Structure of $A^*A - AA^*$

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1. We propose to give a complete answer (Theorem 8) to the following question: Which bounded (necessarily hermitian) operators on a separable complex Hilbert space $\mathcal{H}$ are of the form $A^*A - AA^*$? Here A denotes a bounded operator on $\mathcal{H}$. Expressing A as $M + iN$, where M and N are hermitian, it is seen that the following question is merely another formulation of the one just given: Which (necessarily skew-hermitian) operators on $\mathcal{H}$ are commutators of hermitian operators, *i.e.*, are of the form $AB - BA$, where A and B are hermitian?

It was shown by Wintner [1] that no non-zero scalar multiple of the identity operator is a commutator of hermitian operators. The same result was proved for arbitrary operators by Putnam [2]. A more general result was obtained by Wielandt [3]. Halmos [4] showed that a necessary condition for an operator A to be a commutator is that for no non-zero scalar a and compact operator C it be of the form $aI + C$. It was proved recently by Brown and Pearcy [5] that this condition was also sufficient.

The question of commutators of *hermitian* operators, which is certainly distinct from that of commutators of arbitrary operators, is interesting for at least one reason that the magnitude of $A^*A - AA^*$ can be taken as a measure of the "abnormality" of A . This question was studied by Putnam, who discovered Corollary 1 to Theorem 1 of the present paper, and also by Halmos to whom Theorem 3 is due [6]. A more general result proven by Putnam was that if $C = AB - BA$, where at least one of the two operators A and B is semi-normal, then $\inf \{ |(Cx, x)| : \|x\| = 1 \} = 0$. (An operator A is called *semi-normal* if $A^*A - AA^*$ is either non-negative or non-positive.

While this manuscript was being re-written to make the corrections and alterations suggested by the referee, a paper on commutators of arbitrary operators by Brown, Halmos, and Pearcy [7] has appeared. Although our results were obtained independently, the present paper contains some techniques similar to those employed in [7]; we acknowledge this and reserve the priority credit for [7].

2. Throughout the paper $\mathcal{H}$ will denote a separable complex Hilbert space, infinite-dimensional unless otherwise indicated. By “operator” we shall mean bounded linear operator. By the “span” of a set of vectors in $\mathcal{H}$ will be meant the closure of the linear manifold generated by the set. If K is any operator, $K - a$ will often be used in place of $K - aI$, where a is a scalar and I is the appropriate identity operator. An hermitian operator K will be called non-negative if $(Kx, x) \geq 0$ for every x in $\mathcal{H}$; it will be called positive if the inequality is strict for non-zero x . The adjectives non-positive and negative will be used in an analogous manner. If K is an hermitian operator, then $K > a$ will mean $K - aI$ is a positive operator; $K \geq a$, $K < a$, and $K \leq a$ have obvious meanings. The symbol $\oplus$ will be used to denote direct sums of Hilbert spaces and also of operators. It will also be convenient to use the following additional symbols and definitions.

i) By a *self-commutator* will be meant a (necessarily hermitian) operator on $\mathcal{H}$ which is of the form $A^*A - AA^*$ for some operator A on $\mathcal{H}$. (The definition is due to Halmos who also denoted $A^*A - AA^*$ by $[A]$.)

ii) Let K be an hermitian operator and let a be a real number such that either $K - a$ has infinite-dimensional null space or a is an accumulation point of the spectrum of K . We shall follow Riesz-Nagy [8] in calling a a *limit point* of $Sp(K)$.

iii) Every hermitian operator K can be uniquely expressed as $K^+ \oplus K^-$, where K^+ is non-negative and K^- is negative. (One summand can be defined to be zero on the subspace $\{0\}$ of $\mathcal{H}$ in the cases where K is either non-negative or positive.) The symbols K^+ and K^- will be used consistently.

iv) Let $0 \leq a < b$. Let $\varphi(t; a, b)$ be the real-valued function defined on the real line as follows: $\varphi(t; a, b) = t$ for $a < |t| \leq b$ and $\varphi(t; a, b) = 0$ otherwise. The letter φ in this paper will always denote the above function. If K is an hermitian operator, $\varphi(K; a, b)$ will be used in the usual sense and by $\varphi_0(K; a, b)$ we shall denote the restriction of the operator $\varphi(K; a, b)$ to the closure $\mathcal{H}_{a,b}$ of its range. We shall make use of (and will seldom be explicit about) the fact from the operator calculus that if $0 \leq a < b \leq c < d$, then $\mathcal{H}_{a,b}$ and $\mathcal{H}_{c,d}$ are orthogonal. (If $E(T)$ denotes the resolution of the identity for K , then $\varphi(K; a, b)$ is just $KE(T_{a,b})$, where $T_{a,b} = \{t : a < |t| \leq b\}$.)

Before we conclude this section on notation and definitions, we should mention that all the well-known facts on compact operators and non-negative operators of finite trace will be used freely throughout the paper.

3. We begin with stating a necessary condition for operators to be self-commutators; we shall show later (although quite indirectly indeed) that it is also sufficient.

Theorem 1. *In order for an hermitian operator K on $\mathcal{H}$ to be a self-commutator, it is necessary that there exist an infinite orthonormal sequence $\{e_i\}$ of vectors in $\mathcal{H}$ such that $|\sum_{i=1}^n (Ke_i, e_i)|$ is bounded.*

Proof. We let $L = iK$ and assume $L = MN - NM$, where M and N are

hermitian. There exists an infinite orthonormal sequence $\{e_j\}$ in $\mathcal{H}$, a sequence $\{a_j\}$ of real numbers, and a sequence $\{b_j\}$ of non-negative numbers, such that $Ne_1 = a_1e_1 + b_1e_2$ and $Ne_j = b_{j-1}e_{j-1} + a_je_j + b_je_{j+1}$, $j \geq 2$. (Because N can be expressed as $N_1 \oplus N_2$, where either N_1 is an N -cyclic subspace of $\mathcal{H}$ generated by some vector e_1 , or it is the span of an infinite sequence of eigenvectors of N .) We have

$$\begin{aligned} (Le_j, e_j) &= 2i \operatorname{Im} (MNe_j, e_j) \\ &= 2i \operatorname{Im} [b_{j-1}(Me_{j-1}, e_j) - b_j(Me_j, e_{j+1})], \quad \text{for } j \geq 2, \end{aligned}$$

and

$$(Le_1, e_1) = 2i \operatorname{Im} [-b_1(Me_1, e_2)].$$

Hence

$$\left| \sum_{i=1}^n (Ke_i, e_i) \right| = \left| \sum_{i=1}^n (Le_i, e_i) \right| = 2 |\operatorname{Im} [b_n(Me_n, e_{n+1})]| \leq 2 \|M\| \cdot \|N\|.$$

Corollary 1. *If K is an hermitian operator on $\mathcal{H}$ with $K \geq a$ (or $K \leq -a$) for some positive number a , then K is not a self-commutator.*

Proof. If $\{e_i\}$ is any orthonormal sequence, we have $\sum_{i=1}^n (Ke_i, e_i) \geq na$ ($\leq -na$).

Corollary 2. *If an hermitian operator K on $\mathcal{H}$ can be expressed as $K_1 \oplus K_2$, where the domain of K_1 is finite-dimensional and K_2 satisfies the conditions of Corollary 1, then K is not a self-commutator.*

Proof. We shall use the fact that if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, if $\mathcal{H}_1$ is of finite dimension m , and if $\{e_i\}$ is an orthonormal sequence in $\mathcal{H}$ with $e_i = f_i + g_i$, $f_i \in \mathcal{H}_1$, $g_i \in \mathcal{H}_2$ for every j , then $\sum_{i=1}^{\infty} \|f_i\|^2 \leq m$. Now assume $K_2 \geq a > 0$, and let $\mathcal{H}_i$ be the domain of K_i , $i = 1, 2$. Let $\{e_i\}$ be any orthonormal sequence in $\mathcal{H}$ with f_i and g_i as described above. Choose n such that $\|f_i\|^2 < a/2(a + \|K_1\|)$ for $j \geq n$. Then

$$\begin{aligned} (Ke_j, e_j) &= (K_1f_j, f_j) + (K_2g_j, g_j) \geq a \|g_j\|^2 - \|K_1\| \cdot \|f_j\|^2 \\ &= a(1 - \|f_j\|^2) - \|K_1\| \cdot \|f_j\|^2 \geq a/2 \quad \text{for } j \geq n. \end{aligned}$$

4. We now turn to sufficient conditions. Several such conditions will be given, which, together with the above theorem, will lead to our main result.

Lemma 1. *If K is a compact hermitian operator of zero nullity on $\mathcal{H}$, and if $\{\epsilon_i\}$ is any sequence of positive numbers with $\epsilon_1 \geq \|K\|$, then there exists a sequence $\{K_i\}$ of hermitian operators, each having an infinite-dimensional domain, such that*

$$K = \sum_{i=1}^{\infty} \oplus K_i \text{ and } \|K_i\| \leq \epsilon_i, \quad j = 1, 2, \dots$$

Proof. Let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of K . Let $\{c_i\}$ be the corresponding sequence of eigenvalues. It can be assumed that the e_i have been so arranged as to render the sequence $\{|c_i|\}$ non-increasing. Since $\lim c_i = 0$, the sequence $\{c_i\}$ can be rearranged into a double sequence $\{c_{ki}\}$ such that, for each k , the sequence $\{|c_{ki}|\}$ is non-increasing and $|c_{k1}| = \sup_i |c_{ki}| \leq \epsilon_k$. Let $\{e_{ki}\}$ be the corresponding rearrangement of $\{e_i\}$, and, for each k , let K_k denote the restriction of K to the span of the vectors e_{ki} , $j = 1, 2, \dots$. Then $\|K_k\| \leq \epsilon_k$ and $K = \sum_{k=1}^{\infty} \oplus K_k$.

Lemma 2. *Let K be any hermitian operator on $\mathcal{H}$, and let a be a limit point of $Sp(K)$. Let $\{\epsilon_i\}$ be any sequence of positive numbers with $\epsilon_1 \geq \|K - a\|$. Then, there exists a sequence $\{K_i\}$ of hermitian operators, each having an infinite-dimensional domain, such that*

$$K = \sum_{i=1}^{\infty} \oplus K_i \text{ and } \|K_i - a\| \leq \epsilon_i, \quad j = 1, 2, \dots$$

Proof. By considering the operator $K - a$ the general case is reduced to that in which $a = 0$. Assuming $a = 0$, we shall distinguish three cases.

i) If K has infinite-dimensional null space, the lemma is trivially true: Let $\mathcal{H}_0$ denote the closure of the range of K and write the null space of K as $\sum_{i=1}^{\infty} \oplus \mathcal{H}_i$, where each $\mathcal{H}_i$ is infinite-dimensional. The subspaces $\mathcal{H}_i'$ defined by $\mathcal{H}_1' = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\mathcal{H}_j' = \mathcal{H}_j$, $j \geq 2$, are all infinite-dimensional and $K = \sum_{i=1}^{\infty} \oplus K_i$, where K_i is the restriction of K to $\mathcal{H}_i'$ and $K_i = 0$ for $j \geq 2$. In the remainder of the proof we shall assume that 0 is an accumulation point of the spectrum of K .

ii) Assume for every $b > 0$ there exists $a > 0$ such that $\varphi(K; a, b)\mathcal{H}$ is infinite-dimensional. Let $a_1 = \epsilon_1$ and, for each j , let a_{j+1} be a positive number $\leq \min(j^{-1}, \epsilon_{j+1})$ such that $\mathcal{H}_j = \varphi(K; a_{j+1}, a_j)\mathcal{H}$ is infinite-dimensional. Now $\mathcal{H} = \mathcal{H}_0 \oplus \sum_{i=1}^{\infty} \oplus \mathcal{H}_i$, where $\mathcal{H}_0$ is the null space of K , and $K = K_0 \oplus \sum_{i=1}^{\infty} \oplus K_i$, where $K_0 = 0$ and $K_i = \varphi_0(K; a_{j+1}, a_j)$. Incorporating K_0 into K_1 we obtain the desired result, because $\|\varphi_0(K; a_{j+1}, a_j)\| \leq a_j$, $j = 1, 2, \dots$.

iii) Assume there exists $b > 0$ such that $\varphi(K; a, b)\mathcal{H}$ is finite-dimensional for every $a > 0$. This implies that $\varphi_0(K; 0, b)$ is compact; since its domain is infinite-dimensional by hypothesis, Lemma 1 is applicable and we have $\varphi_0(K; 0, b) = \sum_{i=1}^{\infty} \oplus K_i$, where each K_i has infinite-dimensional domain and $\|K_i\| \leq \epsilon_i$. The proof is completed by observing that

$$K = \varphi_0(K; b, \|K\|) \oplus \sum_{i=1}^{\infty} \oplus K_i$$

and that

$$\|\varphi_0(K; b, \|K\|) \oplus K_1\| \leq \|K\|.$$

Theorem 2. Let $\mathfrak{H} = \sum_{i=1}^{\infty} \oplus \mathfrak{H}_i$, where the $\mathfrak{H}_i$ all have the same (finite or infinite) dimension. Let Q be an isometry on $\mathfrak{H}$ with $Q\mathfrak{H}_i = \mathfrak{H}_{i+1}$, $i = 1, 2, \dots$. For each j , let K_j be an hermitian operator on $\mathfrak{H}_j$ such that the operator $S_n = \sum_{i=1}^n Q^{*i-1} K_i Q^{i-1}$ defined on $\mathfrak{H}_1$ is non-negative for every n (or non-positive for every n). Assume, furthermore, that $\|S_n\|$ is bounded. Then the operator $K = \sum_{i=1}^{\infty} \oplus K_i$ is a self-commutator.

Proof. Let $S_n \geq 0$, $n = 1, 2, \dots$. For each j , define an operator F_j on $\mathfrak{H}_j$ by $F_j = (Q^{j-1} S_j Q^{*j-1})^{1/2}$. Let $F = \sum_{i=1}^{\infty} \oplus F_i$ and $A = QF$. Every $\mathfrak{H}_i$ is invariant under $A^*A - AA^* = F^2 - QF^2Q^*$, and the restriction of the latter to $\mathfrak{H}_i$ is $F_i^2 - QF_{i-1}^2Q^* = K_i$. It follows that $A^*A - AA^* = K$.

The case in which $S_n \leq 0$ is treated similarly by considering

$$F_j = (-Q^{j-1} S_j Q^{*j-1})^{1/2}.$$

Corollary. Let $\mathfrak{H} = \sum_{i=1}^{\infty} \oplus \mathfrak{H}_i$, where the $\mathfrak{H}_i$ have the same dimension. Let K_j be an hermitian operator on $\mathfrak{H}_j$ with $K_j \geq 0$ for every j (or $K_j \leq 0$ for every j). Assume $\sum_{i=1}^{\infty} \|K_i\| = a < \infty$. Then the operator $\sum_{i=1}^{\infty} \oplus K_i$ is a self-commutator.

Proof. If Q is any isometry of the type described in the statement of the theorem, then the operators $Q^{*i-1} K_i Q^{i-1}$ on $\mathfrak{H}_1$ are all non-negative (all non-positive, resp.) and so is their partial sum S_n ; we also have

$$\|S_n\| \leq \sum_{i=1}^n \|K_i\| \leq a.$$

Theorem 3. Every hermitian operator K on $\mathfrak{H}$ whose null space is infinite-dimensional is a self-commutator.

Proof. Assume, first, that K is either non-negative or non-positive. By Lemma 2 we have $K = \sum_{i=1}^{\infty} \oplus K_i$ with $\sum_{i=1}^{\infty} \|K_i\| \leq \|K\| + a$, where a is any pre-assigned positive number and where each K_i has infinite-dimensional domain. Since $K_i \geq 0$ for every j ($K_i \leq 0$ for every j , resp.) the above corollary applies.

If K is an arbitrary hermitian operator of infinite nullity, then $K = K_1 \oplus K_2$, where K_1 and $-K_2$ are non-negative and both have infinite-dimensional null space. By the preceding paragraph, we have $K_i = A_i^* A_i - A_i A_i^*$, $i = 1, 2$, for some operators A_1 and A_2 . Hence $K = A^*A - AA^*$ with $A = A_1 \oplus A_2$.

Theorem 4. Let K be a non-negative (or non-positive) operator on $\mathfrak{H}$ whose spectrum contains 0 as an accumulation point. Then K is a self-commutator.

The proof is similar to that given in the first paragraph of the proof of Theorem 3.

Theorem 5. Every compact hermitian operator is a self-commutator.

Proof. Write the operator as $K = K^+ \oplus K^-$ and let $\mathfrak{H}^+ \oplus \mathfrak{H}^-$ be the corresponding decomposition of $\mathfrak{H}$. If $\mathfrak{H}^+$ and $\mathfrak{H}^-$ are both infinite-dimensional,

then K^+ and K^- are both self-commutators by Theorems 3 and 4, and so is K . Otherwise assume, without loss of generality, that $\mathfrak{H}^-$ is of finite dimension $n > 0$. Then K^+ can be expressed as $K_1 \oplus K_2$, where K_i is defined on an infinite-dimensional subspace of $\mathfrak{H}^+$, $i = 1, 2$, such that K_1 has finite trace with $\text{tr } K_1 \leq \inf \{ \|s\| : s \in \text{Sp}(K^-) \}$. Now K_2 is a self-commutator, so that it would suffice to show that $K^- \oplus K_1$ is also a self-commutator. Express K_1 as $\sum_{i=1}^{\infty} \oplus L_i$, where each L_i has n -dimensional domain. Denoting K^- by L_0 , let Q be an isometry on $\mathfrak{H}$ mapping the domain of L_i onto that of L_{i+1} , $j = 0, 1, 2, \dots$. We observe that

$$\left\| \sum_{i=1}^n Q^{*i} L_i Q^i \right\| \leq \sum_{i=1}^n \|L_i\| \leq \text{tr } K_1, \quad \text{for every } n \geq 1,$$

so that the operator $S_n = \sum_{i=0}^n Q^{*i} L_i Q^i$ (on the domain of L_0) is non-positive for every $n \geq 0$, and $\|S_n\| \leq 2 \|K^-\|$. The proof is then completed by Theorem 2.

Theorem 6. *Let K be an hermitian operator on $\mathfrak{H}$ and let 0 be a limit point of $\text{Sp}(K)$. Then K is a self-commutator.*

Proof. In view of Theorems 3 and 5, we can assume that 0 is an accumulation point of $\text{Sp}(K)$ and that $\text{Sp}(K)$ has a non-zero limit point a . It would suffice to prove the theorem for $a > 0$. We consider three cases.

i) Assume the spectrum of K^+ contains 0 as an accumulation point. Let $0 < b < a$ and $K_1 = \varphi_0(K^+; 0, b)$. Write $K^+ = K_1 \oplus K_2$, where, by hypothesis, K_i has infinite-dimensional domain, $i = 1, 2$. The number a is a limit point of $\text{Sp}(K_2)$. Let $K_2 = \sum_{i=1}^{\infty} \oplus L_i$ be an expression for K_2 obtained from Lemma 2 with $\sum_{i=2}^{\infty} \|L_i - a\| \leq a - b$. Then $L_i \geq b$ for $j \geq 2$. Choose n so large as to yield $(n-1)b \geq \|K^-\|$. The spectrum of the operator $M = K_1 \oplus \sum_{i=n+1}^{\infty} \oplus L_i$ contains 0 as an accumulation point, and, therefore $M = \sum_{i=1}^{\infty} \oplus M_i$, where each M_i has infinite-dimensional domain and $\sum_{i=1}^{\infty} \|M_i\| < \infty$. Setting $N_i = L_i$, $j = 1, 2, \dots, n$; $N_{n+1} = K^-$; and $N_i = M_{i-n-1}$, $j = n+2, n+3, \dots$, we obtain a new expression $\sum_{i=1}^{\infty} \oplus N_i$ for K to which Theorem 2 is applicable: Let Q be an isometry on $\mathfrak{H}$ sending the domain of N_i onto that of N_{i+1} , $j = 1, 2, \dots$. Since $Q^{*i-1} N_i Q^{i-1} \geq b$ for $2 \leq j \leq n$, it follows from the choice of n that

$$S_m = \sum_{i=1}^m Q^{*i-1} N_i Q^{i-1} \geq 0, \quad \text{for every } m.$$

The inequalities $\|S_m\| \leq \sum_{i=1}^{\infty} \|N_i\| < \infty$ complete the proof.

ii) Assume 0 is not an accumulation point of $\text{Sp}(K^+)$; then it is an accumulation point of $\text{Sp}(K^-)$. Also assume that K^- is compact, so that it can be written as $K_1 \oplus K_2$, where K_i has infinite-dimensional domain, $i = 1, 2$, and $\| \text{tr } K_1 \| \leq \inf \{ \|s\| : s \in \text{Sp}(K^+) \}$. Since K_2 is a self-commutator, it would be sufficient to show that $K_1 \oplus K^+$ is a self-commutator. To this end, further express K_1 as $\sum_{i=1}^{\infty} \oplus L_i$, where each L_i has infinite-dimensional domain.

Then the form $K^+ \oplus \sum_{i=1}^{\infty} \oplus L_i$ for $K_1 \oplus K^+$ satisfies the conditions of the corollary to Theorem 2.

iii) The proof of the theorem will be completed if it is proved for the case in which $Sp(K^+)$ and $Sp(K^-)$ both have non-zero limit points. But this will be accomplished by the following result.

Theorem 7. *Let K be an hermitian operator on $\mathfrak{H}$ whose spectrum has two limit points $a_1 > 0$ and $a_2 < 0$. Then K is a self-commutator.*

Proof. The number a_1 is a limit point for $Sp(K^+)$ and a_2 a limit point for $Sp(K^-)$. Let

$$K^+ = \sum_{i=1}^{\infty} \oplus L_i \quad \text{and} \quad K^- = \sum_{i=1}^{\infty} \oplus M_i$$

(according to Lemma 2) with $\|L_i - a_1\| \leq \delta_i$, $\|M_i - a_2\| \leq \delta_i$, and $\sum_{i=1}^{\infty} \delta_i = \delta < \infty$. The remainder of the proof will consist in writing K as $\sum_{i=1}^{\infty} \oplus N_i$, where the sequence $\{N_i\}$ is an appropriate rearrangement of the L_i and M_i , to which Theorem 2 would be applicable.

We first construct a sequence of integers as follows. Let n_1 be the minimal integer n for which $na_1 \geq |a_2| + 2\delta$; then let n_2 be the maximal n for which $n_1a_1 + na_2 \geq 2\delta$. Having constructed n_{2j-1} and n_{2j} , let n_{2j+1} be the minimal integer n with

$$\left(n + \sum_{k=1}^j n_{2k-1}\right)a_1 + \left(\sum_{k=1}^j n_{2k}\right)a_2 \geq |a_2| + 2\delta;$$

then let n_{2j+2} be the maximal n with

$$\left(\sum_{k=1}^{j+1} n_{2k-1}\right)a_1 + \left(n + \sum_{k=1}^j n_{2k}\right)a_2 \geq 2\delta.$$

Every n_j thus constructed is positive and the partial sums of the series $n_1a_1 + n_2a_2 + n_3a_1 + \dots$ are bounded below by 2δ and above by $a + |a_2| + 2\delta$. The operators L_i and M_i can now be arranged into a sequence $\{N_i\}$ as follows. We first take the first n_1 members of $\{L_i\}$ followed by the first n_2 members of $\{M_i\}$; then the next n_3 members of $\{L_i\}$ followed by the next n_4 members of $\{M_i\}$; and so on. Now $K = \sum_{i=1}^{\infty} \oplus N_i$, where N_i has domain $\mathfrak{H}_i$. Let Q be an isometry on $\mathfrak{H}$ with $Q\mathfrak{H}_i = \mathfrak{H}_{i+1}$, $j = 1, 2, \dots$, and let $S_n = \sum_{i=1}^n Q^{*i-1}N_iQ^{i-1}$. A comparison of the operators S_n to the partial sums of the series $n_1a_1 + n_2a_2 + n_3a_1 + \dots$ will show that $0 \leq S_n \leq |a_2| + 4\delta$, $n = 1, 2, \dots$. Theorem 2 completes the proof.

We are now ready to state the main theorem which is a consequence of Theorems 1, 6, 7, and Corollary 2 to Theorem 1.

Theorem 8. *Let K be any hermitian operator on the infinite-dimensional, separable Hilbert space $\mathfrak{H}$. Then the following statements are all equivalent.*

i) *There exists an operator A on $\mathfrak{H}$ such that $A^*A - AA^* = K$.*

- ii) *There exist two hermitian operators M and N on $\mathfrak{H}$ such that $MN - NM = iK$.*
- iii) *There exists an infinite orthonormal sequence $\{e_i\}$ of vectors in $\mathfrak{H}$ such that $|\sum_{i=1}^n (Ke_i, e_i)|$ is bounded.*
- iv) *$Sp(K)$ has at least one non-negative limit point and at least one non-positive limit point.*
- v) *K is not of the form $K_1 \oplus K_2$, where K_1 has finite-dimensional domain and where either $K_2 \geq a > 0$ or $K_2 \leq a < 0$ for some real number a .*

In conclusion, it is perhaps worth mentioning that if $\mathfrak{H}$ is finite-dimensional, then an hermitian operator on $\mathfrak{H}$ is a self-commutator if and only if its trace is zero.

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