*The Initial-Boundary Value Problem for a Non-Linear Hyperbolic Equation in Relativistic Quantum Mechanics**

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Abstract. Considerable attention has been focused on the equation

$$u_{,tt} - \Delta u + u |u|^\gamma = f \quad (\gamma > 0)$$

in the last four years. This paper is concerned with the corresponding initial-boundary value problem. Existence theorems for weak and strict solutions and continuity of weak solutions with respect to the data of the problem are established.

1. Introduction. Non-linear hyperbolic equations of the special form

$$(1.1) \quad u_{,tt} - \Delta u + u |u|^\gamma = f \quad (\gamma > 0),$$

have been of considerable interest to mathematicians and physicists in recent years. The case $\gamma = 2$ in 4-dimensional space-time is important in relativistic quantum mechanics. The function u represents some field quantity and the inhomogeneous term f represents sources of energy in the field. Possible interactions of the field with itself are described by the non-linear term. The initial-boundary value problem for equation (1.1) has the following (classical) formulation. It is required to find a function $u = u(x, t)$ which satisfies (1.1) for $x \in \Omega$, $t > 0$, and the initial conditions

$$(1.2) \quad u(x, 0) = u_0(x), \quad x \in \Omega,$$

$$(1.3) \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

and the boundary condition

$$(1.4) \quad u(x, t) = 0, \quad t \geq 0, \quad x \in \partial\Omega.$$

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In physics, condition (1.4) describes the physical nature of some obstacle (represented by $\partial\Omega$); the initial state of the field is described by (1.2)–(1.3). Here Ω is any domain in n -dimensional real Euclidean space R_n ; $\partial\Omega$ denotes the boundary of Ω ; Δ is the Laplacian. In case $\Omega = R_n$ we ignore (1.4) and have the pure initial value, or Cauchy, problem.

For $n = 3$, K. Jörgens [7] has obtained a classical solution of the Cauchy problem for a certain class of equations of the form

$$(1.5) \quad u_{tt} - \Delta u + uF'(u^2) = 0,$$

which includes equation (1.1) when $f = 0$. In equation (1.5), F' is the derivative of a function F satisfying certain growth and definiteness conditions. Jörgens' results have been generalized to operator differential equations of the form

$$(1.6) \quad \frac{d^2u}{dt^2} + Au + M(u) = 0,$$

by F. E. Browder [2]. Here A is a positive densely defined self-adjoint linear operator on a Hilbert space H ; $M(u)$ is a non-linear function from the domain of $A^{1/2}$ to H satisfying certain growth, Lipschitzian, and definiteness conditions. Browder's results include the existence and uniqueness of strict solutions of equations (1.6); they specialize to give Jörgens' theorem as well as theorems for equations of the form (1.5) in n -dimensional space-time. I. E. Segal [15] subsumes Jörgens' theory of equation (1.5) under a suitable extension of the Hille–Phillips semi-group approach to the non-linear case. The general theory presented there is based on the underlying equation

$$\frac{du}{dt} = Au + K(u),$$

where A is a skew-adjoint operator in a Hilbert space and K is a given non-linear everywhere-defined locally Lipschitzian operator. In [16] Segal combines his approach in [15] with compactness arguments to establish the existence of a global weak solution of the Cauchy problem for equation (1.5) with $F(s) = m^2s + g^2s^p/p$ in n -dimensional space-time (m, g are constants and p is an integer ≥ 2). An important variant of the Cauchy problem is the scattering problem and W. Strauss [19] has utilized Jörgens' results in an abstract treatment of this problem.

In this paper I shall consider weak solutions of equation (1.1) which satisfy the initial conditions (1.2)–(1.3) and the boundary condition (1.4) in generalized senses. The principal references are [6], [8], and [12]. Modifying the fundamental construction procedure developed by E. Hopf [6] for the Navier–Stokes equations, J. L. Lions [12] constructed a weak solution of the initial-boundary value problem (1.1)–(1.4) when γ is any positive even integer. For the important case $\gamma = 2$ and $n = 3$ he also established uniqueness. Most of the concepts introduced in subsequent sections are due to various authors cited in the references. The techniques are in the main due to Lions. Roughly speaking, the main difference

in this presentation is analogous to the modification of Hopf's method due to A. A. Kiselev and O. A. Ladyzhenskaya [8].

The next section deals with the mathematical preliminaries necessary to give a precise formulation of the problem. Section 3 contains the basic definitions of weak and strict solutions of the initial-boundary value problem (1.1)–(1.4). Existence theorems for weak and strict solutions are presented, respectively, in Sections 4 and 7. Section 5 contains an energy equality. Continuity of weak solutions with respect to the data of the problem is established in Section 6.

2. Preliminaries. We begin by introducing in a well known manner certain function spaces which will be used constantly in the sequel. Let Ω be any domain (open connected set) in n -dimensional real Euclidean space R_n and let $x = (x_1, x_2, \dots, x_n)$ be a point in Ω . For a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, with non-negative integer components, we define the differentiation operator:

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}},$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, \quad D_i = \frac{\partial}{\partial x_i}.$$

We denote by $C^k(\Omega)$ the class of k times continuously differentiable real-valued functions on Ω and set $C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$. The subset of $C^\infty(\Omega)$ consisting of functions with compact support in Ω is denoted by $C_0^\infty(\Omega)$. If $\varphi \in C^k(\Omega)$, we set

$$\|\varphi\|_k = \left[\sum_{|\alpha| \leq k} \int |D^\alpha \varphi(x)|^2 dx \right]^{\frac{1}{2}};$$

the domain of integration is Ω and dx is Lebesgue measure in R_n . The subset of $C^k(\Omega)$ consisting of functions φ with $\|\varphi\|_k < \infty$ is denoted by $\tilde{C}^k(\Omega)$. It is clear that $\tilde{C}^k(\Omega)$ is an inner product space with respect to the inner product

$$(\varphi, \theta)_k = \sum_{|\alpha| \leq k} \int D^\alpha \varphi(x) D^\alpha \theta(x) dx.$$

Note that $\|\varphi\|_k^2 = (\varphi, \varphi)_k$.

Definition 2.1. $H_k = H_k(\Omega)$ is the closure of $\tilde{C}^k(\Omega)$ under the norm $\|\cdot\|_k$.

Definition 2.2. $H_k^\circ = H_k^\circ(\Omega)$ is the closure of the subspace $C_0^\infty(\Omega)$ in H_k .

The spaces H_k (and hence H_k°) are Hilbert spaces (Friedrichs [4], Sobolev [18]). $H_0^\circ = H_0$; it consists of all real-valued functions which are square summable on Ω in the sense of Lebesgue. H_k consists of all functions $u \in H_0$ with strong, or generalized, derivatives $D^\alpha u \in H_0$ for $|\alpha| \leq k$. For $k \geq 1$, H_k° is a proper subspace of H_k ; it consists of functions in H_k which, along with their derivatives of order $\leq k - 1$, vanish—in a sense which can be made precise—on the boundary $\partial\Omega$ of Ω .

As usual, we denote by $L^p = L^p(\Omega)$, $1 \leq p < \infty$, the Banach space of all real-valued, Lebesgue measurable functions $v = v(x)$ in Ω , with norm

$$|v|_p = \left[\int |v(x)|^p dx \right]^{1/p}.$$

Observe that $L^2 = H_0$ and $|u|_2 = \|u\|_0$.

For brevity in notation, we write just $|u|$ instead of $|u|_2$ or $\|u\|_0$ and simply (u, v) instead of $(u, v)_0$. Also, for convenience, we set:

$$((u, v)) = (u, v)_1 - (u, v) = \sum_{i=1}^n \int (D_i u)(D_i v) dx, \quad \|u\|^2 = ((u, u)).$$

In the sequel we frequently shall use the following special case of Sobolev's inequality.

Sobolev's inequality. *If $u \in H_1^\alpha \cap L^q$, $q \geq 2$, then*

$$(2.1) \quad |u|_p^{\alpha+\beta} \leq \text{Const. } |u|_q^\alpha \|u\|^\beta,$$

for $q \leq p \leq \infty$ when $\alpha = n(1/p - 1/2) + 1 \geq 0$ and $\beta = n(1/q - 1/p)$ with the exception that $p < \infty$ when $\alpha = 0$.

Remark 2.1. A simple consequence of Sobolev's inequality is that for $2 \leq p \leq 2n/(n-2)$ we have $H_1^\alpha \cap H_2 \subset H_1^\alpha \subset L^p$, also in the topological sense, with the single exception of $p = \infty$ in case $n = 2$. In fact, setting $q = 2$ in (2.1) gives $\alpha + \beta = 1$ and since $2 < p < 2n/(n-2)$ implies $0 < \beta < 1$ we have

$$(2.2) \quad |u|_p \leq \text{Const. } |u|^{1-\beta} \|u\|^\beta;$$

whence

$$|u|_p^2 \leq \text{Const. } |u|^{2(1-\beta)} \|u\|^{2\beta} \leq \text{Const. } [(1-\beta) |u|^2 + \beta \|u\|^2]$$

by Hölder's inequality. Hence $|u|_p \leq \text{Const. } \|u\|_1$, which shows that $H_1^\alpha \subset L^p$ and that the topology of H_1^α is stronger than the topology induced there by L^p . The cases $p = 2$ and $p = 2n/(n-2)$ correspond respectively to $\beta = 0$ and $\beta = 1$ in inequality (2.2) and the result $|u|_p \leq \text{Const. } \|u\|_1$ is immediate.

Next set $I = \{t : 0 \leq t < T < \infty\} = [0, T)$ and let H denote a separable Hilbert space with norm $|\cdot|_H$. The class of k times continuously differentiable functions $\varphi : I \rightarrow H$ (i.e. functions defined for $t \in I$ with values $\varphi(t) \in H$) is denoted by $C^k(I, H)$; the subset of $C^k(I, H)$ consisting of functions with compact support in I (respectively $I^\circ = (0, T)$) is denoted by $C_0^k(I, H)$ (respectively $C_0^k(I^\circ, H)$). The definitions of continuity and derivative for functions with values in H closely parallels the case of real-valued functions [5, Ch. III].

We denote by $L^1(I, H)$ the space of (equivalence classes of) functions $u : I \rightarrow H$ which are (strongly) measurable and Bochner integrable on I ; it is a Banach space under the norm

$$|u|_{L^1(I, H)} = \int_0^T |u(t)|_H dt.$$

(See [5, Ch. III] for definitions of these concepts.) The adjoint space $[L^1(I, H)]^*$ is linearly isometric to the Banach space $L^\infty(I, H)$ consisting of (equivalence classes of) functions $u : I \rightarrow H$ which are (strongly) measurable with norm

$$\|u\|_{L^\infty(I, H)} = \operatorname{ess. sup}_{t \in I} \|u(t)\|_H$$

(cf. [1]). Since $L^1(I, H)$ is separable, the sphere

$$\{u \in L^\infty(I, H) : \|u\|_{L^\infty(I, H)} \leq \text{Const.}\}$$

is sequentially compact in the weak* topology of $L^\infty(I, H)$.

Definition 2.3. A function $u \in L^1(I, H)$ is differentiable in $L^1(I, H)$ with derivative $v \in L^1(I, H)$ if

$$\int_0^T (u(t), \varphi'(t))_H dt = - \int_0^T (v(t), \varphi(t))_H dt,$$

for all $\varphi \in C_0^1(I^\circ, H)$.

Remark 2.2. Definition 2.3 implies that u is equivalent in $L^1(I, H)$ to a continuous function; indeed, C. H. Wilcox [20, Theorem 2.2] has shown that

$$u(t) - u(\tau) = \int_\tau^t v(s) ds \quad (\text{in } H) \quad \text{for almost all } t, \tau \in I.$$

Finally, we denote by $L^2(I, H)$ the subset of $L^1(I, H)$ consisting of functions $u : I \rightarrow H$ with $\int_0^T \|u(t)\|_H^2 dt < \infty$; it is a Hilbert space with respect to the inner product

$$(u, v)_{L^2(I, H)} = \int_0^T (u(t), v(t))_H dt.$$

3. A formulation of the initial-boundary value problem. This section contains a precise definition of what is meant by a weak solution of the initial-boundary value problem (1.1)–(1.4). This concept is defined relative to a certain class of “test functions”:

$$\mathcal{F} = \{\varphi : \varphi \in L^1(I, H_1^\circ), \quad \varphi' \in L^1(I, H_0), \quad \text{and} \\ \varphi(t) \equiv 0 \text{ in a (variable) neighborhood of } t = T\}.$$

The following conditions will be imposed on the inhomogeneous term and the exponent γ in the sequel:

$$(3.1) \quad \begin{cases} f \in L^2(I, H_0) \\ 0 < \gamma \leq \frac{2}{n-2} \quad (\gamma < \infty \text{ when } n = 2). \end{cases}$$

Actually the results obtained in this paper also hold when $\gamma = 0$; however, since equation (1.1) is linear in this case its theory is included in the work of

various authors. Complete results on initial-boundary value problems for linear hyperbolic partial differential equations of the second order have been obtained by C. H. Wilcox [20], [21]. Regularity theorems for the weak solutions in [20] are derived in [21]; in particular, simple sufficient conditions guaranteeing the existence of a global classical solution are given. Consult the bibliography in [20] for works of other principal contributors to the linear theory.

Definition 3.1. Suppose that hypotheses (3.1) hold along with the conditions $u_0 \in H_1^\circ$, $u_1 \in H_0$ on the initial data. A function u is a weak solution of the initial-boundary value problem (1.1)–(1.4) if

$$(3.2) \quad u \in L^\infty(I, H_1^\circ)$$

$$(3.3) \quad u' \in L^\infty(I, H_0)$$

$$(3.4) \quad \int_0^\tau \{-(u'(t), \varphi'(t)) + ((u(t), \varphi(t))) + (u(t)u^\gamma(t), \varphi(t))\} dt \\ = (u_1, \varphi(0)) + \int_0^\tau (f(t), \varphi(t)) dt, \quad \varphi \in \mathcal{F}$$

$$(3.5) \quad u(0) = u_0.$$

Remark 3.1. The functions u^γ and uu^γ are defined in the natural way. For almost all $t \in I$, we have $u(t) \in L^p$ for $2 \leq p \leq 2n/(n-2)$ (except $p = \infty$ when $n = 2$) by Remark 2.1. If $U(x, t)$ is any representative function in L^p of the equivalence class $u(t)$, we define $u^\gamma(t)$ and $u(t)u^\gamma(t)$ respectively to be the equivalence classes with representative functions

$$|U(x, t)|^\gamma \text{ in } L^{p/\gamma} \text{ and } U(x, t) |U(x, t)|^\gamma \text{ in } L^2 = H_0.$$

Remark 3.2. In view of Remark 2.2, functions $\varphi \in \mathcal{F}$ or u satisfying (3.2) and (3.3) are equivalent (in $L^1(I, H_0)$) to continuous functions defined on I with values in H_0 ; in particular, $u(0)$ and $\varphi(0)$ are well defined.

Remark 3.3. It is possible to redefine $u'(t)$ on a subset of I with measure zero in such a way that

$$(u'(\tau), v) + \int_0^\tau \{((u(t), v)) + (u(t)u^\gamma(t), v)\} dt = (u_1, v) + \int_0^\tau (f(t), v) dt$$

holds for $\tau \in I$ and $v \in H_1^\circ$. This result (along with (3.3)) implies that

$$u'(t) \rightarrow u_1 \text{ weakly in } H_0 \text{ as } t \rightarrow 0.$$

However, we shall not make use of this result and accordingly omit its proof. It is mentioned only to indicate the manner in which solutions in the sense of Definition 3.1 assume the initial data u_1 .

Remark 3.4. The condition that $u(t) \in H_1^\circ$ for almost all $t \in I$ is a generalization of the boundary condition (1.4); in fact, functions in $H_1^\circ(\Omega)$ vanish in a

certain sense—which can be made precise—on the boundary $\partial\Omega$ of Ω . A further remark on this point is made after Definition 3.2.

Remark 3.5. Definition 3.1 is equivalent to the classical formulation (1.1)–(1.4) in case the solution and data are (identifiable with) smooth functions of (x, t) in $\Omega \times I$ or of x in Ω respectively.

We next turn to the concept of a strict solution.

Definition 3.2. Suppose that hypotheses (3.1) hold along with the conditions $u_0 \in H_1^\circ$, $u_1 \in H_0$ on the initial data. A function u is a strict solution of the initial-boundary value problem (1.1)–(1.4) if

$$(3.6) \quad u(t) \in H_1^\circ \cap H_2$$

$$(3.7) \quad u'(t) \in H_1^\circ$$

$$(3.8) \quad u''(t) \in H_0$$

$$(3.9) \quad u''(t) - \Delta u(t) + u(t)u^\gamma(t) = f(t) \quad (\text{in } H_0) \quad \text{for all } t \in I.$$

Remark 3.6. In case $n = 2$ or 3 , a well known imbedding theorem of S. L. Sobolev [18] states that a function in $H_2(\Omega)$ is continuous (after correction on a null set) provided that the domain Ω satisfies certain regularity requirements. Moreover, L. Nirenberg [13, Lemma 8, p. 656] has shown that a function in H_1° which is continuous in $\bar{\Omega}$ vanishes identically on the boundary $\partial\Omega$ of Ω . (Here again a regularity condition is imposed on Ω .)

4. The existence theorem. In this section we demonstrate the existence of solutions of the initial-boundary value problem. For $\gamma = 2$ and $n = 3$, the existence of weak solutions in (essentially) the sense of Definition 3.1 has already been proved by J. L. Lions [12]. Indeed, his existence theorem is valid for arbitrary n any positive even integer γ (compare with (3.1)) provided that the initial data u_0 is in $H_1^\circ \cap L^{2+\gamma}$. Lions has also established uniqueness in case $n = 3$ and $\gamma = 2$.

The class of initial data considered in our existence theorem is more restrictive than previously mentioned data but nevertheless is still broad. The advantage is that a weak solution with considerably stronger properties than those required in Definition 3.1 is obtained. These solutions satisfy an energy equality (Sec. 5) and are continuous with respect to the data (Sec. 6); in particular, they are unique.

The additional conditions imposed on the data of the problem are:

$$(4.1) \quad \begin{cases} u_0 \in H_1^\circ \cap H_2 \\ u_1 \in H_1^\circ \\ f' \in L^2(I, H_0). \end{cases}$$

Theorem 4.1. Under hypotheses (3.1) and (4.1) there exists a function u satisfying the conditions:

$$(4.2) \quad u, u' \in L^\infty(I, H_1^\circ),$$

$$(4.3) \quad u'' \in L^\infty(I, H_0),$$

$$(4.4) \quad \int_0^T \{ (u''(t), \varphi(t)) + ((u(t), \varphi(t))) + (u(t)u'(t), \varphi(t)) \} dt \\ = \int_0^T (f(t), \varphi(t)) dt, \quad \varphi \in L^1(I, H_1^\circ).$$

Moreover, the following representations are valid:

$$(4.5) \quad u(t) = u_0 + (H_1^\circ) \int_0^t u'(\tau) d\tau,$$

$$(4.6) \quad u(t) = u_0 + tu_1 + (H_0) \int_0^t \left[(H_0) \int_0^\tau u''(s) ds \right] d\tau.$$

The integrals in (4.5) and (4.6) are Bochner integrals; the notation indicates whether the range space is H_0 or H_1° .

Remark 4.1. The function u is a weak solution in the sense of Definition 3.1. Relation (4.4) clearly holds for $\varphi \in \mathcal{F}(\mathcal{F} \subset L^1(I, H_1^\circ))$; moreover, the representation (4.6) implies that $(u'(t), \varphi(t))$ is an absolutely continuous function of t whence

$$-(u_1, \varphi(0)) = \int_0^T d(u'(t), \varphi(t)) = \int_0^T (u''(t), \varphi(t)) dt + \int_0^T (u'(t), \varphi'(t)) dt.$$

Substitution of this result in (4.4) gives relation (3.4). The remaining requirements in Definition 3.1 are clearly met.

Remark 4.2. From relations (4.2)–(4.3) and (4.5)–(4.6) it follows that

$$\|u(t) - u(\tau)\|_1 \leq \left| \int_\tau^t \|u'(s)\|_1 ds \right| \leq \text{Const. } |t - \tau|, \\ |u'(t) - u'(\tau)| \leq \left| \int_\tau^t |u''(s)| ds \right| \leq \text{Const. } |t - \tau|,$$

for all $\tau, t \in I$; in particular for $\tau = 0$, $u(0) = u_0$, and $u'(0) = u_1$.

The solution is constructed by the Galerkin approximation procedure. We first introduce a fixed sequence $\{\psi_k\} \subset H_1^\circ \cap H_2$ which spans $H_1^\circ \cap H_2$ (i.e. finite linear combinations of the ψ_k are dense in $H_1^\circ \cap H_2$). Next, we assume as an m^{th} approximation the expansion

$$(4.7) \quad u_m = u_m(t) = \sum_{k=1}^m \lambda_k^m(t) \psi_k,$$

where the functions $\lambda_k^m = \lambda_k^m(t)$ ($k = 1, 2, \dots, m$) are twice differentiable

real-valued functions for $t \in I$ and are determined in such a way that u_m satisfies the system

$$(4.8) \quad (u_m''(t), \psi_k) + ((u_m(t), \psi_k)) + (u_m(t)u_m'(t), \psi_k) \\ = (f(t), \psi_k), \quad k = 1, 2, \dots, m.$$

Indeed, for each fixed positive integer m , the existence of functions $\lambda_k^m = \lambda_k^m(t)$ ($k = 1, 2, \dots, m$) with $\lambda_k^m(0)$ and $\dot{\lambda}_k^m(0)$ ($\dot{\cdot} = d/dt$) equal to arbitrary pre-assigned constants such that u_m defined by relation (4.7) satisfies the system (4.8) follows from the standard local existence theorems of ordinary differential equations. Strictly speaking, these local theorems only guarantee the existence of the λ_k^m on some interval $[0, T_m]$; however, as we shall show later, each interval $[0, T_m]$ can be taken as $[0, T] = I$.

In certain respects it is convenient to regard the approximate solutions as real-valued functions on the cylinder $\Omega \times I$:

$$u_m = u_m(x, t) = \sum_{k=1}^m \lambda_k^m(t) \psi_k(x),$$

by choosing a representative $\psi_k(x)$ of $\psi_k \in H_1^\circ \cap H_2$. It should be observed that the function $t \rightarrow u_{mt}(\cdot, t)$ from I to $H_1^\circ \cap H_2$ coincides with u_m' ; that is, $u_m'(t) \equiv u_{mt}(\cdot, t)$ for $t \in I$. The particular point of view adopted in a given argument will be clear from the context.

The initial data $\lambda_k^m(0)$, $\dot{\lambda}_k^m(0)$ is chosen in such a way that as $m \rightarrow \infty$ we have

$$(4.9) \quad u_{m0} = u_m(0) = \sum_{k=1}^m \lambda_k^m(0) \psi_k \rightarrow u_0 \quad \text{in } H_1^\circ \cap H_2$$

and

$$(4.10) \quad u_{m1} = u_m'(0) = \sum_{k=1}^m \dot{\lambda}_k^m(0) \psi_k \rightarrow u_1 \quad \text{in } H_1^\circ.$$

Such a choice is possible in view of (4.1) and the fact that $\{\psi_k\}$ spans $H_1^\circ \cap H_2$ which is dense in H_1° .

We turn next to the derivation of two basic energy estimates. These estimates enable us to establish the existence of the approximate solutions u_m on the interval I (i.e. the existence of the functions $\lambda_k^m = \lambda_k^m(t)$ ($k = 1, 2, \dots, m$) on I) and the convergence, in a certain sense, of a suitable subsequence to a weak solution of the initial-boundary value problem.

Lemma 4.1. *The following energy estimates hold for $t \in I$:*

$$(4.11) \quad |u_m'|^2 + \|u_m\|^2 + \frac{2}{2+\gamma} |u_m|_{2+\gamma}^{2+\gamma} \leq A e^t,$$

$$(4.12) \quad |u_m''|^2 + \|u_m'\|^2 \leq B e^{Ct}.$$

Here A, B, C are constants which depend only on the initial data u_0, u_1 , the inhomogeneous term f , and T ; in particular, they are independent of m .

Proof. If we multiply each of the equations (4.8) by the corresponding λ_k^m and then add all m equations, there results

$$(u_m'', u_m') + ((u_m, u_m')) + (u_m u_m^\gamma, u_m') = (f, u_m').$$

We next rewrite the last relation in the form

$$\frac{d}{dt} |u_m'|^2 + \frac{d}{dt} ||u_m||^2 + \frac{2}{2+\gamma} \frac{d}{dt} |u_m|_{2+\gamma}^{2+\gamma} = 2(f, u_m')$$

and then integrate over $[0, \tau]$, $0 < \tau < T_m$, to obtain

$$(4.13) \quad |u_m'|^2 + ||u_m||^2 + \frac{2}{2+\gamma} |u_m|_{2+\gamma}^{2+\gamma} \\ = |u_{m1}|^2 + ||u_{m0}||^2 + \frac{2}{2+\gamma} |u_{m0}|_{2+\gamma}^{2+\gamma} + 2 \int_0^\tau (f, u_m') dt.$$

Relation (4.13) holds as long as relations (4.7) and (4.8) hold; that is, as long as the solution λ_k^m ($k = 1, 2, \dots, m$) of the system (4.8) of ordinary differential equations lasts which is at least on the time interval $(0, T_m)$. We shall now show that we can take $T_m = T$. If we set

$$C_m = |u_{m1}|^2 + ||u_{m0}||^2 + \frac{2}{2+\gamma} |u_{m0}|_{2+\gamma}^{2+\gamma},$$

then it follows from relation (4.13) that for all τ satisfying $0 \leq \tau < T_m$, we have

$$|u_m'|^2 + ||u_m||^2 + \frac{2}{2+\gamma} |u_m|_{2+\gamma}^{2+\gamma} \leq C_m + \int_0^\tau |f|^2 dt + \int_0^\tau |u_m'|^2 dt.$$

It is now a simple consequence of Gronwall's inequality that

$$(4.14) \quad |u_m'|^2 + ||u_m||^2 + \frac{2}{2+\gamma} |u_m|_{2+\gamma}^{2+\gamma} \leq \left[C_m + \int_0^\tau |f|^2 dt \right] e^\tau, \quad 0 \leq \tau < T_m.$$

It now follows from standard results in the theory of ordinary differential equations that the solution λ_k^m ($k = 1, 2, \dots, m$) of the system (4.8) exists in I ; that is, we can take $T_m = T$. From relations (4.9) and (4.10) it is clear that $|u_{m1}| \rightarrow |u_1|$ and $||u_{m0}|| \rightarrow ||u_0||$ as $m \rightarrow \infty$. Also, by Remark 2.1, the topology of H_1^* is stronger than the topology of $L^{2+\gamma}$, whence $|u_{m0}|_{2+\gamma} \rightarrow |u_0|_{2+\gamma}$. Thus $\sup C_m < \infty$ and relation (4.11) with $A = \sup C_m + \int_0^T |f|^2 dt$ follows immediately from relation (4.14).

We next differentiate equations (4.8) to obtain

$$(4.15) \quad (u_m''', \psi_k) + ((u_m', \psi_k')) + (\gamma + 1)(u_m^\gamma u_m', \psi_k) \\ = (f', \psi_k) \quad (k = 1, 2, \dots, m).$$

Multiplying each of the equations (4.15) by the corresponding λ_k^m and then adding all m equations, there results

$$(u_m''', u_m') + ((u_m', u_m')) + (\gamma + 1)(u_m^\gamma u_m', u_m') = (f', u_m'),$$

which can be written in the form

$$(4.16) \quad \frac{d}{dt} |u_m''|^2 + \frac{d}{dt} ||u_m'|^2 = 2(f', u_m'') - 2(\gamma + 1)(u_m^\gamma u_m', u_m').$$

The right side of relation (4.16) is dominated by

$$2 |f'| |u_m''| + 2(\gamma + 1) |u_m^\gamma u_m'| |u_m'| \leq |u_m''|^2 + 2 |f'|^2 + 2(\gamma + 1)^2 |u_m^\gamma u_m'|^2,$$

whence

$$(4.17) \quad \frac{d}{dt} \{ |u_m''|^2 + ||u_m'|^2 \} \leq |u_m''|^2 + 2(\gamma + 1)^2 |u_m^\gamma u_m'|^2 + 2 |f'|^2.$$

Turning next to estimating the term $|u_m^\gamma u_m'|^2$, we consider three cases.

Case 1: $0 < \gamma < \frac{2}{n-1} \quad \text{and} \quad n \geq 2.$

Setting $r = (2 + \gamma)/2\gamma$ and defining s by $1/r + 1/s = 1$ we have, by Hölder's inequality, the estimate

$$\begin{aligned} |u_m^\gamma u_{mt}|^2 &= \int u_m^{2\gamma} u_{mt}^2 dx \leq \left[\int u_m^{2\gamma r} dx \right]^{1/r} \left[\int u_{mt}^{2s} dx \right]^{1/s} \\ &= |u_m|_{2+\gamma}^{2\gamma} |u_{mt}|_{2s}^2. \end{aligned}$$

Also, by Sobolev's inequality, we have

$$|u_{mt}|_{2s} \leq \text{Const.} |u_{mt}|^{1-\beta} ||u_{mt}||^\beta \quad \left(\beta = \frac{n\gamma}{2+\gamma} < 1 \right),$$

so that

$$|u_m^\gamma u_{mt}|^2 \leq \text{Const.} |u_m|_{2+\gamma}^{2\gamma} |u_{mt}|^{2(1-\beta)} ||u_{mt}||^{2\beta}.$$

Since $1/\beta$ and $1/(1 - \beta)$ are conjugate exponents, it follows from Hölder's inequality that

$$(4.18) \quad |u_m^\gamma u_{mt}|^2 \leq ||u_{mt}||^2 + \text{Const.} |u_m|_{2+\gamma}^{2\gamma/(1-\beta)} |u_{mt}|^2.$$

Case 2: $2 < \gamma < \infty \quad \text{and} \quad n = 2.$

By Schwarz's inequality,

$$|u_m^\gamma u_{mt}|^2 \leq |u_m|_{4\gamma}^{2\gamma} |u_{mt}|_4^2$$

and by Sobolev's inequality, we have

$$|u_{mt}|_4 \leq \text{Const.} |u_{mt}|^{1/2} ||u_{mt}||^{1/2},$$

and

$$|u_m|_{4\gamma} \leq \text{Const.} |u_m|_{2+\gamma}^{(2+\gamma)/4\gamma} ||u_m||^{(3\gamma-2)/4\gamma}.$$

Thus we have

$$(4.19) \quad \begin{aligned} |u_m^\gamma u_{m,t}|^2 &\leq \text{Const.} |u_m|_{2+\gamma}^{(2+\gamma)/2} ||u_m||^{(3\gamma-2)/2} |u_{m,t}| ||u_{m,t}|| \\ &\leq ||u_{m,t}||^2 + \text{Const.} |u_m|_{2+\gamma}^{2+\gamma} ||u_m||^{3\gamma-2} |u_{m,t}|^2. \end{aligned}$$

$$\text{Case 3:} \quad \frac{2}{n-1} \leq \gamma \leq \frac{2}{n-2} \quad \text{and} \quad n \geq 3.$$

First determine exponents r and s by the relations

$$2r = n \quad \text{and} \quad \frac{1}{r} + \frac{1}{s} = 1.$$

Then, by Hölder's inequality, we have

$$|u_m^\gamma u_{m,t}|^2 \leq |u_m|_{n\gamma}^{2\gamma} |u_{m,t}|_{2s}^2.$$

Also, by Sobolev's inequality, we have

$$|u_{m,t}|_{2s} \leq \text{Const.} ||u_{m,t}||,$$

and

$$|u_m|_{n\gamma} \leq \text{Const.} |u_m|_{2+\gamma}^{\alpha/(\alpha+\beta)} ||u_m||^{\beta/(\alpha+\beta)},$$

where

$$\alpha = 1 + \gamma - \frac{n}{2} \geq 0 \quad \text{and} \quad \beta = n \left(\frac{1}{2+\gamma} - \frac{1}{n\gamma} \right) \geq 0.$$

It follows that

$$(4.20) \quad |u_m^\gamma u_{m,t}|^2 \leq \text{Const.} |u_m|_{2+\gamma}^{2\gamma\alpha/(\alpha+\beta)} ||u_m||^{2\gamma\beta/(\alpha+\beta)} ||u_{m,t}||^2.$$

In view of the energy estimate (4.11) which has already been established, the norms $|u_m|_{2+\gamma}$, $||u_m||$, and $|u_{m,t}|$ are uniformly bounded with respect to m and $t \in I$. Therefore, the right side of each of the estimates (4.18)–(4.20) is dominated by a function of the form

$$\text{Const.} ||u_{m,t}||^2 + \text{Const.}$$

It now follows from relation (4.17) that

$$\frac{d}{dt} \{ |u_m''|^2 + ||u_m'|^2 \} \leq C \{ |u_m''|^2 + ||u_m'|^2 \} + \tilde{C} + 2 |f'|^2,$$

where C and $\tilde{C}$ are constants independent of m and $t \in I$. From this differential inequality it follows that

$$(4.21) \quad |u_m''|^2 + ||u_m'|^2 \leq \left\{ |u_m''(0)|^2 + ||u_{m,1}|^2 + \int_0^t 2 |f'|^2 ds + \tilde{C}t \right\} e^{Ct}.$$

Now the sequence $||u_{m,1}|| \rightarrow ||u_1||$ as $m \rightarrow \infty$ by relation (4.10) and hence is a

bounded sequence. Thus, if we can show that the norms $|u_m''(0)|$ are bounded, then the energy estimate (4.12) holds with

$$B = \sup |u_m''(0)|^2 + \sup ||u_{m1}||^2 + \int_0^T 2 |f'|^2 dt + \tilde{C}T.$$

Setting $t = 0$ in equations (4.8) we have

$$(u_m''(0), \psi_k) + ((u_{m0}, \psi_k)) + (u_{m0}u_{m0}^\gamma, \psi_k) = (f(0), \psi_k), \quad k = 1, 2, \dots, m.$$

Making the substitution $((u_{m0}, \psi_k)) = -(\Delta u_{m0}, \psi_k)$ (recall that $u_{m0}, \psi_k \in H_1^\circ \cap H_2$) in each of the preceding equations, multiplying by the corresponding $\tilde{\lambda}_k^m(0)$, and then adding all m equations, there results

$$|u_m''(0)|^2 = (f(0) + \Delta u_{m0} - u_{m0}u_{m0}^\gamma, u_m''(0)).$$

Thus we have the estimate

$$(4.22) \quad |u_m''(0)| \leq |f(0)| + |\Delta u_{m0}| + |u_{m0}u_{m0}^\gamma|.$$

But

$$|u_{m0}u_{m0}^\gamma| = \left[\int u_{m0}^{2+2\gamma} dx \right]^{1/2} = |u_{m0}|_{2+2\gamma}^{1+\gamma},$$

where

$$2 + 2\gamma \leq \frac{2n}{n-2}$$

and therefore, by Remark 2.1, $H_1^\circ \subset L^{2+2\gamma}$ in the topological sense so that the sequence $|u_{m0}|_{2+2\gamma} \rightarrow |u_0|_{2+2\gamma}$ as $m \rightarrow \infty$. Since we also have $u_{m0} \rightarrow u_0$ in $H_1^\circ \cap H_2$, the norms $|\Delta u_{m0}|$ are bounded too and it follows from relation (4.22) that $\sup |u_m''(0)| < \infty$. Thus the proof of Lemma 4.1 is complete.

We next extract a subsequence which converges—in a sense made precise in the following lemma—to a weak solution.

Lemma 4.2. *There exists a subsequence of approximate solutions satisfying the conditions:*

$$(4.23) \quad u_m(t) \rightarrow u(t) \text{ in the weak topology of } H_0, t \in I,$$

$$(4.24) \quad u_m(t) \rightarrow u(t) \text{ in the weak topology of } H_1^\circ, t \in I,$$

$$(4.25) \quad u_m' \rightarrow u' \text{ in the weak* topology of } [L^1(I, H_1^\circ)]^*,$$

$$(4.26) \quad u_m'' \rightarrow u'' \text{ in the weak* topology of } [L^1(I, H_0)]^*,$$

$$(4.27) \quad u_m u_m^\gamma \rightarrow u u^\gamma \text{ in the weak* topology of } [L^1(I, H_0)]^*.$$

Moreover, representations (4.5) and (4.6) hold.

Proof. The desired subsequence is obtained after several refinements of the original sequence; however, for brevity in notation, all subsequences are still

denoted by $\{u_m\}$. An immediate consequence of the energy estimate (4.11) is that the sequences $u_{m,t}$, $D_j u_m$ ($j = 1, \dots, n$) are uniformly bounded in $L^2(\Omega \times I)$. Thus, by Rellich's lemma (cf. [3, p. 489]), for any bounded subdomain $\tilde{\Omega} \subset \Omega$ it is possible to extract a subsequence which converges in $L^2(\tilde{\Omega} \times I)$; moreover, we may suppose also that the convergence is pointwise a.e. in $\tilde{\Omega} \times I$. By the classical diagonal procedure it is possible to select a subsequence such that:

$$(4.28) \quad u_m \rightarrow U \text{ in } L^2(K \times I) \text{ for every compact } K \subset \Omega,$$

$$(4.29) \quad u_m(x, t) \rightarrow U(x, t) \text{ a.e. in } \Omega \times I.$$

From the energy estimates (4.11) and (4.12) it follows immediately that:

$$\{u'_m\} \text{ is a bounded sequence in } L^\infty(I, H_1^\circ), \text{ and}$$

$$\{u''_m\} \text{ is a bounded sequence in } L^\infty(I, H_0).$$

Consequently there exist elements

$$\tilde{u} \in L^\infty(I, H_1^\circ), \quad \tilde{\tilde{u}} \in L^\infty(I, H_0),$$

and a subsequence such that

$$u'_m \rightarrow \tilde{u} \text{ in the weak* topology of } [L^1(I, H_1^\circ)]^*, \text{ and}$$

$$u''_m \rightarrow \tilde{\tilde{u}} \text{ in the weak* topology of } [L^1(I, H_0)]^*.$$

These last two relations along with (4.9), (4.10), and the obvious equations:

$$(4.30) \quad (u_m(t), v)_1 = (u_{m0}, v)_1 + \int_0^t (u'_m(s), v)_1 ds, \quad v \in H_1^\circ;$$

$$(4.31) \quad (u_m(t), v) = (u_{m0}, v) + t(u_{m1}, v) + \int_0^t \int_0^s (u''_m(\eta), v) d\eta ds, \quad v \in H_0;$$

imply that $u_m(t) \rightarrow u(t)$ weakly in H_1° for every $t \in I$ and also $u_m(t) \rightarrow u_*(t)$ weakly in H_0 for every $t \in I$ where u and u_* satisfy:

$$(4.32) \quad (u(t), v)_1 = (u_0, v)_1 + \int_0^t (\tilde{u}(s), v)_1 ds, \quad v \in H_1^\circ;$$

$$(4.33) \quad (u_*(t), v) = (u_0, v) + t(u_1, v) + \int_0^t \int_0^s (\tilde{\tilde{u}}(\eta), v) d\eta ds, \quad v \in H_0.$$

Next we show that $u_*(t) \in H_1^\circ$ and is identically equal to $u(t)$. For any $\epsilon > 0$, $t \in I$, and $v \in H_0$ with $|v| = 1$, choose an integer N for which $|(u_m - u_*, v)| < \epsilon$ when $m > N$ (we omit the t for convenience). Since the sequence $\{u_m\}$, $w_m = u_{N+m}$, converges weakly in H_1° to u , a theorem of Banach and Saks (cf. [14, p. 80]) guarantees the existence of a subsequence $\{w_{m_k}\}$ whose arithmetic means $W_k = (1/k)\{w_{m_1} + w_{m_2} + \dots + w_{m_k}\}$ converge (strongly) in H_1° to u .

The simple estimate

$$\begin{aligned} |(u - u_*, v)| &\leq |(u - W_k, v)| + |(W_k - u_*, v)| \\ &\leq \|u - W_k\|_1 + \frac{1}{k} \sum_{j=1}^k |(u_{N+m_j} - u_*, v)| \\ &< \|u - W_k\|_1 + \epsilon \end{aligned}$$

holds for all k whence $|(u - u_*, v)| \leq \epsilon$. In view of the arbitrariness of ϵ and v it follows that $|u - u_*| = 0$; consequently, since the natural imbedding of H_1 into H_0 is one-to-one, $\|u - u_*\|_1 = 0$ also.

Next observe that

$$|u_m(t)u_m^\gamma(t)| = \left[\int u_m^{2+2\gamma}(x, t) dx \right]^{1/2} = |u_m(t)|_{2+2\gamma}^{1+\gamma},$$

where $2 + 2\gamma \leq 2n/(n - 2)$ and therefore, by Remark 2.1, $|u_m(t)|_{2+2\gamma} \leq \text{Const. } \|u_m(t)\|_1$. Moreover, from relation (4.30) and Schwarz's inequality we obtain

$$|(u_m(t), v)_1| \leq \left\{ \|u_{m0}\|_1 + \int_0^t \|u'_m(s)\|_1 ds \right\} \|v\|_1$$

and therefore

$$(4.34) \quad \|u_m(t)\|_1 \leq \|u_{m0}\|_1 + \int_0^t \|u'_m(s)\|_1 ds \leq \text{Const.}$$

(independent of m and t) by relations (4.9), (4.11), and (4.12). Thus $\{u_m u_m^\gamma\}$ is a bounded sequence in $L^\infty(I, H_0)$ and consequently there exists an element $\tilde{U} \in L^\infty(I, H_0)$ and a subsequence such that

$$u_m u_m^\gamma \rightarrow \tilde{U} \text{ in the weak* topology of } [L^1(I, H_0)]^*.$$

Finally in view of relations (4.28) and (4.29) one can show that the function $t \rightarrow U(\cdot, t)$ is in $L^\infty(I, H_0)$, $U(\cdot, t) = u_*(t) (\equiv u(t))$ in H_0 for almost all $t \in I$, and $\tilde{U}(t) = U(\cdot, t)U^\gamma(\cdot, t)$ in H_0 for almost all $t \in I$.

In view of the definition of the Bochner integral, relation (4.32) is equivalent to

$$(u(t), v)_1 = (u_0, v)_1 + \left((H_1^\circ) \int_0^t \tilde{u}(s) ds, v \right)_1, \quad v \in H_1^\circ;$$

that is, $u(t) = u_0 + (H_1^\circ) \int_0^t \tilde{u}(s) ds$. By a basic property of the Bochner integral

$$h^{-1}\{u(t+h) - u(t)\} = (H_1^\circ)h^{-1} \int_t^{t+h} \tilde{u}(s) ds \rightarrow \tilde{u}(t) \text{ a.e. in } I$$

in the H_1 norm (cf. [5, p. 88]). Thus $u'(t) = \tilde{u}(t)$ a.e. in I as elements in H_1° . A similar argument establishes the representation (4.6) and completes the proof of Lemma 4.2.

Proof of Theorem 4.1. With the exception of conclusion (4.4) the proof follows immediately from Lemma 4.2.

A dense set in $L^1(I, H_1^\circ)$ is the set of step functions: $\sum_{i=1}^N \chi_i(t) v_i$, $v_i \in H_1^\circ$, N arbitrary, χ_i the characteristic function of an interval $[a_i, b_i) \subset I$. Thus step functions of the special form

$$(4.35) \quad \Phi(t) = \sum_{k=1}^N c_k \chi_k(t) \psi_k \quad (c_k \text{ constant})$$

are also dense in $L^1(I, H_1^\circ)$. From relation (4.8) it is clear that

$$(u_m''(t), \Phi(t)) + ((u_m(t), \Phi(t))) + (u_m(t) u_m^\gamma(t), \Phi(t)) = (f(t), \Phi(t))$$

holds for m sufficiently large. Integrating over I and letting $m \rightarrow \infty$ there results

$$\int_0^T \{(u''(t), \Phi(t)) + ((u(t), \Phi(t))) + (u(t) u^\gamma(t), \Phi(t))\} dt = \int_0^T (f(t), \Phi(t)) dt$$

by Lemma 4.2. Since this result holds for all Φ of the form (4.35) it holds for arbitrary $\varphi \in L^1(I, H_1^\circ)$; that is, relation (4.4) holds and the proof is complete.

5. The energy equality. In this section an energy equality is established which results in an immediate estimate on the energy at any time t in terms of the data of the problem.

Theorem 5.1. *Suppose u is a weak solution in the sense of Definition 3.1 such that $u, u' \in L^\infty(I, H_1^\circ)$ and $u'' \in L^\infty(I, H_0)$. Then the energy of the field u at time t , defined by the equation*

$$\varepsilon(t) = \frac{1}{2} \left\{ |u'(t)|^2 + \|u(t)\|^2 + \frac{2}{2+\gamma} |u(t)|_{2+\gamma}^{2+\gamma} \right\},$$

satisfies the identity

$$(5.1) \quad \varepsilon(t) = \varepsilon(0) + \int_0^t (f(\tau), u'(\tau)) d\tau.$$

Corollary 5.1. *In case there are no sources of energy in the field (i.e., $f = 0$), then $\varepsilon(t) \equiv \varepsilon(0)$.*

Also, from Schwarz's inequality and Gronwall's lemma, we easily obtain

Corollary 5.2. *(Energy estimate).*

$$(5.2) \quad \varepsilon(t) \leq \left\{ \varepsilon(0) + \int_0^t |f(\tau)|^2 d\tau \right\} e^t.$$

Proof of Theorem 5.1. In view of Remark 2.2, the representation

$$u(t) = u_0 + t u'(0) + (H_0) \int_0^t \left[(H_0) \int_0^\tau u''(s) ds \right] d\tau$$

holds ($u(0) = u_0$ by (3.5)). We next show $u'(0) = u_1$.

For any $\tau, \varepsilon > 0$ satisfying $\tau + \varepsilon < T$, let $\theta = \theta(t)$ be a real-valued $C^1(I)$ function whose graph has the form indicated below.

Next observe that for any $v \in H_1^0$ the function $\theta v \in \mathcal{F}$ whence from (3.4) there results

$$(5.3) \quad - \int_{\tau}^{\tau+\varepsilon} \theta'(t) \langle u'(t), v \rangle dt + \int_0^{\tau+\varepsilon} \theta(t) \{ \langle (u(t), v) \rangle + \langle u(t)u^\gamma(t), v \rangle \} dt \\ = \langle u_1, v \rangle + \int_0^{\tau+\varepsilon} \theta(t) \langle f(t), v \rangle dt.$$

By the continuity of u' we have

$$- \int_{\tau}^{\tau+\varepsilon} \theta'(t) \langle u'(t), v \rangle dt = - \langle u'(\eta), v \rangle \int_{\tau}^{\tau+\varepsilon} \theta'(t) dt = \langle u'(\eta), v \rangle,$$

for some $\eta \in [\tau, \tau + \varepsilon]$. Substituting the last equation into relation (5.3) and then letting $\varepsilon \rightarrow 0$ gives

$$\langle u'(\tau), v \rangle + \int_0^{\tau} \{ \langle (u(t), v) \rangle + \langle u(t)u^\gamma(t), v \rangle \} dt = \langle u_1, v \rangle + \int_0^{\tau} \langle f(t), v \rangle dt.$$

In view of the arbitrariness of τ and v it follows that $u'(0) = u_1$. Thus the following representations hold:

$$(5.4) \quad u(t) = u_0 + (H_1^0) \int_0^t u'(\tau) d\tau,$$

$$(5.5) \quad u(t) = u_0 + tu_1 + (H_0) \int_0^t \left[(H_0) \int_0^{\tau} u''(s) ds \right] d\tau.$$

Next observe that $\theta u' \in \mathcal{F}$; whence substitution of $\theta u'$ for φ in equation (3.4) is permissible and there results

$$\int_0^T \{ -\theta(u', u'') - \theta' |u'|^2 + \theta \langle (u, u') \rangle + \theta \langle uu^\gamma, u' \rangle \} dt = |u_1|^2 + \int_0^T \theta \langle f, u' \rangle dt.$$

In view of the representations (5.4) and (5.5) the real-valued functions $|u'(t)|^2$ and $\|u(t)\|^2$ are absolutely continuous with derivatives $2\langle u'(t), u''(t) \rangle$ and $2\langle u(t), u'(t) \rangle$, respectively, almost everywhere in I . It follows that

$$(5.6) \quad \int_0^T \theta \{ \langle u', u'' \rangle + \langle (u, u') \rangle + \langle uu^\gamma, u' \rangle \} dt = \int_0^T \langle f, u' \rangle dt.$$

The representation (5.5) implies the existence (cf. [5, p. 71]) of a real-valued function $U(x, t)$ which satisfies the conditions:

$$(5.7) \quad \begin{cases} U(x, t) \text{ is measurable in } \Omega \times I, \\ U(x, t) \text{ and } U_t(x, t) \text{ are absolutely continuous functions of } t \text{ for} \\ \quad \text{each fixed } x \in \Omega, \text{ and} \\ U(\cdot, t) = u(t), U_t(\cdot, t) = u'(t) \text{ for each } t \in I. \end{cases}$$

By Fubini's theorem, we have

$$\int_0^\tau \int_\Omega \frac{\partial}{\partial t} |U(x, t)|^{2+\gamma} dx dt = \int_\Omega \{|U(x, \tau)|^{2+\gamma} - |U(x, 0)|^{2+\gamma}\} dx$$

and it follows that $\int_\Omega |U(x, t)|^{2+\gamma} dx$ has, a.e. in I , the derivative

$$\int_\Omega \frac{\partial}{\partial t} |U(x, t)|^{2+\gamma} dx = (2 + \gamma) \int_\Omega U(x, t) |U(x, t)|^\gamma U_t(x, t) dx.$$

Thus $|u(t)|_{2+\gamma}^{2+\gamma}$ is an absolutely continuous function on I :

$$|u(\tau)|_{2+\gamma}^{2+\gamma} = |u_0|_{2+\gamma}^{2+\gamma} + (2 + \gamma) \int_0^\tau \langle u(t)u^\gamma(t), u'(t) \rangle dt, \quad \tau \in I.$$

Hence

$$\varepsilon(t) = \frac{1}{2} \left\{ |u'(t)|^2 + \|u(t)\|^2 + \frac{2}{2+\gamma} |u(t)|_{2+\gamma}^{2+\gamma} \right\}$$

is absolutely continuous on I and we find, from relation (5.6) and integration by parts,

$$-\varepsilon(0) - \int_\tau^{\tau+\epsilon} \varepsilon(t) \theta'(t) dt = \int_0^{\tau+\epsilon} \theta(t) (f(t), u'(t)) dt.$$

Now let $\epsilon \rightarrow 0$ in the last relation to obtain (5.1).

6. The continuous dependence of weak solutions on the data. The principal result in this section is that a small variation in the data u_0, u_1, f only results in a small change in the solution. This is one of the necessary conditions that the initial-boundary value problem for equation (1.1) is well posed.

Theorem 6.1. *Suppose that u and v are solutions in the sense of Definition 3.1 which correspond, respectively, to data u_0, u_1, f and v_0, v_1, g such that*

$$(6.1) \quad u, u', v, v' \in L^\infty(I, H_1^0) \quad \text{and} \quad u'', v'' \in L^\infty(I, H_0).$$

If $w = u - v, w_0 = u_0 - v_0, w_1 = u_1 - v_1, h = f - g$ then

$$|w'(t)|^2 + \|w(t)\|^2 \leq \left\{ \|w_0\|_1^2 + |w_1|^2 + \int_0^t |h(\tau)|^2 d\tau \right\} e^{(A+\gamma)t}.$$

Here A is a constant which depends only on the data u_0, v_0, f and g and T .

Remark 6.1. (Uniqueness). In case u and v are solutions which correspond to the same data we have $w_0 = w_1 = h = 0$ and consequently $u(t) \equiv v(t)$.

Remark 6.2. For the important special case $\gamma = 2$ and $n = 3$, J. L. Lions [12] has already shown that weak solutions in the sense of Definition 3.1 are unique.

Proof of Theorem 6.1. Since u and v are solutions it follows from (3.4) that

$$(6.2) \quad \int_0^T \{-(w', \varphi') + ((w, \varphi)) + (uw^\gamma - v^\gamma, \varphi)\} dt \\ = (w_1, \varphi(0)) + \int_0^T (h, \varphi) dt, \quad \varphi \in \mathcal{F}.$$

Now let $U = U(x, t)$ and $V = V(x, t)$ be real-valued functions with U satisfying (5.7) and V satisfying analogous conditions relative to v . On the measurable subset E of $\Omega \times I$ on which $W = U - V$ is non-zero we define $\tilde{W} = \tilde{W}(x, t)$ by $U(x, t) |U(x, t)|^\gamma - V(x, t) |V(x, t)|^\gamma = W(x, t) \tilde{W}(x, t)$ and then extend $\tilde{W}$ by 0 to $\Omega \times I$. Clearly $\tilde{W}$ is measurable in $\Omega \times I$, non-negative, and—on the set E —satisfies $\tilde{W} = (\gamma + 1) |\xi|^\gamma$ where ξ is between U and V . Consequently, $\tilde{W}^{1/\gamma}$ belongs to every L^p class which has both U and V as members. Thus (since $u(t) = U(\cdot, t)$ and $v(t) = V(\cdot, t)$ are in H_1° for $t \in I$) we have $\tilde{W}^{1/\gamma}(\cdot, t) \in L^p$ for $2 \leq p \leq 2n/(n - 2)$ (except $p = \infty$ when $n = 2$) by Remark 2.1. Consequently, the function $\tilde{w}$ defined by $\tilde{w}(t) = \tilde{W}(\cdot, t)$ for $t \in I$ satisfies

$$(6.3) \quad \sup_{t \in I} |\tilde{w}(t)|_p \leq \text{Const.}, \quad \frac{2}{\gamma} \leq p \leq \frac{2n}{\gamma(n - 2)}.$$

Next observe that $\theta w' \in \mathcal{F}$ (θ is defined in Sec. 5); whence substitution of $\theta w'$ for φ in equation (6.2) is permissible and there results

$$\int_0^T \{-\theta(w', w'') - \theta' |w'|^2 + \theta((w, w')) + \theta(w\tilde{w}, w')\} dt \\ = |w_1|^2 + \int_0^T \theta(h, w') dt.$$

Representations which are analogous to (5.4)–(5.5) hold for w whence the real-valued functions $|w'(t)|^2$ and $||w(t)||^2$ are absolutely continuous with derivatives $2(w'(t), w''(t))$ and $2((w(t), w'(t)))$, respectively, a.e. in I . A simple calculation shows that

$$\int_0^T \theta\{(w', w'') + ((w, w'))\} dt = \int_0^T \theta(h - w\tilde{w}, w') dt.$$

Now let ϵ tend to zero. First observe that

$$\begin{aligned} 2 \int_0^\tau \theta \{ (w'', w') + ((w, w')) \} dt &= \int_0^{\tau+\epsilon} \theta \frac{d}{dt} \{ |w'|^2 + ||w||^2 \} dt \\ &= - \{ |w_1|^2 + ||w_0||^2 \} - \int_\tau^{\tau+\epsilon} \theta' \{ |w|^2 + ||w||^2 \} dt \\ &= - \{ |w_1|^2 + ||w_0||^2 \} + \{ |w'(s)|^2 + ||w(s)||^2 \}, \end{aligned}$$

for some $s \in [\tau, \tau + \epsilon]$ and therefore tends, as $\epsilon \rightarrow 0$, to

$$|w'(\tau)|^2 + ||w(\tau)||^2 - |w_1|^2 - ||w_0||^2.$$

Also, as $\epsilon \rightarrow 0$, we obviously have

$$\int_0^\tau \theta(h - w\tilde{w}, w') dt \rightarrow \int_0^\tau (h - w\tilde{w}, w') dt.$$

It follows that for every $\tau \in I$, we have

$$(6.4) \quad |w'(\tau)|^2 + ||w(\tau)||^2 = |w_1|^2 + ||w_0||^2 + 2 \int_0^\tau (h - w\tilde{w}, w') dt.$$

Next we determine a bound for the term $(w\tilde{w}, w')$. First note that formally we have

$$|(w\tilde{w}, w')| \leq |w|_\alpha |\tilde{w}|_p |w'| \quad \text{when} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}$$

by Hölder's inequality, which, by Sobolev's inequality, is dominated by

$$(6.5) \quad \text{Const. } |w|^{1-\beta} ||w||^\beta |\tilde{w}|_p |w'|,$$

provided that $\beta = n(1/2 - 1/q) = n/p \leq 1$. In order to justify this estimate we must of course show that the various norms are finite and that the exponents p, q satisfy the above restrictions. We consider three cases in which it is to be understood that q and β are determined by p from the relations $1/p + 1/q = 1/2$ and $\beta = n/p$.

$$\text{Case 1:} \quad 0 < \gamma < \frac{2}{n-1} \quad \text{and} \quad n \geq 2.$$

Taking $p = (2 + \gamma)/\gamma$ we have $n < p < 2n/\gamma(n-1)$ whence $\beta < 1$ and $|\tilde{w}|_p < \infty$ by relation (6.3).

$$\text{Case 2:} \quad 2 \leq \gamma < \infty \quad \text{and} \quad n = 2.$$

For any p satisfying $2 < p < \infty$ we have $\beta < 1$ and $|\tilde{w}|_p < \infty$ by (6.3).

$$\text{Case 3:} \quad \frac{2}{n-1} \leq \gamma \leq \frac{2}{n-2} \quad \text{and} \quad n \geq 3.$$

Taking $p = n$ we have $1 + 2/\gamma \leq p \leq 2n/\gamma(n-2)$ and again $|\tilde{w}|_p < \infty$ by (6.3).

Thus a bound of the form (6.5) always holds for $(w\bar{w}, w')$ which, in view of (6.3) and Hölder's inequality, implies that

$$|(w\bar{w}, w')| \leq \text{Const.} \{ (1 - \beta) |w|^2 + \beta ||w||^2 + |w'|^2 \}.$$

It is now a simple consequence of the preceding inequality, equation (6.4), and the obvious estimates $\frac{1}{2} |w|^2 \leq |w - w_0|^2 + |w_0|^2$, $|2(h, w')| \leq |h|^2 + |w'|^2$, that for some constant A we have

$$\begin{aligned} |w'(\tau)|^2 + ||w(\tau)||^2 &\leq |w_1|^2 + ||w_0||^2 + \int_0^\tau |h|^2 dt \\ &\quad + A \int_0^\tau \{ |w_0|^2 + |w - w_0|^2 + |w'|^2 + ||w||^2 \} dt. \end{aligned}$$

Setting $E(t) = |w'(t)|^2 + ||w(t)||^2$ and $F(t) = |w(t) - w_0|^2$ we have

$$E(\tau) \leq E(0) + \int_0^\tau |h(t)|^2 dt + A \int_0^\tau \{ |w_0|^2 + F(t) + E(t) \} dt.$$

Denoting the right side of the last relation by $Az(\tau)$ we have

$$\begin{aligned} \frac{d}{d\tau} \{ z(\tau) e^{-A\tau} \} &= e^{-A\tau} \{ A^{-1} |h(\tau)|^2 + |w_0|^2 + F(\tau) + E(\tau) - Az(\tau) \} \\ &\leq A^{-1} |h(\tau)|^2 + e^{-A\tau} |w_0|^2 + e^{-A\tau} F(\tau). \end{aligned}$$

By integration we obtain

$$z(t) e^{-At} - A^{-1} E(0) \leq A^{-1} \int_0^t |h(\tau)|^2 d\tau + A^{-1} |w_0|^2 + \int_0^t e^{-A\tau} F(\tau) d\tau,$$

which implies

$$(6.6) \quad e^{-At} E(t) \leq ||w_0||_1^2 + |w_1|^2 + \int_0^t |h(\tau)|^2 d\tau + A \int_0^t e^{-A\tau} F(\tau) d\tau.$$

From the representation $w(\tau) = w_0 + (H_0) \int_0^\tau w'(s) ds$ it follows that

$$|w(\tau) - w_0| \leq \int_0^\tau |w'(s)| ds \leq \tau^{1/2} \left\{ \int_0^\tau |w'(s)|^2 ds \right\}^{1/2};$$

consequently,

$$(6.7) \quad F(\tau) \leq T \int_0^\tau E(s) ds.$$

Substituting this bound for $F(\tau)$ into relation (6.6) and interchanging the order of integration we find

$$\begin{aligned} e^{-At} E(t) &\leq ||w_0||_1^2 + |w_1|^2 + \int_0^t |h(\tau)|^2 d\tau + AT \int_0^t E(s) \int_s^t e^{-A\tau} d\tau ds \\ &\leq ||w_0||_1^2 + |w_1|^2 + \int_0^t |h(\tau)|^2 d\tau + T \int_0^t e^{-As} E(s) ds. \end{aligned}$$

Applying Gronwall's lemma there results

$$(6.8) \quad E(t) \leq \left\{ \|w_0\|_1^2 + |w_1|^2 + \int_0^t |h(\tau)|^2 d\tau \right\} e^{(A+T)t}.$$

Relations (6.7) and (6.8) obviously imply that

$$(6.9) \quad |w(t) - w_0|^2 \leq \left\{ \|w_0\|_1^2 + |w_1|^2 + \int_0^t |h(\tau)|^2 d\tau \right\} e^{(A+T)t}.$$

It remains to show that the constant A can be estimated in terms of the data u_0, u_1, f and v_0, v_1, g . First note that if $2/\gamma \leq p \leq 2n/\gamma(n-2)$ (except $p = \infty$ when $n = 2$) and $q = \gamma p$, then

$$\begin{aligned} |\bar{w}(t)|_p^{1/\gamma} &= \left[\int \{ \tilde{W}^{1/\gamma}(x, t) \}^q dx \right]^{1/q} \\ &\leq (1 + \gamma)^{1/\gamma} \left[\int \{ |U(x, t)| + |V(x, t)| \}^q dx \right]^{1/q} \\ &\leq (1 + \gamma)^{1/\gamma} [|u(t)|_q + |v(t)|_q], \end{aligned}$$

where $2 \leq q \leq 2n/(n-2)$ (except $q = \infty$ when $n = 2$). Thus Remark 2.1 shows that

$$|\bar{w}(t)|_p^{1/\gamma} \leq \text{Const.} [\|u(t)\|_1 + \|v(t)\|_1].$$

With the aid of the energy estimate (5.2) for u it is easy to bound the norm $\|u(t)\|_1$ (independently of t) in terms of the data u_0, u_1, f . An analogous statement holds of course for the solution v . Thus the constant A , which is essentially the constant in relation (6.3), can be estimated in terms of the data.

7. The existence of strict solutions. By specializing regularity results for elliptic equations (which have been established by a number of authors) we shall prove that the weak solution of the initial-boundary value problem constructed in Section 4 is in fact a strict solution.

Theorem 7.1. *Under hypotheses (3.1) and (4.1) there exists a strict solution of the initial-boundary value problem; that is, a solution in the sense of Definition 3.2.*

Proof. Let u be the function which satisfies relations (4.2)–(4.6). (Its existence is guaranteed by Theorem 4.1; uniqueness by Remark 6.1). Let χ denote the characteristic function of the subinterval $[0, \tau] \subset I$ and let $v \in H_1^0$. Setting $\varphi = \chi v$ in equation (4.4) there results

$$(7.1) \quad \int_0^\tau \{ (u''(t), v) + ((u(t), v)) + (u(t)u'(t), v) \} dt = \int_0^\tau (f(t), v) dt.$$

With the exception of $(u''(t), v)$ the inner products in equation (7.1) are con-

tinuous on I . Also, on the Lebesgue set $\mathcal{L}$ of $u'' \in L^\infty(I, H_0)$ one has

$$\frac{d}{d\tau} \left\{ (H_0) \int_0^\tau u''(t) dt \right\} = u''(\tau) \quad (\text{in } H_0).$$

Thus by differentiation of (7.1) there results

$$(7.2) \quad (u''(\tau), v) + ((u(\tau), v)) + (u(\tau)u^\gamma(\tau), v) = (f(\tau), v), \quad \tau \in \mathcal{L}.$$

Next we show that after correction of u'' on (at most) the null set $I - \mathcal{L}$, equation (7.2) holds for all $\tau \in I$. For an arbitrary $t \in I$ let $\{\tau_i\}$ be a sequence satisfying:

$$\tau_i \in \mathcal{L}, \quad \tau_i \rightarrow t, \quad \text{and} \quad u''(\tau_i) \rightarrow u^*(t) \text{ weakly in } H_0.$$

The fact that $u^*(t)$ is well defined follows from (7.2). Replacing τ by τ_i in relation (7.2) and then letting $j \rightarrow \infty$ there results

$$(u^*(t), v) + ((u(t), v)) + (u(t)u^\gamma(t), v) = (f(t), v).$$

Clearly $u^*(t) = u''(t)$ (in H_0) when $t \in \mathcal{L}$. Redefining u'' to agree with u^* on $I - \mathcal{L}$ we may suppose that relation (7.2) holds for $\tau \in I$. If we set $F(t) = u''(t) + u(t)u^\gamma(t) - f(t)$ then we have

$$(7.3) \quad ((u(t), v)) + (F(t), v) = 0, \quad t \in I, \quad v \in H_1^0.$$

Thus $u(t)$ ($t \in I$ fixed) is a weak solution of the Poisson equation $\Delta w = F$. By known regularity results concerning weak solutions of elliptic equations (e.g. [13, Lemma 3, p. 663]), the fact that $F(t) \in H_0$ implies that $u(t) \in H_2$. It is now a simple consequence of equation (7.3) that

$$(\Delta u(t), v) = (F(t), v), \quad t \in I, \quad v \in H_1^0.$$

This result is equivalent to (3.9) and completes the proof.

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