On Reynolds Operators in Finite-Dimensional Algebras

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1. Definitions. Let $\mathfrak{a}$ be a commutative algebra with identity e over a field F of characteristic not equal to two. A REYNOLDS operator in $\mathfrak{a}$ is an everywhere defined linear operator of $\mathfrak{a}$ into itself satisfying the REYNOLDS identity (after the British engineer OSBORNE REYNOLDS)

$$(1) \quad R(fg) = (Rf)(Rg) + R[(f - Rf)(g - Rg)], \quad f, g \in \mathfrak{a}.$$

The origin of this identity, as well as its numerous applications to operator theory, probability and turbulence theory, will be discussed elsewhere. The reader is referred to the works of Mme. MARIE-LOUISE DUBREIL-JACOTIN (*cf.* [6], [7] and [8]) for recent surveys of the literature related to REYNOLDS operators.

A related concept is that of a *smoothing operator*; such an operator A is an everywhere defined linear operator in the algebra $\mathfrak{a}$ which satisfies the identity (the so-called averaging identity)

$$(2) \quad A[f(Ag)] = (Af)(Ag), \quad f, g \in \mathfrak{a}.$$

In a lattice-ordered algebra (*cf.* [4]) an order-preserving smoothing operator is called an *averaging operator*. For a survey of the theory of averaging operators, see [3]; the relationship with the notion of conditional expectation in probability theory is discussed in [12]. Order and lattice properties will not be considered in this work. Smoothing operators have been studied rather completely, and can be considered as known (*cf.* [3]). Smoothing operators in the algebra of continuous functions $C(S)$ over a compact Hausdorff space have been completely classified (*cf.* [2], [3] and [10]). Averaging operators in the algebra of measurable functions have also been studied (*cf.* [12]).

The following counter-example, due to MOLINARO (*cf.* [6]), shows that in general the REYNOLDS identity does not imply the averaging identity. Let $\mathfrak{a}$

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be the algebra of all bounded real continuous function on the real line, and let

$$(Rf)(x) = e^{-x} \int_{-\infty}^x f(t)e^t dt.$$

Then R is a REYNOLDS operator, but it is evidently not a smoothing operator.

The purpose of this work is to show that *in a finite-dimensional algebra $C(S)$ of real functions on a finite set S (or, equivalently, a vector space under componentwise product) every REYNOLDS operator is a smoothing operator*. This result, together with the comparatively simple characterization of smoothing operators (also derived independently below), gives a complete classification of REYNOLDS operators in a finite-dimensional algebra $C(S)$. A partial result in this direction was recently obtained by MOLINARO (cf. [11]), who showed that in such an algebra an order-preserving REYNOLDS operator is an averaging operator. Using similar methods J. ARBAULT (cf. [1]) was able to dispense with the condition of positivity. The methods used in the present work are altogether different and independent from those of MOLINARO and ARBAULT.

The exposition is self-contained. Some of the propositions below have been established independently by Mme. DUBREIL-JACOTIN and her school. Whenever possible, the proofs given below generalize or are closest to being generalizable to the case $C(S)$ of all continuous functions on a compact Hausdorff space.

2. Identities. We collect below some useful properties of general REYNOLDS operators which do not require any finiteness assumptions.

Proposition 2.1. *The following identities are equivalent to the REYNOLDS identity:*

$$(3) \quad R(fRg + gRf) = RfRg + R(RfRg)$$

$$(4) \quad 2R(fRf) = (Rf)^2 + R(Rf)^2.$$

Proof. (3) follows from (1) upon expanding and cancelling. (4) follows from (1) by setting $f = g$. One obtains (1) from (4) by polarization, replacing f by $f + g$ and expanding.

Proposition 2.2. *Every idempotent ($R^2 = R$) REYNOLDS operator is a smoothing operator.*

Proof. Set $g = Rh$ in the REYNOLDS identity.

Proposition 2.3. *If f is a fixpoint of R (i.e., $Rf = f$), then*

$$(5) \quad R(fg) = fRg, \quad g \in \mathfrak{a}.$$

Proof. Immediate from the REYNOLDS identity.

Proposition 2.4. *If $Rg = 0$ then*

$$R(gRf) = 0, \quad f \in \mathfrak{a}.$$

Proof. Immediate from (3).

Proposition 2.5. *If $Rg = e$ (e is the identity of $\mathfrak{a}$) then*

$$R(gRf) = R^2f, \quad f \in \mathfrak{a}.$$

Proof. Use identity (3) to get

$$R(fe + gRf) = Rf + R(eRf)$$

from which the conclusion is immediate.

Proposition 2.6. *Let $\mathfrak{a}$ be a commutative algebra with identity having no non-zero nilpotent elements. If for a given h in $\mathfrak{a}$, and for $n \geq 1$, $R^n h = 0$, then $R^2 h = 0$.*

Proof. It can be assumed that $n \geq 3$. Setting $g = R^{n-1}h$ and $f = R^{n-2}h$ in Proposition 2.4 gives $R((R^{n-1}h)^2) = 0$. Now set $f = R^{n-2}h$ in identity (4) and obtain

$$\begin{aligned} 2R(R^{n-2}hR^{n-1}h) &= (R^{n-1}h)^2 + R(R^{n-1}h)^2 \\ &= (R^{n-1}h)^2. \end{aligned}$$

Setting $g = R^{n-1}h$ and $f = R^{n-3}h$ in Proposition 2.6 we see that the left side vanishes. Hence $(R^{n-1}h)^2 = 0$; but this implies that $R^{n-1}h = 0$ in view of the assumption.

Corollary 2.7. *Under the same assumptions on R , if f is a fixpoint of R and $R^n g = f$, then $R^2 g = f$.*

Proof. Set $h = g - f$ in the preceding statement.

Proposition 2.8. *For $n \geq 1$, the range $R^n(\mathfrak{a})$ is a subalgebra of $\mathfrak{a}$.*

Proof. Obvious from the REYNOLDS identity.

Proposition 2.9. *Let $F = \{f \mid Rf = f\}$ be the subspace of fixpoints of R . Then F is a subalgebra of $\mathfrak{a}$.*

Proof. Immediate from the REYNOLDS identity (1).

3. Reduction. From now on it will be always assumed that $\mathfrak{a} = C(S)$ is the algebra of all real-valued functions on a finite S .

Recall that a subalgebra $\mathfrak{a}'$ of $C(S)$ can be characterised as follows. Let $N \subset S$ be the set of all points $i \in S$ such that $f(i) = 0$ for all $f \in \mathfrak{a}'$. Then $\mathfrak{a}'$ is isomorphic to a subalgebra $\mathfrak{a}''$ of $C(S - N)$ such that for $i \in S - N$, $f(i) \neq 0$ for some $f \in \mathfrak{a}''$. It follows that one can find a unique partition $S_1, S_2, \dots, S_r$ of the set $S - N$ with the property that $\mathfrak{a}''$ is the algebra of all functions such that $f(i) = f(j)$ if i and j belong to the same partition S_k . In particular $\mathfrak{a}''$ is isomorphic to the algebra $C(S')$, where $S' = \{S_1, S_2, \dots, S_k\}$. The partition $N, S_1, S_2, \dots, S_k$ is called below the partition *associated with* the sub-algebra $\mathfrak{a}'$. Note that the identity element e belongs to a subalgebra $\mathfrak{a}'$ if and only if the component N of the partition associated with $\mathfrak{a}'$ vanishes.

We denote by e_i the idempotent function such that $e_i(j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. If f belongs to $\alpha = C(S)$, we denote by $S(f)$ the *carrier* (support) of f , that is, the set of all i in S for which $f(i) \neq 0$.

A *minimal* REYNOLDS operator is one for which the identity element e of α is the only fixpoint.

Proposition 3.1. *If the REYNOLDS operator R is invertible, then R is the identity operator.*

Proof. Since R is one-to-one onto, we can find h such that $Rh = e$, the identity element. From Proposition 2.5 it follows that $R(hRf) = R^2f$ for all f . Since R^{-1} exists we obtain $hRf = Rf$ for all Rf . Hence $h = e$ and $Re = e$. Now in identity (3) choose g and f so that $Rg = e_i$, $Rf = e_j$ with $i \neq j$. Then $RfRg = 0$. Operating with R^{-1} on both sides of (3) we obtain $fRg + gRf = fe_i + ge_j = 0$. Multiplying both sides by e_i we get $e_i fe_i = fe_i = f(i)e_i = 0$ for all $i \neq j$. Hence only the j th component of f may be different from zero, or $f = k_j e_j$ with $k_j \neq 0$ since R is invertible. Similarly, $g = k_i e_i$, $k_i \neq 0$, and $Re_i = k_i^{-1} e_i$ for all i . Now $e = \sum_{i=1}^n k_i^{-1} e_i$ is left fixed by R ; hence $k_i^{-1} = 1$ for all i , and $Re_i = e_i$ for all i , *q.e.d.*

Proposition 3.2. *Let $F = \{f \mid Rf = f\}$ be the subalgebra of fixpoints of R (cf. Proposition 2.9). Then $F = R^2(\alpha)$.*

Proof. Let $\alpha_0 = \alpha$, $\alpha_{n+1} = R(\alpha_n)$. Then α_n ($n = 0, 1, 2, \dots$) is a descending sequence of subalgebras. It follows that for some n_0 , $\alpha_{n_0} = \alpha_{n_0+1} = \dots$. Clearly $F \subset \alpha_{n_0}$; on the other hand, α_{n_0} is isomorphic to $C(S')$, as remarked at the beginning of this section, and the restriction of R to α_{n_0} is an onto map. It follows from the preceding Proposition 3.1 that $Rg = g$ for all g in α_{n_0} , and hence that $\alpha_{n_0} = F$. Thus $R^n g$ is a fixpoint of R for all g in α ; but this implies by Corollary 2.7 that $R^2 g$ is a fixpoint. Hence $n_0 = 2$, *q.e.d.*

Corollary 3.3. *The operator R satisfies the identity $R^2(I - R) = 0$. In particular, R^2 is an idempotent, and the only points in the spectrum of R (considered as a linear operator in the complexification of the algebra $C(S)$) are 0 and 1.*

Proposition 3.4. *Let f be an idempotent of α (that is, $f^2 = f$) fixed under R . If $S(g) \subset S(f)$ then $S(Rg) \subset S(f)$.*

Proof. We have $fg = g$ and by Proposition 2.3, $Rg = R(fg) = fRg$, *q.e.d.*

Corollary 3.5. *If f is as above, then the subalgebra $C(S(f))$ is invariant under R .*

Theorem 1. (Reduction theorem). *Let $N, S_1, \dots, S_k$ be the partition associated with the subalgebra F of fixpoints of the REYNOLDS operator R . Then each of the subalgebras $C(N), C(S_1), C(S_2), \dots, C(S_k)$ is invariant under R . Thus every REYNOLDS operator in $C(S)$ is the direct sum (in the algebra sense) of a nilpotent REYNOLDS operator and of minimal REYNOLDS operators.*

Proof. It follows from the classification of subalgebras given above that the

idempotent f_i whose support is the set S_i is fixed under R . From the preceding Proposition 3.4 it is seen that the subalgebras $C(S_1), \dots, C(S_k)$ are invariant under R . Evidently the idempotent f_i acts as an identity in the subalgebra $C(S_i)$; thus R is a minimal REYNOLDS operator when restricted to $C(S_i)$. It remains to be shown that $C(N)$ is invariant under R , where N is the "null set" of the fixpoint algebra F . Let f be the idempotent whose support is $S - N$. If $S(g) \subset N$ then $gf = 0$. By Proposition 2.3, $R(fg) = 0 = fR(g)$ since f is fixed under R . Therefore the support of Rg is disjoint from the support of f ; in other words, $S(Rg) \subset N$, *q.e.d.*

4. Structure. The main result is now established first with minimal REYNOLDS operators and then extended to the general case. We first show that there are no nilpotent REYNOLDS operators.

Proposition 4.1. *If R is a nilpotent REYNOLDS operator, then $R = 0$.*

Proof. By Proposition 2.6, $R^2 = 0$. The proof is complete if it can be shown that $Re_i = 0$ for each i , where $e_i(j) = \delta_{ij}$. Since $R^2e_i = 0$ we have by Proposition 2.4 that $R(Re_iRf) = 0$ for all f , in particular $R((Re_i)^2) = 0$. Setting $f = e_i$ in identity (4) we obtain $2R(e_iRe_i) = (Re_i)^2$. But $e_iRe_i = ke_i$, where $k = (Re_i)(i)$ (that is, the value of the function Re_i at the point i). Hence $2kRe_i = (Re_i)^2$. Taking the values of the functions on both sides at the point i we obtain $2k^2 = k^2$, whence $k = 0$. This implies that $(Re_i)^2 = 0$ and hence that $Re_i = 0$, *q.e.d.*

Corollary 4.2. *Every REYNOLDS operator is a direct sum (in the algebra sense) $R = 0 \oplus R_1 \oplus \dots \oplus R_k$, where 0 is the zero operator on $C(N)$ and R_i are minimal REYNOLDS operators on $C(S_i)$ (cf. Theorem 1).*

Corollary 4.3. *If the carrier of R^2e is S , then $Re = e$, and conversely.*

Proof. R^2e is a fixpoint of R , by Proposition 3.2. Therefore the component N in Theorem 1 is the null set, and, as remarked in the beginning of Section 3, the identity e belongs to the algebra F of fixpoints. The converse is trivial.

Theorem 2. *Every minimal REYNOLDS operator is idempotent.*

Proof. It suffices to show that the range of R is spanned by the identity element e . By Proposition 3.2, $R^2e_i = ke_i$; it suffices to prove that $Re_i = ke_i$. If $k = 0$ then the proof given to Proposition 4.1 can be repeated to show that $Re_i = 0$. If $k \neq 0$ then set $g = k^{-1}Re_i$ in Proposition 2.5 and obtain $k^{-1}R(Re_iRf) = R^2f$ for all f ; in particular the choice $f = Re_i$ gives $k^{-1}R[(Re_i)^2] = R^2e_i = ke_i$, or equivalently $R[(Re_i)^2] = k_i^2e_i$.

In identity (4) set $f = e_i$ and obtain $2R(e_iRe_i) = (Re_i)^2 + k^2e_i$. Let $k_i = (Re_i)(j)$ (the value of Re_i at the point j). Then the last identity becomes $2k_iRe_i = (Re_i)^2 + k^2e_i$; evaluating this at the point i we obtain $2k_i^2 = k_i^2 + k^2$, that is, $k_i^2 = k^2$. Evaluating the same identity at the point $j \neq i$ we obtain $2k_ik_j = k_i^2 + k^2$. This gives the quadratic equation $k_j^2 - 2k_ik_j + k_i^2 = 0$ for k_i , whence

$k_j = k_i$ for all j . Hence all the components of Re_i coincide, and $Re_i = k_i e$, *q.e.d.*

The statements of Theorems 1 and 2 together yield

Theorem 3. *Every REYNOLDS operator in a finite-dimensional algebra $C(S)$ is idempotent.*

From Corollary 4.3 we infer

Theorem 4. *The identity element e is fixed under R if and only if the carrier of Re is all of S .*

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