infty} \frac{e^{b\sigma(a)}}{\sigma(a)} da = C_1(\xi), \quad \langle f, g \rangle = \int_{-\infty}^{\infty} f(a)g(a) da$$

that

$$\begin{aligned} C_1(\xi)y(t) &= \frac{e^{b\sigma(t)}}{\sigma(t)} \frac{\mu(\xi)}{(2\pi)^{1/2}} + \frac{e^{b\sigma(t)}}{\sigma(t)} < \frac{1}{\sigma}, \\ \theta &> -\frac{\theta(t)}{\sigma(t)} C_1(\xi) - \frac{be^{b\sigma(t)}}{\sigma(t)} \int_{-\infty}^t \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \theta(\tau) \left\{ \int_{-\infty}^{\tau} \frac{e^{b\sigma(a)}}{\sigma(a)} da \right\} d\tau \\ &\quad - \frac{be^{b\sigma(t)}}{\sigma(t)} \int_{\infty}^t \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \theta(\tau) \left\{ \int_{\tau}^{\infty} \frac{e^{b\sigma(a)}}{\sigma(a)} da \right\} d\tau. \end{aligned}$$

Put $\omega(t) = [C_1(\xi)]^{-1}$ times the sum of last two integrals above then we have

$$y(t) = \frac{1}{C_1} \frac{e^{b\sigma(t)}}{\sigma(t)} \frac{\mu(\xi)}{(2\pi)^{1/2}} + \frac{1}{C_1} \frac{e^{b\sigma(t)}}{\sigma(t)} < \frac{1}{\sigma}, \quad \theta > -\frac{\theta(t)}{\sigma(t)} - \omega.$$

Now we have the trivial estimate

$$\begin{aligned} \int_{-\infty}^{\infty} \sigma^3 |y|^2 dt &\leq \frac{4}{C_1^2} \frac{|\mu|^2}{2\pi} \int_{-\infty}^{\infty} \sigma(t) e^{2b\sigma(t)} dt + 4 \int_{-\infty}^{\infty} \sigma^3 |\omega|^2 dt \\ &\quad + \frac{4 \left| \left\langle \frac{1}{\sigma}, \theta \right\rangle \right|^2}{C_1^2} \int_{-\infty}^{\infty} \sigma(t) e^{2b\sigma(t)} dt + 4 \int_{-\infty}^{\infty} \sigma |\theta|^2 dt. \end{aligned}$$

Noting that

$$\left| \left\langle \frac{1}{\sigma}, \theta \right\rangle \right|^2 \leq \int_{-\infty}^{\infty} \frac{dt}{\sigma^3(t)} \cdot \int_{-\infty}^{\infty} \sigma(t) |\theta|^2 dt,$$

and putting

$$C_2(\xi) = \int_{-\infty}^{\infty} \sigma(t) e^{2b\sigma(t)} dt, \quad C_3(\xi) = \int_{-\infty}^{\infty} \frac{dt}{\sigma^3(t)}$$

we have

$$\begin{aligned} \int_{-\infty}^{\infty} \sigma^3 |y|^2 dt &\leq \frac{2}{\pi} \frac{C_2}{C_1^2} |\mu|^2 + 4 \int_{-\infty}^{\infty} \sigma^3 |\omega|^2 dt \\ &\quad + \frac{4C_3C_2}{C_1^2} \int_{-\infty}^{\infty} \sigma |\theta|^2 dt + 4 \int_{-\infty}^{\infty} \sigma |\theta|^2 dt, \end{aligned}$$

or,

$$\int_{-\infty}^{\infty} \sigma^3 |y|^2 dt \leq \frac{2}{\pi} \frac{C_2}{C_1^2} |\mu|^2 + 4 \int_{-\infty}^{\infty} \sigma^3 |\omega|^2 dt + 4 \left(\frac{C_3C_2}{C_1^2} + 1 \right) \int_{-\infty}^{\infty} \sigma |\theta|^2 dt.$$

We would like an estimate of the type

$$\int_{-\infty}^{\infty} \sigma^3 |\omega|^2 dt \leq C \int_{-\infty}^{\infty} \sigma |\theta|^2 dt.$$

Formulating θ in terms of ω have that

$$\begin{aligned} \langle \sigma(t) e^{-b\sigma(t)} \omega \rangle_t &= \frac{bt}{C_1 \sigma(t)} e^{-b\sigma(t)} \theta(t) \cdot \int_t^{\infty} \frac{e^{b\sigma(a)}}{\sigma(a)} da \\ &\quad + \frac{bte^{-b\sigma(t)}}{C_1 \sigma(t)} \theta(t) \cdot \int_{-\infty}^t \frac{e^{b\sigma(a)}}{\sigma(a)} da = \frac{bte^{-b\sigma(t)}}{\sigma(t)} \cdot \theta(t). \end{aligned}$$

Hence

$$\theta(t) = \frac{\sigma(t)}{bt} e^{b\sigma(t)} \langle \sigma(t) e^{-b\sigma(t)} \omega \rangle_t,$$

and the inequality becomes

$$\int_{-\infty}^{\infty} \sigma^3 |\omega|^2 dt \leq C \int_{-\infty}^{\infty} \frac{\sigma^2}{t^2} e^{2b\sigma} |(\sigma e^{-b\sigma} \omega)_t|^2 dt,$$

or letting $z(t) = \sigma e^{-b\sigma} \omega$ we have

$$\int_{-\infty}^{\infty} \sigma e^{2b\sigma} |z|^2 dt \leq C \int_{-\infty}^{\infty} \frac{\sigma^3}{t^2} e^{2b\sigma} |z_t|^2 dt.$$

It is easy to see that the function $z = \sigma e^{-b\sigma} \omega$ satisfies the hypothesis of lemma 4.2.3. Hence we conclude that

$$\int_0^{\infty} |z|^2 e^{2b\sigma} \sigma dt \leq C \int_0^{\infty} |z'|^2 e^{2b\sigma} \frac{\sigma^3}{t^2} dt + C' \xi^{3/2} e^{2b\xi} |z(0)|^2.$$

But

$$\begin{aligned}
|z(0)|^2 &= \frac{b^2}{C_1^2} \left| \int_{-\infty}^0 \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \theta(\tau) \left\{ \int_{-\infty}^{\tau} \frac{e^{b\sigma(a)}}{\sigma(a)} da \right\} d\tau \right. \\
&\quad \left. + \int_{\infty}^0 \frac{\tau}{\sigma(\tau)} e^{-b\sigma(\tau)} \theta(\tau) \left\{ \int_{\tau}^{\infty} \frac{e^{b\sigma(a)}}{\sigma(a)} da \right\} d\tau \right|^2 \\
&\leq \frac{2b^2}{C_1^2} \int_{-\infty}^0 \frac{\tau^2}{\sigma^3(\tau)} e^{-2b\sigma(\tau)} \left| \int_{-\infty}^{\tau} \frac{e^{b\sigma(a)}}{\sigma(a)} da \right|^2 d\tau \int_{-\infty}^0 \sigma |\theta|^2 d\tau \\
&\quad + \frac{2b^2}{C_1^2} \int_0^{\infty} \frac{\tau^2}{\sigma^3(\tau)} e^{-2b\sigma(\tau)} \left| \int_{\tau}^{\infty} \frac{e^{b\sigma(a)}}{\sigma(a)} da \right|^2 d\tau \cdot \int_0^{\infty} \sigma |\theta|^2 d\tau,
\end{aligned}$$

and since

$$\left| \int_{\tau}^{\infty} \frac{e^{b\sigma(a)}}{\sigma(a)} da \right|^2 \leq \int_{\tau}^{\infty} e^{2b\sigma(a)} \frac{a}{\sigma(a)} da \cdot \int_{\tau}^{\infty} \frac{da}{a\sigma(a)} \leq C e^{2b\sigma(\tau)} \int_{\tau}^{\infty} \frac{da}{a^2} = C e^{2b\sigma(\tau)} \cdot \frac{1}{\tau^3},$$

hence

$$\int_{-\infty}^0 \frac{\tau^2}{\sigma^3(\tau)} e^{-2b\sigma(\tau)} \left| \int_{-\infty}^{\tau} \frac{e^{b\sigma(a)}}{\sigma(a)} da \right|^2 d\tau \leq C \frac{1}{\zeta^3} \int_{-\infty}^0 \frac{d\tau}{\tau(1+\tau^2)} = C' \frac{1}{\zeta^3}.$$

Similar reasoning provides an estimate for the first term above.

Recall that

$$C_1^2(\xi) \geq C e^{2b\xi} \cdot \xi^{-1},$$

hence

$$|z(0)|^2 \leq C e^{-2b\xi} \cdot \xi^{-2} \int_{-\infty}^{\infty} \sigma |\theta|^2 dt.$$

Thus,

$$\int_0^{\infty} \sigma^3 |\omega|^2 dt \leq C \int_0^{\infty} \sigma |\theta|^2 dt + C' \zeta^{-1/2} \int_{-\infty}^{\infty} \sigma |\theta|^2 dt.$$

The same inequality holds for the interval $-\infty \leq t \leq 0$, and since $\zeta \geq 1$ we have

$$\int_{-\infty}^{\infty} \sigma^3 |\omega|^2 dt \leq C'' \int_{-\infty}^{\infty} \sigma |\theta|^2 dt$$

as was to be shown. Hence we have the inequality

$$\int_{-\infty}^{\infty} \sigma^3 |y|^2 dt \leq \frac{2}{\pi} \frac{C_2}{C_1^2} |\mu|^2 + 4 \left(\frac{C_3 C_2}{C_1^2} + C_4 \right) \int_{-\infty}^{\infty} \sigma |\theta|^2 dt,$$

where C_4 is the constant defined immediately above. Now we investigate $C_1(\xi)$ and $C_2(\xi)$. Let $y = \sigma(\tau) - \zeta$, hence

$$C_1 = 2e^{b\zeta} \int_0^\infty \frac{e^{by}}{(y)^{1/2}(y+2\zeta)^{1/2}} dy.$$

However, we have

$$\frac{1}{\zeta^{1/2}(y+2)^{1/2}} \leq \frac{1}{(y+2\zeta)^{1/2}} \leq \frac{1}{(2\zeta)^{1/2}}$$

hence there exist constants k_1 and k_2 such that

$$\frac{k_1}{\zeta^{1/2}} \leq \frac{1}{(y+2\zeta)^{1/2}} \leq \frac{k_2}{\zeta^{1/2}}$$

for all y such that $0 \leq y < \infty$. Hence

$$\frac{2k_1 e^{b\zeta}}{\zeta^{1/2}} \int_0^\infty \frac{e^{by}}{(y(y+2))^{1/2}} dy \leq C_1(\xi) \leq \frac{2k_2 e^{b\zeta}}{\zeta^{1/2}} \int_0^\infty \frac{e^{by}}{(y(y+2))^{1/2}} dy,$$

or

$$\frac{K_1 e^{b\zeta}}{\zeta^{1/2}} \leq C_1(\xi) \leq \frac{K_2 e^{b\zeta}}{\zeta^{1/2}}.$$

Similarly, if we put $y = \sigma(\tau) - \zeta$ then it follows that

$$C_2(\xi) = 2e^{2b\zeta} \int_0^\infty e^{2by} \cdot \frac{(y+\zeta)^2}{(y(y+2\zeta))^{1/2}} dy.$$

However,

$$\frac{(y+\zeta)^2}{(y+2\zeta)^{1/2}} \leq (y+2\zeta)^{3/2} \leq \zeta^{3/2} \cdot (y+2)^{3/2}$$

and letting $y = k\zeta$ we have

$$\frac{(y+\zeta)^2}{(y+2\zeta)^{1/2}} = \zeta^{3/2} \frac{(k+1)^2}{(k+2)^{1/2}}, \quad 0 \leq k < \infty,$$

and since $(k+1)^2/(k+2)^{1/2} > 0$ and behaves like $k^{3/2}$ for large k , clearly this function has a positive minimum say K_1 . Hence we have

$$K_1 \zeta^{3/2} \leq \frac{(y+\zeta)^2}{(y+2\zeta)^{1/2}} \leq \zeta^{3/2} (y+2)^{3/2}$$

so that

$$2K_1 e^{2b\zeta} \zeta^{3/2} \int_0^\infty \frac{e^{2by}}{(y)^{1/2}} dy \leq C_2(\xi) \leq 2e^{2b\zeta} \zeta^{3/2} \int_0^\infty \frac{e^{2by}}{(y)^{1/2}} (y+2)^{3/2} dy,$$

hence

$$K_1 e^{2b\zeta} \zeta^{3/2} \leq C_2(\xi) \leq K_2 e^{2b\zeta} \zeta^{3/2}.$$

Estimating $C_3(\xi)$ we observe that the substitution $t = \zeta k$ yields

$$C_3(\xi) = \frac{1}{\zeta^2} \int_{-\infty}^{\infty} \frac{dk}{(1 + k^2)^{3/2}} = K_3 \frac{1}{\zeta^2}.$$

Hence we arrive at the following inequality:

$$\int_{-\infty}^{\infty} \sigma^3(t) |y|^2 dt \leq K_1 \zeta^{5/2} |\mu|^2 + K_2 \zeta^{1/2} \int_{-\infty}^{\infty} \sigma(t) |\theta|^2 dt.$$

From section 4.3 and lemma 4.4.2 it follows that if the factor $\zeta^{1/2}$ is suppressed in the right hand side of the above inequality then we would have the estimate

$$\|u\|_2 \leq K[\|B_1 u\|_1 + \|r_{00}(u)\|_2]$$

for all $u \in C_0^\infty(H)$ with $B_1 u \in C_0^\infty(E)$ where $r_{00}(u)$ is the restriction to $\mathcal{C}$ of u . Moreover, estimates similar to those appearing in section 4.3 could be derived by methods similar to those used in that section. Unfortunately, however, the above estimate appears to be sharp so that the bothersome factor of $\zeta^{1/2}$ cannot be dropped. Hence it is clear that if an estimate of type we seek does indeed exist the Dirichlet condition on the emergent cycle will not provide it when our norms are employed.

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